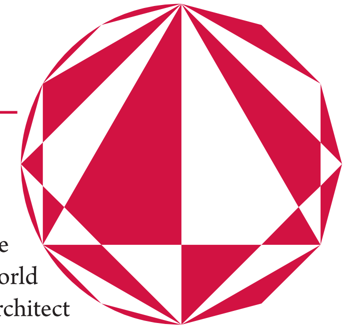
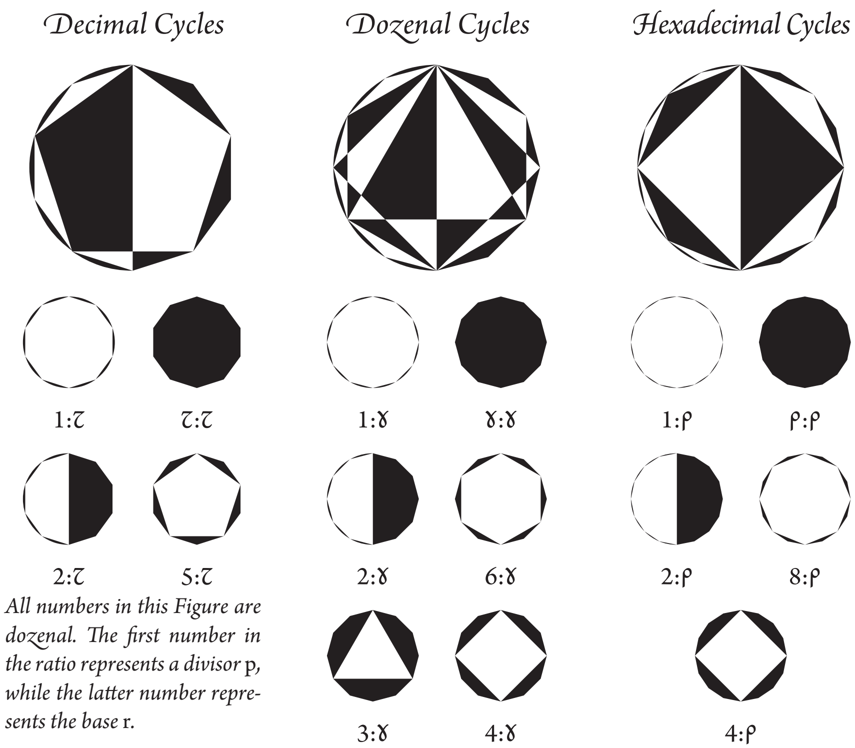
cyclic resonances

by michael de vlieger



In architecture school, the student receives a healthy dose of "visual training," an instruction that helps the future architect rely on graphic methods to analyze the world around him or her, to review his or her own ideas. The architect is trained to use visual methods, avoiding lengthy calculations when drafting. (This was especially so before the computer changed how drawings are produced). In this spirit I present a method of visualizing the relationship of integers with their divisors. The resultant diagrams are perhaps emblematic of an integer's versatility, especially when the integers are small.

The cyclic resonance diagrams (CRD) work much like a common clock, starting our work from the top of the circle. If we are evaluating the integer *r*, then we will need to divide the circle into *r* equal parts. If *r* equals a dozen, then the circle will be divided just like the clock on the wall. The only exception is that we'd start with zero rather than "twelve" at the top. We aren't "confined" to using twelve: we can use any integer and build cyclic resonance diagrams that illustrate that number's integer properties. An example of this can be found below.



All numbers in this Figure are dozenal. The first number in the ratio represents a divisor p*, while the latter number represents the base* r.

