Working Second Draft

Turbulent Candidates

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Abstract

This paper describes several conditions that pertain to the appearance of terms in OEIS A244052, "Highly regular numbers $a(n)$ defined as positions of records in A010846." The sequence is arranged in "tiers" T wherein each member n has equal values $\omega(n) =$ OEIS A001221(n). These tiers have primorials $p_T\#$ = OEIS A002110(T) as their smallest member. Each tier is divided into "levels" associated with integer multiples $kp_T\#$ with $1 \le k < p_{(T+1)}$. Thus all the terms of OEIS A060735 are also in A244052. Members of A244052 that are not in A060735 are referred to as "turbulent terms." This paper focuses primarily on the nature of these terms. We give parameters for their likely appearance in tier T as well as methods of efficiently constructing them. We compare the sequence $a(n)$ produced by this necessary-but-insufficient set of conditions with actual data from A244052 and show that the sequence $a(n)$ can serve as prevalidated candidates for testing for inclusion in A244052. Finally, the paper explores computation methods related to $a(n)$ and A244052.

1. Introduction

This paper presumes an understanding of the concepts of prime and composite numbers and the Unique Factorization Theorem that pertains to the prime decomposition of nonzero integers. Also presumed is the notion of divisibility and coprimality, of the greatest common divisor (i.e., the highest common factor) of two integers, that 1 is the "empty product" and a divisor of all integers that is simultaneously coprime to all integers.

Hereinafter we shall refer to the Online Encyclopedia of Integer Sequences as "OEIS." This is a peer-reviewed cataloged and searchable repository of integer sequences. We shall refer to the OEIS when integer sequences of interest appear in the database.

Multiplicity notation. Let n be a positive nonzero integer. We can write n as the product of primes thus:

(1.1) $n = {p_1}^{e_1} \times {p_2}^{e_2} \times \ldots \times {p_k}^{e_k}$
with $p_1 < p_2 < \ldots < p_k$

The numbers e are multiplicities of primes p that are factors of n. This is the standard form of prime decomposition of n. If the primes that divide n are small and the set compact, we can conveniently "abbreviate" the standard form by using a sort of positional notation such that position k signifies PRIME(k), with the value e_k written in position k. For example, $n = 75 = 3 \times 5^2$, thus we would write 210, with zero holding the place of 2 (as another way of writing 75 is $2^0 \times 3^1 \times 5^2$). This notation is embodied in OEIS A054841. We need not conjoin the multiplicities e (since once $e > 9$, the notation would not be able to express n properly in decimal) nor would the order of places need be bigendian. We could instead express $75 = 2^0 \times 3^1 \times 5^2$ as $\{0, 1, 2\}$. We shall refer to this notation simply as "multiplicity notation". Since all cases in this paper involve numbers whose factors have multiplicities that never approach 9, we will simply concatenate the digits, thus rendering $75 = \{0, 1, 2\}$ as "012".

In multiplicity notation, a zero represents a prime totative $q <$ GPF(n) = A006530(n), the greatest prime factor of n. Indeed the multiplicity notation A054841(n) of any number n does not express the infinite series of prime totatives $q >$ GPF(n). We could also write A054841(75) = 01200000…, with an infinite number of zeroes following the 2. Because it is understood that the notation "012" implies all the multiplicities "after" the 2 are 0.

Regulars. Consider two positive nonzero integers m and n. The number m is said to be "regular to" or "a regular of" n if and only if $m \mid n^e$ with integer $e \ge 0$. This paper is concerned with $1 \le m \le n$ but clearly $m > n$ also can divide n^e and be called "regular to" n.

It is plain from this definition that regular m may divide n^1, thus divisors $m = d \mid n$ are a special case of regular number with respect to n. We can say that if $d \mid n$ then d is regular to n.

Another way to think of regular m of n is this. The number m is said to be "regular to" or "a regular of" n if all prime divisors p of m also divide n and no primes q coprime to n divide m. The empty product 1 is regular to n since there are no primes q coprime to n that divide 1.

We know that if $m \mid n^e$ then $m \mid n^{(e+1)}$ since incrementing the integer power of n does not reduce the set of distinct prime divisors. This implies there is a least exponent ρ such that $m \mid n^\rho$ that would have $m \mid n^{(\rho-1)}$ false. In the case of divisors d, we have two cases. For $d = 1$, $1 \mid n^0$ thus $\rho = 0$. For $d > 1$, $d \mid n^1$, thus $\rho = 1$. Let us call the least exponent ρ the "richness" of regular m with respect to n.

We can "search" for by using a `While` statement and iterating ρ. There is a way to definitely compute ρ. Let's consider some facts about regular m of n in the range $1 \le m \le n$.

$$1 \mid 1$$

The only number in the range of $n = 1$ is 1 itself; since 1 divides all integers, 1 is regular to itself. Thus $1 \mid 1^0$ and the maximum $\rho = 0$ for regular m in the range of n for $n = 1$.

$$1 \mid n^0$$

We know that divisors d are regular and that 1 is a divisor of all integers n. Therefore 1 is regular and $1 \mid n^1$. However, we know that all numbers divide themselves thus $1 \mid 1$ and we know $n^0 = 1$. Therefore $1 \mid n^0$ for all integers n. Thus for $m = 1$, $\rho = 0$ for all n.

$$d \mid n^1$$

The definition of a divisor d of n is that $d \mid n$. We know that $n = n^1$. Thus for all divisors d of n, $\rho = 1$ by definition. This suggests that $\rho > 1$ for nondivisor regular m, i.e., for m that are "semidivisors" of n. This implies, perhaps obviously, that the maximum power ρ of n such that all regular $m \mid n$ is 1.

$$p \mid n^1 \text{ and } m \mid p^1$$

Consider prime p regular to $n > 1$. It is clear that p must divide n. Thus richess $\rho = 1$ for prime p regular to $n > 1$. (Note there are no $p \le 1$ regular to 1).

Consider n = prime p. Then all regular $1 \le m \le p$ must divide p. To be sure, the only regulars in this range are 1 and p. Thus the richess $\rho = 1$ for all regular $m > 1$ of prime p, since all regular m of prime p are divisors of p.

$$m \mid n^1 \text{ for } n = p^k$$

Consider $1 \le m \le p^k$, i.e., regulars m in the range of a perfect prime power with $k > 1$. Since there is one distinct prime divisor p of perfect prime power p^k, the only regular numbers in this range are the powers $p^e \le p^k$, i.e., $\{1, p, p^2, \ldots, p^{(k-1)}, p^k\}$ with $0 \le e \le k$. It is clear that all such powers divide p^k, thus all regular $1 \le m \le p^k$ are divisors of p^k. Therefore $\rho = 1$ for all regular $m > 1$ of prime p.