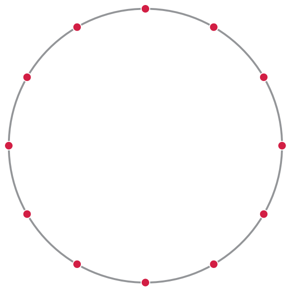


Fig. 0: *The Circumcircle.*

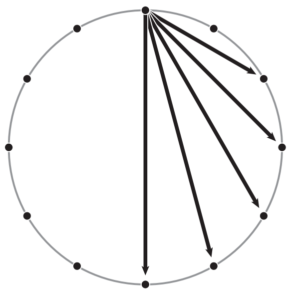


Fig. 1: *Available Avenues.*

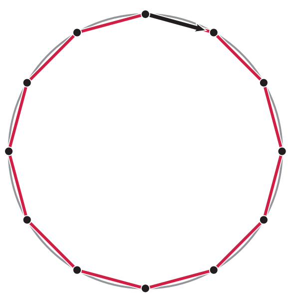


Fig. 2: *Stepping by 1's.*

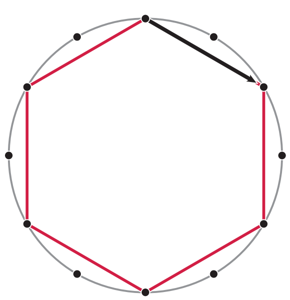


Fig. 3: *Stepping by 2's.*

Construction.

The construction of the cyclic diagram begins with a circle. This circle will serve as the "circumcircle" within which all the polygons that make up the diagram will be inscribed. Figure 0 shows the circumcircle with *r* points placed at equal distances. In this example, *r* equals twelve, so each of the dozen points lie thirty degrees apart from one another. Let's consider the topmost point the "zero point", just as on a common clock face.

There is a natural symmetry to the diagrams, as each component polygon uses the zero point as a common vertex. Thus, only half of the circumcircle and points need to be studied. The cycles established on the right side are simply reflected on the left side of the diagram. This twofold symmetry applies to odd integers as well as even. We'll return to symmetry a bit later.

With the symmetry in mind, consider those points that lie on a vertical axis through and to the right of the center of the circumcircle. Each point on the circle under consideration lies at a unique angle to the zero point. The path to each point represents a possible avenue for the generation of a regular, non-reentrant cyclic polygon (see Fig. 1). The bounding lines of such a polygon must not cross.

Intervals *s* and Polygons *p*.

We'll now successively join the points on the circumcircle to attempt to produce regular cyclic polygons. This activity is closely related both to modular mathematics and the integer properties of *r*. We are interested in regular cyclic polygons because these illustrate relationships between the integer *r*, and all integers $s < r$. The joining process represents successive addition mod *r*. The step *s* is the interval we use to join points. A polygon drawn using *s* will only be non-reentrant and close in one cycle (within one full revolution from the zero point) if *s* is a divisor of *r*. When *s* is a divisor of *r*, joining points separated at an interval of *s* will produce a regular non-reentrant cyclic polygon with *p* sides. Thus the number of sides *p* is the reciprocal divisor of *s* with respect to *r*: $s \times p = r$.

Figure 2 illustrates the relationship between $s = 1$ and $p = r$. Beginning with the point closest to the zero point, we can draw a line between each point and generate a regular cyclic polygon with *r* sides. The joining of points using an interval $s = 1$ is equivalent to successively adding 1 mod *r*. The polygon *p* is complete after *r* iterations; in the case of $s = 1$, *p* is equal to *r*, illustrating that $1 \times r = r$. Thus the number 1 is a divisor of *r*. The reciprocal divisor of *r* is *r* itself. This is the "unity-identity" pair of divisors $\{1, r\}$. Every integer possesses the unity-identity set of divisors. Also, 1 is a totative, "relatively prime" to *r*, meaning that the number 1 produces a regular cyclic polygon only after the number of iterations $i = r$. Thus, 1 is both a divisor and a totative of *r*.

Figure 3 illustrates the case s = 2. A hexagon (number of sides $p = 6$) results from drawing points 2 intervals apart. Solving the equation $s \times p = r$: $2 \times 6 = 10;$. Thus 2 and 6 are reciprocal divisors of one dozen, forming a divisor pair $\{2, 6\}$. This polygon *p* was produced when the number of iterations $i = p$; *i* and *p* are equal when *s* is a divisor of *r*.

Figure 4 shows the production of a square when s = 3. The square is a regular cyclic polygon with a number of sides $p = 4$. Three and four are reciprocal divisors of one dozen, forming a third divisor pair $\{3, 4\}$.

Figure 5 illustrates that a triangle ($p = 3$) is produced when the interval $s = 4$. This restates and reinforces the third divisor pair $\{3, 4\}$. Again, *i* and *p* are equal for $s = 3$ or 4, because those integers are divisors of *r* = twelve.

When $s = 5$, as shown by Figure 6, a non-reentrant cyclic polygon is not produced. The first step from the zero point joins 0 and 5. The second step joins 5 and ↊. The sum of 5 and ↊ is greater than r, making a polygon which closes within one cycle impossible. Thus 5 is not a divisor of one dozen. There is no integer *p* that, when multiplied by $s = 5$, will produce *r* = one dozen. We can produce a regular closed "star" polygon using the interval s = 5 when the number of iterations $i = r$. Five is a totative of one dozen (i.e. five is coprime to twelve).

Figure 7 illustrates a "line" results when s = 6. This is a special case peculiar to the diagrams. Technically, the "line" is a "polygon" with 2 sides called a "digon". It is formed by joining the zero point to the point at 6, then returning to zero. Despite its appearance, it confirms what Figure 3 introduces: there is a pair of divisors {2, 6} for *r* = one dozen.

The Completed Diagram

Since we've arrived at $s = r/2$, we've exhausted all possible avenues and can produce a cyclic diagram by overlaying all the results which produced regular convex cyclic polygons (see Figure 8). The left side of the diagram can be regarded as reflections of the right side.

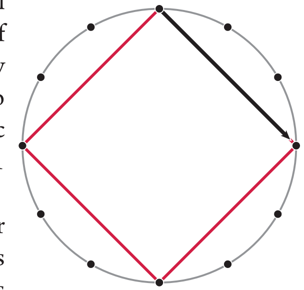


Fig. 4: *Stepping by 3's.*

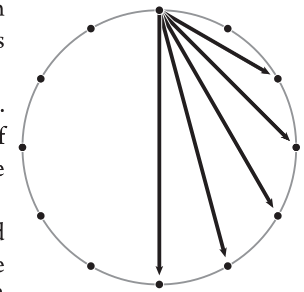


Fig. 5: *Stepping by 4's.*

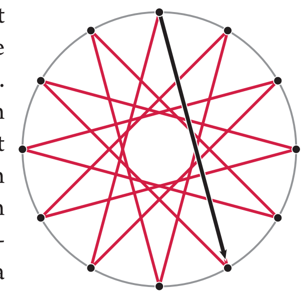


Fig. 6: *Stepping by 5's.*

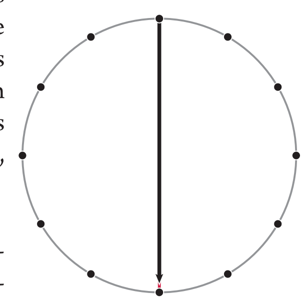


Fig. 7: *Stepping by* $6 = r/2$.