We can generalize and say that $m \mid n^1$ for n with $\omega(n) = 1$, i.e., for n with one distinct prime divisor p. This implies that, across all regulars in the range of n, the maximum value of ρ such that $m \mid n^\rho$ for n prime or $n = p^k$ is 1. We can consider the set of divisors a 1-rank tensor that is the power range of the prime divisor p bounded by n, i.e., p raised to the powers $0 \le e \le k$.

$$\text{Semidivisor } m \mid n^\rho \text{ with } \rho > 2$$

It is easy to show that there is at least 1 nondivisor regular m (semidivisor m) in the range $1 \le m \le n$ for composite $n > 4$, and that such m must be composite, since prime m must either divide or be coprime to any n. The only composite regular is $m = 4$ for $n = 4$, $4 \mid 4$ and 4 is the smallest composite number, thus there are no semidivisors for $n = 4$. This implies that, across all regulars in the range of composite $n > 4$, the maximum value of ρ exceeds 1, while for $n = 4$, maximum $\rho = 1$.

Nondivisor regular m have at least one prime $p \mid n$ whose multiplicity in m exceeds that in n. An example of this is $m = 4$, $n = 10$; GCD$(4, 10) = 2$, but the multiplicity of 2 in 4 is 2, while that in 10 is 1. Therefore, 4 does not divide 10 despite the fact it is regular to 10. Another way to look at this is that nondivisor regular m are "too rich" in the prime p to divide n. Suppose the prime decomposition of semidivisor $m = p^i \times q^j$ while $n = p^k \times q^\ell \times r$, with $p < q < r$ prime and $i > j > k > \ell$. Considering only the primes common to both m and n, richness can be computed thus:

(1.2) $\rho = \text{MAX}(\text{CEILING}(i/k), \text{CEILING}(j/\ell))$.

<u>Regular counting function</u>. Let $r(n)$ = OEIS A010846(n) be a function that counts the number of regulars of n. We can call $r(n)$ the "regular counting function," akin to the divisor counting function. There are several practical methods of counting the regulars of n that we will explore in greater depth in Section 4. Values of $r(n)$ for $1 \le n \le 540$ appear in Appendix A1.

<u>A244052</u>. Let $a(n)$ = OEIS A244052(n) be an integer sequence in which those numbers b that "set records" for the value of $r(b)$. In other words, sequence A244052 can be defined thus:

A244052(1) = 1, and
A244052(n) is the least number $k >$ A244052$(n-1)$
such that A010846$(k) >$ A010846(A244052$(n-1)$).
Values of A244052 appear in Appendix B1.

<u>Primorial</u>. A "primorial" is the product of a contiguous and distinct set of the smallest primes starting with $p_1 = 2$. The first primorials are:

$$2 = 2,$$
$$6 = 2 \times 3$$
$$30 = 2 \times 3 \times 5$$
$$210 = 2 \times 3 \times 5 \times 7$$

thus, $p_k\#$ is the product of the smallest k primes, with none of the primes having multiplicity $e > 1$, and the number $\omega(p_k\#)$ of the distinct prime divisors is k, with $\omega(n)$ = OEIS A001221(n). The sequence of primorials are found at OEIS A002110. For the purposes of this paper, we will call $p_k\#$ the "primorial on k."

Primorial $p_k\#$ is the smallest number n to have $\omega(n) = k$ distinct prime factors, since the product is made by multiplying $p_{(k-1)}\#$ with the prime p_k. By definition, there are no interposing primes between $p_{(k-1)}$ and p_k. Multiplying $p_{(k-1)}\#$ with any other prime $p_{(k+i)}$ with integer i positive, not already in the sequence will produce a number $p_{(k-1)}\# \; p_{(k+i)} > p_k\#$. (Multiplying $p_{(k-1)}\#$ with a prime that is a factor of same produces a number with $k - 1$ distinct primes with one prime having multiplicity $e = 2$.)

2. The *T*-Rank Regular Tensor of *n*.

The distinct prime divisors p of n determine the number of "dimensions" in the "matrix" or rank of the tensor of regular numbers m of n. Multiplicity of any prime merely increases the bound imposed on the infinite tensor. The next section expounds upon this.

Using the algorithms mentioned above, we can generate the regular m of n a number of ways. The result is a list. For example, the regulars $1 \le m \le n$ of $n = 10$ are $\{1, 2, 4, 5, 8, 10\}$.

We can approach generating the regulars of n in a way akin to a way divisors might be arranged according to the divisor counting function.

<u>Tensor implied by divisor counting function</u>. Let's look at the divisor counting function, keeping in mind the standard form of prime decomposition of n (Formula 1.1):

(2.1) $\tau(n) = (e_1 + 1) \times (e_2 + 1) \times \ldots \times (e_k + 1)$

We could produce an $\omega(n)$-rank tensor **D** that contains all the divisors of n. Divisor tensor **D** is thus the product of the power ranges **P** of each prime divisor p of n such that all powers are integers that divide n, i.e., $\{1, p, p^2, \ldots, p^e\}$:

(Fig. 2.1)

	p_1^e		
p_2^e	1	p_1	p_1^2
	p_2	$p_1 p_2$	$p_1^2 p_2$

Figure 2.1 shows the divisors of $n = p_1^2 \times p_2$. The rank of the tensor is 2 since $\omega(n) = 2$. It follows from Formula 2.1 that n with 3 distinct prime divisors would produce a 3-rank tensor of divisors, and those with a single prime divisor would produce a 1-rank tensor of divisors, etc. The divisor counting function $\tau(n)$ is the product of the $\omega(n)$ terms $(e_k + 1)$.

Thus we can arrange the divisors of any nonzero positive integer n into $\omega(n)$-dimensional array, i.e., $\omega(n)$-rank tensors. Further examples appear in Figure 2.2. "Rasterized" versions of these tensors appear in OEIS A275055.

p^e	
1	p

Prime $n = p$

		p^e		
1	p^1	p^2	…	p^k

Prime power $n = p^k$

	p_1^e	
p_2^e	1	p_1
	p_2	$p_1 p_2$

(FIG. 2.2)

Semiprime $n = p_1 \times p_2$

	2^e		
3^e	1	2	4
	3	6	12

$n = 12$

p_3^0	p_1^e	
p_2^e	1	2
	3	6

p_3^1	p_1^e	
p_2^e	5	10
	15	30

$n = 30$

THE INFINITE REGULAR ARRAY. We can build a tensor that contains all the regulars $1 \le m \le n$ of n in a related manner. First let's return to the notion that there is an infinite set of regular m of n, and that the set of m need not be bound by n. In this case there is an infinite array **I** of regular m purely dependent on the distinct prime divisors p of n. Numbers n that have the same squarefree kernel OEIS A007947(n) share the same infinite array of regulars. These might be annotated according to A007947(n) thus: $\mathbf{I}_6$ is the infinite array of regulars of $n = \{6, 12, 18, 24, 36, 48, 54, 72, 96, 108, \dots\}$, i.e., numbers n that share the same value of A007947$(n) = 6$. Thus we might note the infinite array of regulars generally $\mathbf{I}_{\text{A007947}(n)}$. Some such arrays appear in the OEIS sorted. For example, $\mathbf{I}_2 = 2^n =$ A000079, $\mathbf{I}_6 =$ the 3-smooth numbers, A003586, $\mathbf{I}_{10} =$ A003592. The infinite regular arrays **I** represent a T-rank tensor that is the tensor product of the infinite prime power ranges $p^{(0\dots\infty)}$ of the prime divisors of A007947(n).

(FIG. 2.3)

	2^e							
3^e	1	2	4	8	16	32	64	…
	3	6	12	24	48	96	192	…
	9	18	36	72	144	288	576	…
	27	54	108	216	432	864	1728	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Figure (2.3) shows the origin and environs of the infinite array of regular numbers of a 3-smooth number n. Observe that we can derive both the set of divisors and the set of regulars of any number n from $\mathbf{I}_{\text{A007947}(n)}$.

THE REGULAR TENSOR AS AN EXTENSION OF **D**. Considering the numbers regulars $1 \le m \le n$ of n bounds the power ranges **P** of the prime divisors p such that no power $p^e > n$. Additionally, the $\omega(n)$-rank tensor **R** of regular $1 \le m \le n$ of n derives from **I** bounded by n. The divisor tensor **D** is the product of power ranges **P** such that all integer powers divide n, while the regular tensor **R** is the product, bounded by n, of power ranges **P** also bounded by n. The first is an orthogonal array, the latter involves an irregular "surface" that is bounded by n, and both are contained in **I**. The geometry of **R** approximates an $\omega(n)$-dimensional right simplex with a curved non-orthogonal sheet that joins the largest values of all distinct prime power ranges **P** bounded by n.

For n with $\omega(n) = 1$, $\mathbf{R} = \mathbf{D}$, i.e., the regular array is the divisor array, since the only regular $1 \le m \le n$ of n that is prime or a perfect prime power are $m \mid n$, i.e., divisors of n.

For n with $\omega(n) = 2$, **R** appears as a roughly "triangular" chart when we have $\omega(n) = 2$. The prime power ranges form the orthogonal sides of the "triangle," while the "hypotenuse" is actually a quasi-curve bounded by n. Figure (2.4) shows three values of n that share the same infinite regular array. Note that the divisors of n occupy a rectangle defined by the empty product at the upper left and n itself at the lower right. The divisors of n appear in a "box" inscribed in the "triangle."

(FIG. 2.4)

	2^e		
3^e	1	2	4
	3	6	

$n = 6$

	2^e			
3^e	1	2	4	8
	3	6	12	
	9			

$n = 12$

	2^e					
3^e	1	2	4	8	16	32
	3	6	12	24		
	9	18	36			
	27					

$n = 36$

For n with $\omega(n) = 3$, **R** appears as a 3 dimensional analog to the "right triangular" charts above, i.e., a "tetrahedron" with three orthogonal surfaces and one irregular sheet bounded by n. The divisors occupy a 3 dimensional "box" inscribed in **R**.

(FIG. 2.5): $n = 30$

p_3^0	p_1^e				
p_2^e	1	2	4	8	16
	3	6	12	24	
	9	18			
	27				

p_3^1	p_1^e		
p_2^e	5	10	20
	15	30	

p_3^2	p_1^e
p_2^e	25

THE GENERAL GEOMETRY OF **R**. Generally, for n with $\omega(n) = 3$, the geometry of **R** is that of a $\omega(n)$-dimensional simplex with orthogonal axes that are distinct prime divisor power ranges **P** bounded by n and one compound curved sheet or surface bounded by n that joins the largest values in all **P**. Inscribed with in **R** is **D**, occupying all $\omega(n)$ dimensions from the origin at the empty product (i.e., 1) diagonally through $\omega(n)$-space to n in the inexact middle of the curved sheet. **D** has all orthogonal axes and is a $\omega(n)$-dimensional parallelotope easily described by algorithm implied by the divisor counting function $\tau(n)$. "Rasterized" or "line-by-line" versions of tensors **R** appear in OEIS A275280.

The geometry of the curved sheet may be amenable to calculus and is beyond the scope of this paper.

"COMPACT" SET OF DISTINCT PRIME DIVISORS. Let's return to the notion of **R** as produced by distinct prime divisor power ranges **P** bounded by n. It follows that when the distinct prime divisors p are more similar, the resultant matrix is closer to a square, since their power ranges **P** bounded by n have a similar number of terms. Let's disregard the fact that when we switch out a smaller for a larger prime the magnitude of n increases and thus expands the bound on **R**, since the infinite regular arrays **I** are necessarily distinct. To compensate, we can examine two examples involving identical values of $\omega(n)$ with comparable magnitudes. Let's look at the effect of the power ranges **P** on the overall shape of **R**.

(FIG. 2.6)

3^e \ 2^e					
1	2	4	8	16	32
3	6	12	24	48	
9	18	36			
27					

$n = 48 = 2^4 \times 3$

23^e \ 2^e					
1	2	4	8	16	32
23	46				

$n = 46 = 2 \times 23$

Figure 2.6 shows the range of $\mathbf{P}_2$ is similar, but that of $\mathbf{P}_3$ and $\mathbf{P}_{23}$ are markedly different. Despite the fact that 46 and 48 are rather similar in magnitude, the factor 3 admits more regulars from $\mathbf{I}_6$ than 23 does from $\mathbf{I}_{46}$. There is some dependency of $r(n)$ on the length of the power ranges **P**, but due to the bound imposed by n, we cannot compute $r(n)$ exactly using those lengths. The number of terms in the distinct prime divisor power ranges **P** bounded by n can be computed thus:

$$\delta_p = 1 + \text{FLOOR}(\log_p n) \qquad (2.2)$$

Suppose we have two primes, $p_1 < p_2$. Then the length of the prime power range bounded by n, $\delta_1 > \delta_2$ of p_1 and p_2 respectively. The fact that the power ranges **P** bounded by n are shorter for larger prime divisors p implies that a "compact" set of prime divisors makes for a more "efficient" configuration of prime factors, that is, more terms in regular tensor **R** for numbers comparable in magnitude.