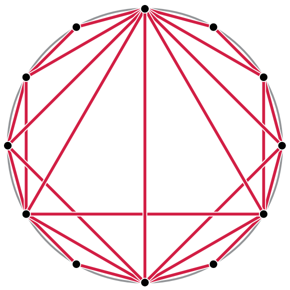


Fig. 8: *Cycles.*

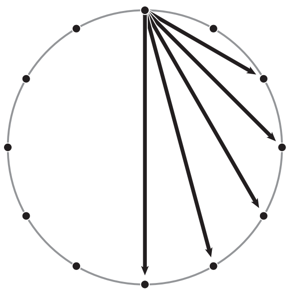


Fig. 9: *Cycles, shaded.*

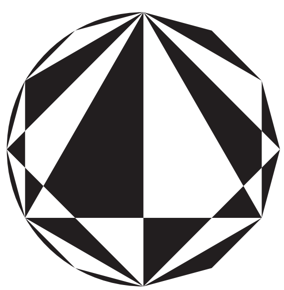


Fig. Շ: *The Cyclic Resonance Diagram.*

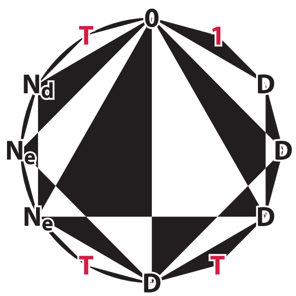


Fig. ᒥ: *Digit classes of the dozen.*

Symmetry and Resonance

The points on the circumcircle are joined by the twelve-sided polygon. This twelve-sided polygon or, speaking generally, the *r*-gon represents the trivial or "unity-identity" divisor pair {1, *r*} common to all integers. The symmetry of the diagram, within this *r*-gon, conveys the symmetry of the properties of integers.

Totatives exhibit an additive symmetry: they are paired such that when the elements of a paired set of totatives are summed, they equal *r*. Thus for every totative pair $\{t_1, t_2\}$, $t_1 + t_2 = r$. Every integer possesses the {1, *r*-1} totative pair, along with additional pairs for every number greater than 6. Twelve has two pairs of totatives: {1, Ɛ} and {5, 7}. The diagram for twelve situates the totatives horizontally across from one another. These are marked in red in Figure Ɛ. (The number 1 appears as a numeral because it is both a divisor and a totative of *r*.) So in the case of the position horizontally across from 5, we can employ the property of symmetry to surmise that *s* = 7 will project the path generated by *s* = 5, as shown by Figure 6, in reverse. The same is true for the position horizontally across from 1, despite the special nature of 1. The path generated by *s* = b is simply that of *s* = 1, shown by Figure 2, in reverse.

The totatives, integers less than *r* which are relatively prime to *r*, are vertices only of the *r*-gon. In this example, the set of totatives are {1, 5, 7, ᒥ}.

Examination of the completed diagram shows the relationships among the divisors of *r*. Figures 10; and 13; illustrate the reciprocal divisor pair {2, 6}, while Figures 11; and 12; illustrate {3, 4}. The non-trivial divisors of *r* outside of the set {1, *r*}, thus the elements of the set {2, 3, 4, 6} each occupy vertices of more than one polygon. If we look at the point at "3 o'clock" (equivalent to 3), we can see that it is a vertex of the twelve-sided polygon. Additionally, 3 is a vertex of a square, a 4-sided polygon.

The divisors of an integer exhibit multiplicative symmetry, which is demonstrated in the relationship between the interval *s* and the resultant polygon *p*. This relationship is reversable: consider how the interval *s* = 4 produces a polygon with *p* = 3 sides as seen in Figure ∂, and how the interval *s* = 3 produces a polygon with *p* = 4 sides in Figure Ɛ. The number 1 is special, again, because it is both a divisor of *r* and also relatively prime, thus a totative of *r*. Figure ᒥ indicates divisors by the letter "D".

The last class of digits, integers less than *r*, are those that are neither divisors nor totatives of *r*. They occupy positions that remain when divisors and totatives are eliminated. Note that twelve is not exemplary, having all the nontotative-nondivisors on the right side of the diagram. Some numbers, like ten, have a nontotative-nondivisors on the right. Four is not a divisor of ten, nor is it relatively prime to ten. It is the product of two divisors (both instances of the divisor 2).

Integers less than *r* which are neither totatives nor divisors, in this example {8, 9, Շ}, form vertices of multiple polygons like the divisors, but themselves do not "generate" the polygons. In the case of 8, this vertex is joined to the triangle generated by an interval *s* = 4; 8 simply is the next point on the triangle after 4. Likewise, Շ is the final point on a hexagon generated by an interval *s* = 2. (There are nuances of relationships within the set of "nontotative nondivisors" that are illustrated in these figures which we will describe in a later article.)

Graphic Treatment.

The diagrams presented here employ a graphic technique called "exclusion". This technique reverses the color of that portion of an object that lies "on top" of any portion of another object. This technique tends to produce the clearest diagrams, though certainly there are other ways to produce these diagrams. The effect of exclusion ultimately amounts to alternating the color of the "slices" defined by the intersection of the sides of various polygons (see Figure 9). Figure Շ shows the simplest manifestation of the diagram. Though the diagrams can be used to analyze the integer properties of a given integer when accompanied by the numbering of each point, the diagrams function well without annotation when compared side by side.