3. The Effect of Multiplication on Tensor R.

Let's consider the effect of multiplication on the tensor **R**. A prime number must either divide or be coprime to n. Therefore we consider the following cases.

CASE 1 involves pn, the product of n and a prime p that divides n. Since p by definition is prime, the product $pn > n$. In Section 2 we saw that numbers n that have the same squarefree kernel OEIS A007947(n) share the same infinite array **I** of regulars. It follows that A007947(n) = A007947(pn), since the prime p already occurs among the prime divisors of n, i.e., $\omega(n) = \omega(pn)$. The multiplicity of p in n increments by 1 and must be greater than 1. No new prime divisors are introduced and none are lost, thus the squarefree kernel of n is that of pn. Since $\omega(n) = \omega(pn)$, the rank of the infinite regular tensors is the same. These facts together imply the new tensor $\mathbf{R}'$ is merely **I** bounded at a larger value pn. Therefore, multiplication of n by a prime $p \mid n$ merely increases the bound and admits more regular terms in the infinite regular array **I** common to both n and pn.

CASE 2 involves qn, the product of n and a prime q coprime to n. Since q by definition is prime, the product $qn > n$. The infinite regular array of n cannot be that of qn, since the distinct prime factors of qn have an additional prime factor q missing in n, i.e., A007947$(n) \neq$ A007947(qn). Furthermore, $\omega(n) < \omega(qn)$, more precisely, $\omega(qn) = \omega(n) + 1$. This implies the new infinite array $\mathbf{I}'$ pertaining to qn has a higher rank than **I** pertaining to n. Specifically, $\mathbf{I}'$ has 1 more dimension than **I**. Since $qn > n$, $\mathbf{I}'$ is bounded at a larger value. The distinct prime power ranges **P** of n appear in $\mathbf{I}'$ and all the terms for each prime divisor p that are less than qn occur in $\mathbf{I}'$ along with those attributable to q. Therefore $r(qn)$ must be significantly larger than $r(n)$.

CASE 3. Suppose we want to conserve the value of $\omega(n)$. Case 1 above conforms to such conservation but Case 2 violates it. We can, however, "bargain away" one prime p_1 for a prime q to have qn/p_1. This way, $\omega(qn/p_1) = \omega(n)$ and the regular tensors of both have the same rank. Now suppose n is a primorial $p_T\#$. This implies that q is larger than all the prime divisors p of $p_T\#$ and $qn/p_1 > n$. Also implicit is the fact that q_1, the smallest prime totative of n, is larger than that of qn/p_1. Further, a number qn/p_1 with primorial n signifies p_1 is the smallest prime totative of qn/p_1. Since $p_1 < q_1$ and $qn/p_1 > n$, $|f(n, p_1)| > |f(n, q_1)|$. This implies that, although $r(qn/p_1)$ may exceed $r(n)$, qn/p_1 certainly is less "efficient" at producing regulars; indeed the magnitude of the larger number might only be overcoming the reduced efficiency. Thus we may see certain numbers like qn/p_1 among the terms of A244052, but they would obviously succeed any primorial.

The preceding cases reveal several important considerations regarding multiples of n and how they relate to n. Firstly, multiplication of n by a divisor d, even if not prime, shares the same $\omega(n)$-dimensional infinite regular array **I** with n, thus **R** is fully contained in $\mathbf{R}'$ of dn, and all the regulars $1 \le m \le n$ of n are regular to dn. Secondly, multiplication of n by a number t coprime to n adds one or more dimensions to the infinite regular array, significantly increasing the number of regulars in $\mathbf{R}'$ of tn, while $tn > n$ and $\omega(tn) > \omega(n)$. Again, **R** is fully contained in $\mathbf{R}'$ of tn, and all the regulars $1 \le m \le n$ of n are regular to tn. However, some numbers regular to tn are nonregular to n. If we impose the constraint that the value of $\omega(n)$ must be conserved, we have a third case that has us lose one prime divisor p_1 of n to gain a prime divisor q coprime to n.

In this case, the regular tensors of both n and qn/p_1 have the same rank. If n is a primorial, the least prime totative q_1 of n, is larger than that of qn/p_1 which must be p_1. Therefore

$$|f(n, p_1)| > |f(n, q_1)|,$$

implying qn/p_1 certainly is less "efficient" than primorial n in generating regulars despite the fact that $r(qn/p_1)$ may exceed $r(n)$. That latter fact may only be an effect attributable to $qn/p_1 > n$.

4. The Regular Counting Function

We can construct the set R of numbers $m \le n$ regular to n. The regular function $r(n)$ = the number of terms in R. There are four practical approaches that can calculate $r(n)$ for $n \le p_9\#$ or less. Perhaps the simplest conceptually is the factor subset approach, i.e., that regular m has no prime divisors q coprime to n. Next is the congruency approach, taking advantage of the fact that regular $m \mid n^e$. One notable method maps a Möbius function across the totatives, or numbers t coprime to b, i.e., GCD$(t, b) = 1$. The most efficient method of computing the number of regulars $r(n)$ is to construct all the regulars m in the tensor **R** rather than test all the numbers $1 \le k \le n$ to see if k is m regular to n.

Following are algorithms that can construct R or compute $r(n)$.

1. Factor Subset Approach.

This approach is predicated on the fact the prime factors p of m are a subset of those of n. This strategy involves factoring n and memoizing the set D of distinct prime divisors p. The result is construction of the set of regular m of n. It is perhaps the most conceptually straightforward method but is not as efficient as other methods of computing $r(n)$. Using this method we have to account for the fact that the empty product $m = 1$ is regular to all n as it may be missed by algorithms that deal with primes.

Example: the prime factors of $12 = \{2, 3\}$. Factor each k in a loop structure or function and check if each distinct prime factor of k is a member of the set D. We could also check if the prime divisors of k are a subset of those of n. If all the primes $p \mid k$ are members then $k = m$ is regular to n. We can use Boolean values for `True` results and multiply the set; if we get a product 1, then $k = m$ is regular to n; if 0, then k is nonregular. Example; $k = 6$ of $n = 12$; 6 has the prime divisors $\{2, 3\}$ and both of these are members of $D = \{2, 3\}$, thus the program would get `{True, True}`, Boolean value is `{1, 1}`, the product is 1, thus $k = m = 6$ is regular to $n = 12$. Another example; $k = 14$, $n = 15$; k has the prime divisors $\{2, 7\}$ and n has $\{3, 5\}$. Thus the program would render `{False, False}`, the Boolean value of which is `{0, 0}`, and the product of these is 0 so $k = 14$ is nonregular to $n = 15$.

The following algorithm constructs all regular m of n:

(CODE 4.1)

```
f[n_] := Function[d,
  {1}~Join~Select[Range[2, n],
    SubsetQ[d, First /@ FactorInteger@ #] &]][
    First /@ FactorInteger@ n]
```

The program requires about 100 seconds to generate $r(p_8\#)$.

2. Congruency of m and n^ρ.

These tests take advantage of the fact that regular $m \mid n^\rho$, therefore we can write a congruency test. It is computationally more expensive to find the very smallest exponent ρ that satisfies the congruency. Thus let's look for computationally simpler "surefire" tests.

See OEIS A280269 for values of ρ across the regulars in the range of n. The sequence OEIS A280274 gives the maximum value of ρ across regulars in the range of n. Examining this data seems to support the conjecture that maximum ρ is relatively small compared to n and even any regular $1 \le m \le n$. For $m = 1$, $\rho = 0$; m prime, $\rho = 1$, and m composite, ρ varies as to multiplicity ratios. Generally, richness ρ varies as to the multiplicity ratio of a certain prime factor p common to m and n (see Formula 1.1)

What is the largest possible multiplicity that can be obtained for any integer $1 \le k \le n$? Since 2 is the smallest prime and thus the multiplicity of 2 can escalate to be the largest among all multiplicities in the same range, the number k with the largest multiplicity in the range must be a power of 2. Indeed it must be the largest power of 2 less than or equal to n. There is no way for the multiplicity of regular $1 \le m \le n$ to exceed $\text{FLOOR}(\text{LOG}_2\, n)$.

We can produce a test for regular $1 \le m \le n$ based on congruency. We know that if $m \mid n^e$, then $m \mid n^{(e+1)}$ since the set of distinct prime divisors of both powers of n has not decreased and the multiplicity of all prime divisors has increased from n^e to $n^{(e+1)}$. Through induction we know we can find a "sure-fire" test based on $m \mid n^e$, so long as $e > \rho$. Since we can obtain no multiplicity of regular $1 \le m \le n$ greater than that of $2^{\text{FLOOR}(\text{LOG}_2 n)}$, we can use the following as a "sure fire" test for finding all regular $1 \le m \le n$:

$$n^{\text{FLOOR}(\text{LOG}_2 n)} \pmod m = 0 \tag{4.1}$$

This method is especially fast if we memoize $n^{\text{FLOOR}(\text{LOG}_2 n)}$:

(CODE 4.2)

```
f[n_] := With[{m = n^Floor@ Log2@ n},
  Count[Range@ n, k_ /; Divisible[m, k]]]
```

The program requires about 4½ seconds to generate $r(p_8\#)$ and 101 seconds to produce $r(p_9\#)$.

Absent an easy way to determine $\text{FLOOR}(\text{LOG}_2\, n)$, we know $m \mid n^m$ for all regular m and $n \ge 1$, since $m > \rho$ in all cases. The method is less than half as fast as above:

$$n^m \pmod m = 0 \tag{4.1.1}$$

The following function constructs the set R of n:

(CODE 4.2.1)

```
f[n_] := Select[Range@ n, PowerMod[n, #, #] == 0 &]
```

The program requires about 10 seconds to generate $r(p_8\#)$ and 224 seconds to produce $r(p_9\#)$. It was used to extend A244052 to 175 terms in November 2016. Term $17p_9\#$, required 1 hour 21 minutes on a 64 bit 2.9 GHz quadcore Intel Xeon laptop with 32 Gb of RAM. In this case, the memory proved insufficient to generate terms much larger than the 162nd. Doubling the memory would seem to only afford validating the 175th term.

3. Difference of Ratios.

Map the following formula across each $m \le n$:

If $(\text{FLOOR}(n^m/m) - \text{FLOOR}((n^m - 1)/m)) = 1$ then m is regular.

The formula avoids factoring numbers to construct the set R of n. It is related to the fact that $m \mid n^m$.

(CODE 4.3)

```
f[n_] := Select[Range@ n,
   (Floor[n^#/#] - Floor[(n^# - 1)/#]) == 1 &]
```

The program requires about 20 seconds to generate $r(p_6\#)$, much less efficient than the previous methods.

4. Recursive GCD Test.

This method takes the $\text{GCD}(a, b)$, setting $a = m$ and $b = n$. While $a > 1$ and $\text{GCD}(a, b) \ne 1$, it sets $a = \text{GCD}(a, b)$ and $b = b/a$. In this way, a number m whose prime divisors p also divide n reduces to 1, while m that have at least one prime divisor q coprime to n reduce to a number greater than 1. The following program is a test that can be used to generate R or $r(n)$.

(CODE 4.4)

```
fQ[a_, b_] :=
  Block[{m = a, n = b},
   While[And[m != 1, ! CoprimeQ[m, n]],
    n = GCD[m, n]; m = m/n]; m == 1]
```

Another manifestation of the same approach as a nested function:

(CODE 4.4.1)

```
f[n_] := Count[Range@ n, k_ /; First@ NestWhile[
  Function[s, {#1/s, s}]@ GCD[#1, #2] & @@ # &, {k, n},
  And[First@ # != 1, ! CoprimeQ @@ #] &] == 1]
```

The programs require about a minute to generate $r(p_8\#)$.

5. Möbius Function across Totatives of n.

We can use the following summation across the totatives $1 \le t < n$ such that $\gcd(t, n) = 1$:

$$\sum_{1 \le t \le n} \mu(t)\ \mathrm{floor}(n/t) \tag{4.2}$$

This function implements the above formula to compute $r(n)$. The manner in which the function arrives at $r(n)$ is quite different. Instead of constructing regular $m \le n$ of n, it arrives at a sum derived from the totatives of n.

(code 4.5)

```
f[n_] := Total@
  Map[MoebiusMu[#] Floor[n/#] &,
   Select[Range[n - 1], CoprimeQ[#, n] &]]
```

The program requires about 15½ seconds to generate $r(p_8\#)$ and 445 seconds to produce $r(p_9\#)$. There are two parts to the algorithm. The first part generates all t such that $\gcd(t, n) = 1$. Though this subset of the range $1 \le m \le n$ is markedly smaller for highly divisible n, sifting through the range to find qualified terms requires additional time. The second part sums the product of the Möbius function of t and the floor function of the ratio n/t across each t.