Because these diagrams are succinct, it becomes possible to compare integers visually. The diagram "Dozenal Cycles" on page 2·1·0 presents the integer twelve along with all the resonances associated with each divisor below it. Compare this to the "Decimal Cycles" and "Hexadecimal Cycles" on the same page. It's evident that ten features fewer divisor relationships than the dozen. Ten has four divisors {1, 2, 5, ᒥ}, and four totatives {1, 3, 7, 9}, while the dozen has six divisors and four totatives. Hexadecimal cycles are often cited as an appealing alternative to decimal or dozenal numeration. Visually comparing the diagram for twelve and that of sixteen makes evident the denser resonances of the dozen. Sixteen has 5 divisors {1, 2, 4, 8, ƿ} and eight totatives (every odd number lesser than *r* = sixteen!) It seems clear, looking at the diagrams, that a greater number of relationships, versatility, and proportionally fewer obstacles (as conveyed by totatives) await us if we were to use the dozen.

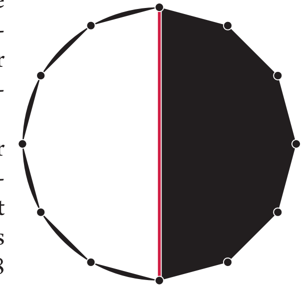


Fig. Ɣ: *The 2–6 Relationship.*

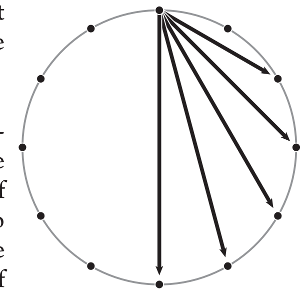


Fig. ∂: *The 3–4 Relationship.*

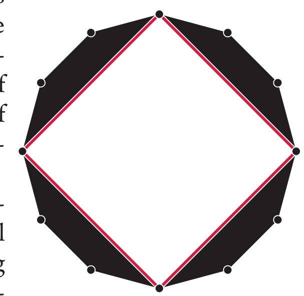


Fig. Ɛ: *The 4–3 Relationship.*

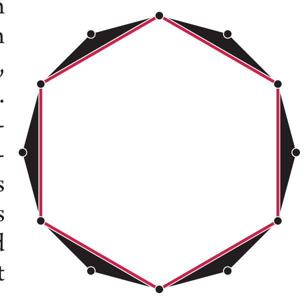
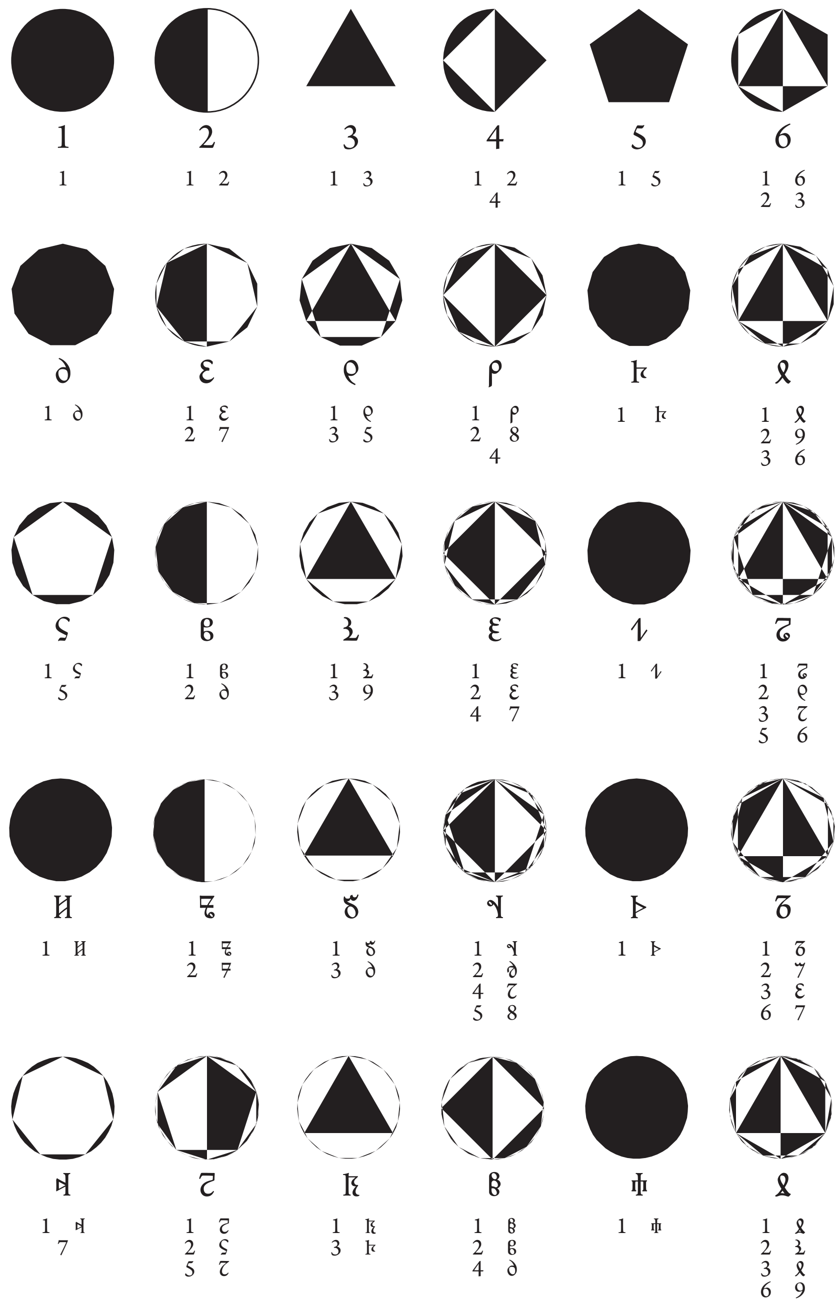
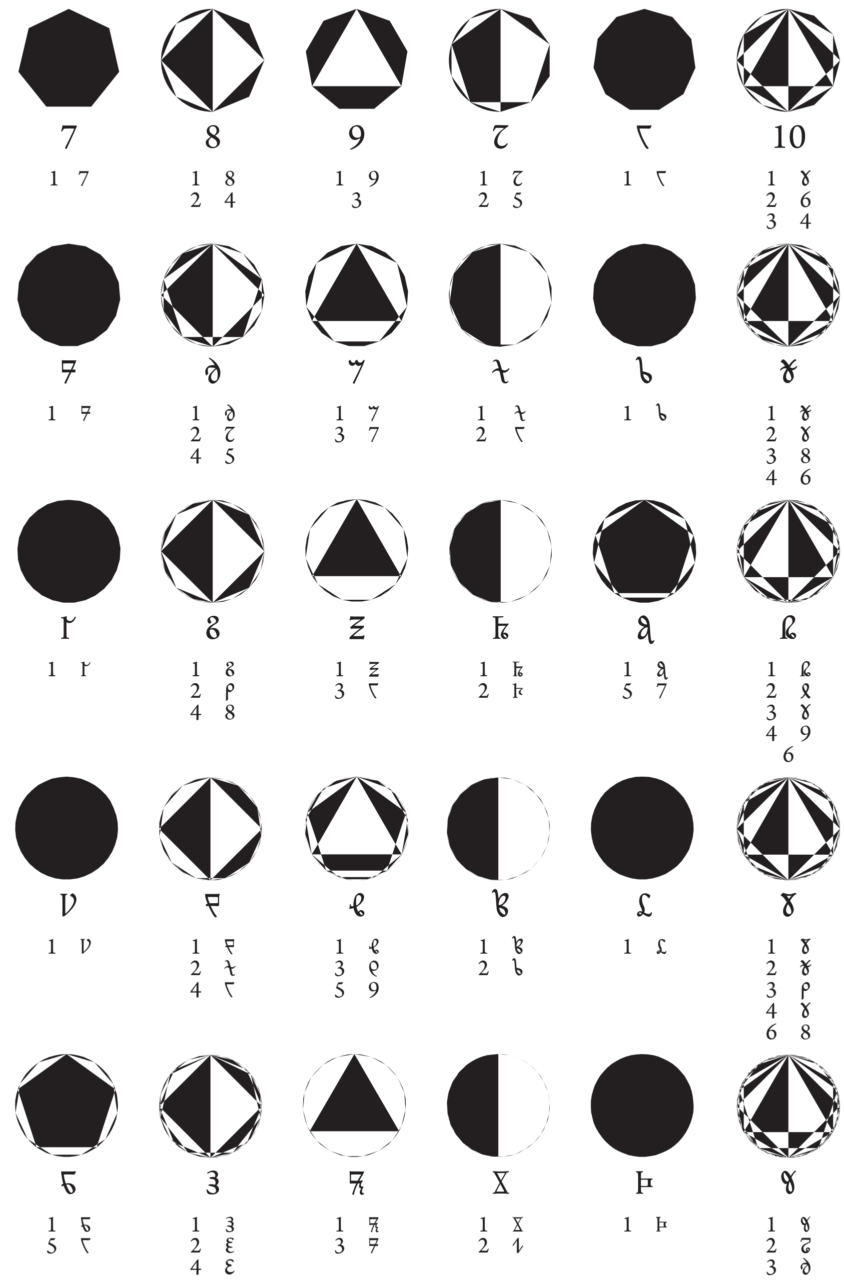


Fig. ϙ: *The 6–2 Relationship.*

Cyclic Resonance Diagrams

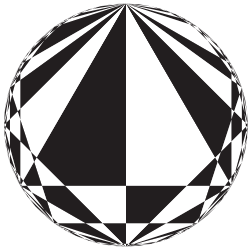


The integer r corresponding to the diagram appears below the diagram. The divisors of r appear below the integer, with reciprocal divisors paired in the same row.

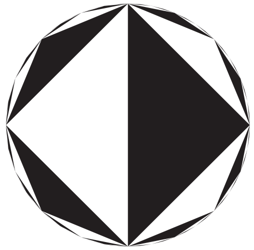


All figures are argam. Chart produced in 2009 by Mike De Vlieger.

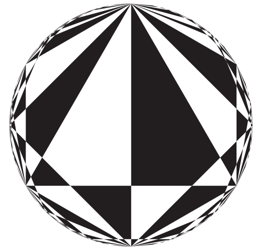
Cyclic Resonance Diagrams of Larger Composite Integers



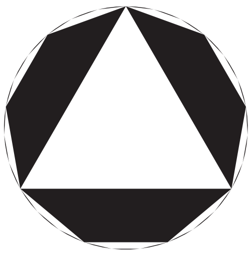
10 (60.)



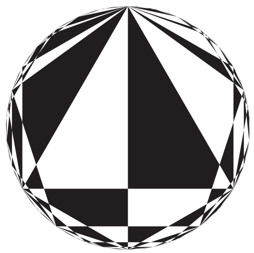
14 (64.)



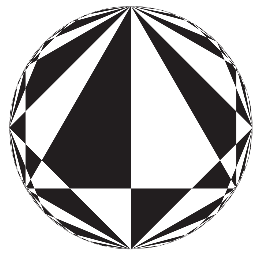
1ɤ (72.)



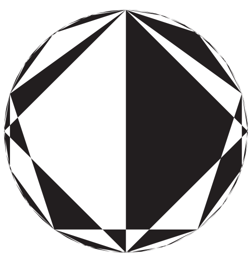
17 (81.)



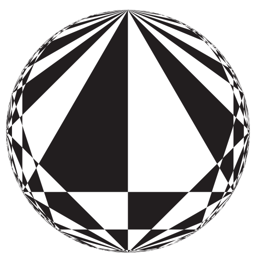
1Ꙅ (90.)



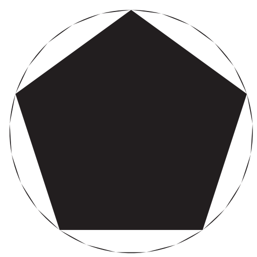
1ꝭ (96.)