6. The Logarithm Constructor Approach.

We can use the property of the tensor **R** of regular numbers as the tensor product of the power ranges of the prime divisors of n to most efficiently compute $r(n)$. In short, we can generate a table of all the products less than n of the powers of the distinct primes of n, then count the products, rather than search all numbers in the range $1 \le k \le n$ for m regular to n. This method is the most efficient because the value of $r(n)$ is rather small compared to n as n increases. This approach involves the construction of **R**.

The most conspicuous problem to which this method has been applied is that of the famous "Hamming number" problem put forth by Edsger W. Dijkstra. In this application, we are asked to create a table of the 60-regular numbers, i.e., the 5-smooth numbers, in their order. In Wolfram Language, Robert G. Wilson V wrote the following script to generate oeis A051037 which lists the 5-smooth numbers in their order:

(code 4.6)

```
f[n_]:= Sort@ Flatten@ Table[ 2^a * 3^b * 5^c,
 {a, 0, Log[2, n]},
 {b, 0, Log[3, n/2^a]},
 {c, 0, Log[5, n/(2^a * 3^b)]}]
```

We can generalize this code to fit any positive integer n merely by obtaining a list of the distinct prime divisors of n. This program writes a statement given such a list:

(code 4.6.1)

```
f[n_] := Function[w,
  Length@ ToExpression@
   StringJoin["With[{n = ", ToString@ n, "},
    Flatten@ Table[", ToString@
     InputForm[Times @@ Map[Power @@ # &, w]], ", ",
        Most@ Flatten@ Map[{#, ", "} &, #], "]]"] &@
  MapIndexed[Function[p,
   StringJoin["{", ToString@ Last@ p, ", 0, Log[",
    ToString@ First@ p, ", n/(", ToString@
     InputForm[Times @@ Map[Power @@ # &,
      Take[w, First@ #2 - 1]]], ")]}"]]@
       w[[First@ #2]] &, w]]@
  Map[{#,
   ToExpression["p" <> ToString@ PrimePi@ #]} &,
    FactorInteger[n][[All, 1]]]
```

Given an input of 2310, for example, the script writes the statement:

```
With[{n = 2310}, Flatten@ Table[
  2^p1 * 3^p2 * 5^p3 * 7^p4,
  {p1, 0, Log[2, n/(1)]},
  {p2, 0, Log[2, n/(2^p1)]},
  {p3, 0, Log[5, n/(2^p1 * 3^p2)]},
  {p4, 0, Log[7, n/(2^p1 * 3^p2 * 5^p3)]}]]
```

and the statement `Length` counts the 68 terms in the table.

This program finds $r(p_8\#)$ in 1¼, $r(p_9\#)$, $r(p_{10}\#)$ in 24, $r(p_{11}\#)$ in 105 seconds. We can project that it would require about 10, 50, and 240 minutes to calculate $r(p_{12}\#)$, $r(p_{13}\#)$, and $r(p_{14}\#)$, respectively. The same algorithm, compiled on other machines, might calculate these figures in very much less time.

An alternative formulation of the same function using `Do` rather than `Table`:

(code 4.6.2)

```
f[n_] := Function[w, ToExpression@
  StringJoin["Module[{k = 0, n = ", ToString@ n, "}, ",
     StringDrop[#, -1], "; k]"] &@
   StringJoin@ Fold[{StringJoin["Do[", First@ #1,
     ToString@ #2, "],"],
   StringJoin[Last@ #1]} &, {"k++, ", ""}, #] &@
     Reverse@ MapIndexed[Function[p,
       StringJoin["{", ToString@ Last@ p, ", 0, Log[",
         ToString@ First@ p, ", n/(", ToString@
         InputForm[Times @@ Map[Power @@ # &,
           Take[w, First@ #2 - 1]]], ")]}"]]@
           w[[First@ #2]] &, w]]@
   Map[{#,
     ToExpression["p" <> ToString@ PrimePi@ #]} &,
     FactorInteger[n][[All, 1]]]
```

This program writes an expression like this for the input 2310:

```
Module[{k = 0, n = 2310},
  Do[
   Do[
    Do[
     Do[
      Do[k++,
       {p5, 0, Log[11, n/(2^p1*3^p2*5^p3*7^p4)]}],
      {p4, 0, Log[7, n/(2^p1*3^p2*5^p3)]}],
     {p3, 0, Log[5, n/(2^p1*3^p2)]}],
    {p2, 0, Log[3, n/(2^p1)]}],
   {p1, 0, Log[2, n/(1)]}]; k]
```

This approach proved to be slightly less efficient than the `Table` method above.

We can take advantage of the fact that numbers n with the same squarefree root oeis A007947(n) have the same infinite regular tensor (see the next section) to efficiently generate $r(n)$ for all such n.

(code 4.6.3)

```
f[T_, k1_: 1, k2_: 1, m_: 1] :=
 Module[{
   P = Times @@ Prime@ Range@ T,
   n = If[k2 == 1, NextPrime@ Prime@ T - 1,
     Min[k2, NextPrime@ Prime@ T - 1]], x, r},
   x = If[m == 1, P, m];
   r[x_] := Function[w, ToExpression@
     StringJoin["With[{n =", ToString@ x,
      "}, Flatten@ Table[", ToString@
       InputForm[Times @@ Map[Power @@ # &, w]],
      ", ", Most@ Flatten@
       Map[{#, ", "} &, #], "]]"] &@
       MapIndexed[
        Function[p,
         StringJoin["{", ToString@ Last@ p,
          ", 0, Log[", ToString@ First@ p,
          ", n/(", ToString@ InputForm[Times @@
          Map[Power @@ # &, Take[w, First@ #2 - 1]]],
            ")]}"]]@ w[[First@ #2]] &, w]]@
```

```
            Map[{#, ToExpression["p" <> ToString@
          PrimePi@ #]} &, FactorInteger[x][[All, 1]]];
  Function[w, Map[Function[k,
   {k x, Length@ TakeWhile[w, # <= k x &]}],
     Range[k1, n]]]@ Sort@ r[x n]]
```

This function takes up to four parameters and requires some knowledge of the turbulent candidate sequence $a(n)$. For instance, from Figure 5.2 we observe in tier $T = 5$ that the number 2730 is the squarefree root of 5460, 8190, 10920, and 13650. Thus we would set the parameters thus:

```
f[5, 1, 5, 2730]
```

and the program yields $r(n)$ for all 5 related terms in the time it takes to calculate the largest.

This concludes the description of various algorithms that leverage several qualities of regular numbers to calculate the number of regular $1 \le m \le n$.