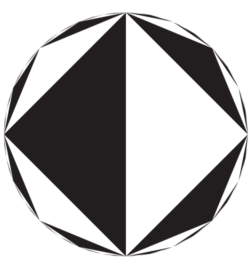
1ዓ (100.)



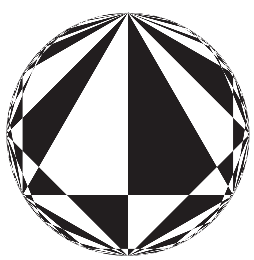
20 (120.)



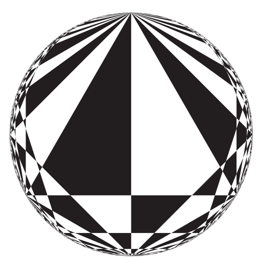
25 (125.)



28 (128.)



2ɤ (144.)



60 (360.)

Larger integers can have especially complex diagrams. The simpler divisors of these numbers are perhaps more evident and discernable than the larger divisors. Integers are given in sexagesimal argam, with decimal in parentheses. Chart by Michael De Vlieger.

The Wider View.

The spread on pages 2·1·5–6 further illustrates the first five dozen integers. Observe the prime numbers; these appear as black polygons, since these are divisible only by themselves and 1. (Two appears as a half black, half white circle in order to accentuate its status as a "digon".) Highly composite numbers are rather easily picked out. This chart was arranged so that a dozen integers appear on each row. This arranges the chart in a way that confines the primes to the dozenal totatives {1, 5, 7, Ɛ}, which correspond to {ɤn±1, ɤn±5}. Even numbers appear as diagrams that appear split in half. Integers divisible by three feature a rather prominent triangle. Those divisible by four feature a prominent square. Even though five is not a factor of twelve, the multiples of five communicate their composition, proudly displaying a pentagon. Multiples of six feature a hexagon. Integer properties displayed for divisors of larger integers are inherited by the larger integers. Primes increasingly resemble black circles, their bounding segments becoming indistinguishable from circles beyond around one dozen five or seven.

Diagrams of Large Integers.

As the integers get larger, the larger divisors of these integers become more difficult to discern. A thirty-sided polygon that is nested within a sixty-sided polygon is a challenge for most folks. Still, the intricacy of such figures points to the greater versatility of the integer five dozen over other integers such as four dozen eleven or other neighbors.

The diagrams on page 2·1·7 illustrate some highly composite numbers and powers of simple primes larger than five dozen. These demonstrate intricacy around their edges that make the diagrams less useful analytically, but still indicative of the lower divisors of the number in question.

Some Conclusions.

The author invites the reader to ponder the diagrams. These diagrams are rather universal since they are produced by geometry. The properties of the integer r are demonstrated automatically through these geometric diagrams. Outside the convention of representing the "digon" or two-sided "polygon" as a half-circle and the general application of the graphic tool of "exclusion", nothing is done to "process" the geometry.

It may be possible to consider these resonance diagrams as logograms, especially of the smaller integers, because the diagrams convey so many intrinsic integer properties at a glance. The great thing about them is that one needs not know how to read a numeral to see the relationships between an integer and its divisors and totatives. Although it may be difficult for a human user to employ them as digits, they could certainly serve as universal digits of an "infinite base".

Comparing highly composite numbers like five dozen with powers of two like five dozen four, we can see the various avenues of versatility emanating from the zero points. It's apparent that sixty features pathways in directions sixty four doesn't cover. We can also observe the apparent diminishing returns that are contributed by increasing the power of various primes in the prime composition of related integers, as seen in the diagrams of 50; $(2^2 \times 3 \times 5)$ and 260; $(2^3 \times 3^2 \times 5)$.

The cyclic resonance diagrams perhaps represent a tool by which we might analyze the properties of integers, especially the small integers. They also offer a visual means of examining and comparing the properties of a range of integers. Finally, they stand as a beautiful representation of the natural symmetry and resonances embodied in each integer.