5. General Structure of A244052(n)

Running the regular counting function $r(n)$ across the positive nonzero integers n and selecting those n whose values of $r(n)$ set records yields the sequence OEIS A244052(n). Let's look at the apparent general structure of the sequence.

There are roughly three major kinds of numbers m in A244052:

1. $m = p_T\#$. The primorials thus arrange the sequence into "tiers" wherein all numbers in a tier $p_T\# \le m < p_{(T+1)}\#$ must have $\omega(m) = T$. The appearance of all primorials in A244052 is supported by the Möbius function version of $r(n)$ and the nature of the geometry of the tensor $\mathbf{R}$ of regular $1 \le m \le n$ as a product of distinct prime factor power tensors bounded by n (see OEIS A275280).
2. $k\,p_T\#$ with $1 \le k < p_{(T+1)}$, i.e., in A060735. The $k\,p_T\#$ organize each tier into "levels."
3. "Turbulent" numbers m occur within a "level" $k\,p_T\# < km < (k+1)\,p_T\#$ such that $\omega(m) = T$. This is a necessary but insufficient condition. These terms are merely "turbulent candidates" that must be tested via the regular counting function to see if the term sets a record.

There are several major features of each tier T:

1. The tier T has primorial $p_T\#$ as its smallest term.
2. All integer multiples $k\,p_T\#$ with $1 \le k < p_{(T+1)}$ begin new "levels" k within each tier T.
3. There is "primary turbulence" in all tiers $T > 0$. These turbulent terms m "echo" in level k in the form km and thus constitute "secondary turbulence," "echo turbulence," or "reverb".
4. Turbulence has the following qualities:
   a. Essentially, a turbulent term or candidate has GPF$(m) > p_T$ and at least one prime totative $q \le p_T$ such that GCD$(m, p_T\#) < p_T\#$.
   b. "Distension" i, meaning the difference in the indexes of the greatest prime factor GPF(m) and GPF$(p_T\#) = p_T$. This is easier to see if m is written in "multiplicity notation", that is, as A054841(m). This notation is a sort of abbreviation of highly composite numbers such that the prime is conveyed by a reverse positional notation, while only the multiplicity is written. In this notation, primorials $p_T\#$ appear as repunits (repeated series of 1s) of length T.
   c. "Depth" j. Let e be the index of the least prime totative (LPT) of m and let T be the index of the GPF$(p_T\#) = p_T$. Then $j = T - i$. This appears in notation as the position of the first zero, e.g., the depth $j = 1$ of $m = 10$, since the second prime, 3, is coprime to 10, but the index of the GPF$(10) = 3$; $3 - 2 = 1$, thus $j = 1$.

   Each tier has a maximum or "cardinal" depth and distension that can be calculated using A020900(T). We'll look more closely at depth, distension, and calculation of these in Section 7. Appendix B and Figure 7.2 show segments of data from $a(n)$ and A244052.
5. We can regard each tier of the sequences as having these structural features, illustrated by example in Figure 5.1:

(FIG. 5.1) Structural features of tier $T = 5$ of $a(n)$ and A244052.

k	Position in A244052	A244052(n)	A054841(a(n))	A010846(a(n))	
1	29	2310	11111	283	} Head = $p_T\#$
	30	2730	111101	295	Primary turbulence: $p_T\# < m < 2p_T\#$ with $\omega(m) = T$
	31	3570	1111001	313	
	32	3990	11110001	322	
		4290	111011	315	
2	33	4620	21111	382	Secondary turbulence: $kp_T\# < km < (k+1)p_T\#$ with $\omega(m) = T$ (minus terms $kp_T\#$ themselves, e.g. 4620, 6930, etc.)
	34	5460	211101	395	
3	35	6930	12111	452	
	36	8190	121101	463	
4	37	9240	31111	505	
	38	10920	311101	519	
5	39	11550	11211	551	
	40	13650	112101	567	
6	41	13860	22111	593	Tail: all m of the form $kp_T\#$ with $1 < k < p_{(T+1)}$
7	42	16170	11121	629	
8	43	18480	41111	660	
9	44	20790	13111	691	
10	45	23100	21211	717	
11	46	25410	11112	743	
12	47	27720	32111	766	

Note the term 4290 shown in light face above is in $a(n)$, but not in A244052. Looking at its regular counting function at far right we see that it fails to set a record. We shall call this "disqualification," as 4290 meets the necessary conditions: squarefree $p_T\# \le m < p_{(T+1)}\#$ with $\omega(m) = T$ but $r(4290) < r(3990)$.

Turbulence exhibits "distension" (the GPF$(m) > p_T$ and there are at least 1 prime totative $q <$ GPF(m). There is a maximum distension called "cardinal distension". In the above example, the cardinal distension $i = 3$. It is plain to see in multiplicity notation (column A054841($a(n)$)); the furthest 1 to the right is 3 places from the last 1 in the multiplicity notation of $p_T\#$ with $T = 5$ (i.e., 2310):

(FIG. 5.2) Cardinal distension i of tier $T = 5$ of $a(n)$ and A244052.

			★		
1	29	2310	11111	283	Cardinal distension: indicates GPF of tier T. $i = 3$; last "1" shifted 3 places to the right
	30	2730	111101	295	
	31	3570	1111001	313	
	32	3990	11110001	322	
			---★		

In the above example, the cardinal depth is $j = 2$. Multiplicity notation makes it easy to see the furthest 0 to the left is 1 place from the end of the "word" "11111" that is the multiplicity notation of 2310:

(FIG. 5.3) Cardinal depth j of tier $T = 5$ of $a(n)$ and A244052.

1	29	**2310**	**11111**◆	**283**	Cardinal depth:
	30	**2730**	**111101**	**295**	indicates LPT of tier T.
	31	**3570**	**1111001**	**313**	$j = 1$; last "0" shifted 1
	32	**3990**	**11110001**	**322**	place to the right

If 4290 were to have qualified, the cardinal depth would have been 2 rather than 1:

4290	**111011**	315	$j = 2$; last "0" shifted 2 places to the right in $a(n)$

Thus the major features of the "highly regular number" sequence A244052 includes tiers T with primorials $p_T\#$ as the smallest term and all terms in the tier with $\omega(n) = T$. There are levels k associated with integer multiples $1 \le k < p_{(T+1)}$ of primorial $p_T\#$. Terms in tier T that are not in A060735 are called turbulence. There are two aspects that help delimit the possible values of turbulence that facilitate efficient construction of a table of candidate terms for A244052, these are distension and depth. Distension describes the greatest prime factor with respect to the GPF of $p_T\#$, i.e., p_T, while depth descripes the smallest prime totative with respect to p_T. Multiplicity notation is a great aid in efficiently constructing the table of turbulent candidate terms.

6. Observations Regarding the Smallest Prime Totative.

The Möbius function method of generating $r(n)$ in Section 4.5 merits examination not merely because it differs from the "intuitive" methodologies associated with the properties of regulars themselves, but because of its implications regarding small prime totatives. Chief among the implications is that small prime totatives wreak havoc against a high value of $r(n)$. This is supported by the tensor constructor method.

Let $f(n, t) = \mu(t)$ FLOOR(n/t) and let q_1 be the smallest prime totative of n. We can determine the following about the behavior of the function f. Firstly, the value of the function $f(n, t)$ applied to prime $t < ½\, n$ is negative and greater than 1, increasingly large as t approaches 2; it is at its most pronounced at $t = q_1 = 2$. For prime $t > ½\, n$ the value is -1.

The smallest totative of any nonzero positive integer n is 1; the value of $f(n, 1) = n$, since $n/1 = 1$ and $\mu(1) = 1$. This value can be seen as the "total possible value" of $r(n)$ that $f(n, t)$ for $t > 1$ modify to arrive at actual $r(n)$. We can see all $t < ½\, n$ as having the greatest effect on the ultimate value of $r(n)$ for the following reasons:

1. The totatives of n are symmetrically arranged about $½\, n$.
2. Nonzero $|f(n, t)| > 1$ for $t < ½\, n$

   while nonzero $|f(n, t)| = 1$ for $t < ½\, n$.
3. The presence of a ratio with a constant numerator in Formula (2.2) implies $f(n, q_1)$ generates the largest negative value against $r(n)$. Let S = the sum of $f(n, t)$ across $½\, n < t < n$. $f(n, q_1) \ge S$, in other words, $|f(n, q_1)|$ is at least as large as S. The set of numbers that have $f(n, q_1) = S$ is finite: $\{3, 4, 6, 8, 12, 18, 24, 30\}$. This reinforces the implication that the smallest totative q_1 of n has the most influence on the value of $r(n)$.

In other words, we can see the smallest prime totatives as incurring the greatest "damage" against potential $r(n)$ as they present a large negative value of $f(n, t)$. The smallest prime totative q_1 is the most significant influence on $r(n)$. This suggests that odd numbers and those with "gaps" or small prime totatives within the range of its prime divisors will tend to have a lower $r(n)$ than comparably-sized even numbers and numbers that are products of a contiguous set of the smallest primes, i.e., primorials. From the properties of the smallest prime totative q_1 of n, we can see that primorials $p_T\#$ may seem to minimize their effect and thus maximize $r(n)$ for numbers n with T distinct prime divisors.

Examination of q_1 alone is incomplete regarding the full effect of the smallest prime totative q_1 on $r(n)$.

We can simulate the effect of transforming a prime divisor $p_k \to q_1$ while holding the magnitude of n. We may use the Möbius approach of Section 4.5 or the constructor approach of 4.6 to achieve the same effect. To simplify, let's only consider primorials $p_n\#$.

The code below calculates the regular function of primorial $p_n\#$ attributable to p_n using the Möbius function approach:

(CODE 6.1)

```
attributableToGPF[n_] :=
 Function[P,
   Total@ Map[MoebiusMu[#] Floor[P/#] &,
     Select[Range@ P,
      And[CoprimeQ[#, Times @@ Prime@ Range[n - 1]],
        Divisible[#, Prime@ n]] &]]][
          Times @@ Prime@ Range@ n]
```

Figure 6.1 shows the full effect of the greatest prime factor p_n of primorial $p_n\#$ on the regular counting function $r(p_n\#)$ as if it were ignored, or deemed coprime to $p_n\#$. The second column shows $r(p_n\#)$. The third column shows $\mu(p_n)$ FLOOR$(p_n\#/p_n)$ while the "balance" column shows the remaining portion of Formula 4.2 on all products $1 \le k\, p_n \le p_n\#$, which are interpreted as coprime to $p_n\#$. We arrive at a number of regulars attributable to p_n in the penultimate column, and regulars not attributable to p_n in the last column.

(FIG. 6.1)

n	r(p_n#)	m(p_n)	+	balance	=	attrib.	not attrib.
1	**2**	-1	+	0	=	-1	1
2	**5**	-2	+	0	=	-2	3
3	**18**	-6	+	0	=	-6	12
4	**68**	-30	+	8	=	-22	46
5	**283**	-210	+	126	=	-84	199
6	**1161**	-2310	+	1937	=	-373	788
7	**4843**	-30030	+	28474	=	-1556	3287
8	**19985**	-510510	+	503782	=	-6728	13257

This study can be generalized to any prime divisor p_k of $p_n\#$.

Consider a function $s(n, k)$ that calculates the regular counting function of primorial $p_n\#$ minus the full contribution of prime p_k with $1 \le k \le n$. This is tantamount to the following adjustment to Formula 4.2:

$$\sum_{1 \le t \le n} \mu(t)\ \mathrm{FLOOR}(n/t) \quad \text{t such that } \mathrm{GCD}(t, n/p_k) = 1 \tag{6.1}$$

The following code generates $s(n, k)$ via Möbius:

		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_10	p_11	p_12
n	r(p_n#)	2	3	5	7	11	13	17	19	23	29	31	37
1	**2**	1											
2	**5**	3	2										
3	**18**	11	9	6									
4	**68**	47	37	27	22								
5	**283**	206	170	130	109	84							
6	**1161**	871	734	583	500	405	373						
7	**4843**	3732	3190	2601	2268	1877	1746	1556					
8	**19985**	15680	13554	11239	9906	8337	7813	7031	6728				
9	**349670**	66141	57713	48461	43107	36738	34601	31405	30155	28111			
10	**1456458**	281949	248018	210504	188649	162498	153691	140428	135239	126722	117035		
11	**6107257**	1186118	1049898	898509	809882	703282	667209	612779	591418	556289	516197	505163	
12	**25547835**	5017655	4465718	3849404	3486968	3049110	2900419	2675418	2586893	2441064	2274216	2228176	2110136

(FIG. 6.2) $s(n, k) = r(n)$ attributable to prime p_k and relevant multiples of p_k, with $1 \le k \le \pi(n)$.

		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_10	p_11	p_12
n	r(p_n#)	2	3	5	7	11	13	17	19	23	29	31	37
1	**2**	**1**											
2	**5**	2	**3**										
3	**18**	7	9	**12**									
4	**68**	21	31	41	**46**								
5	**283**	77	113	153	174	**199**							
6	**1161**	290	427	578	661	756	**788**						
7	**4843**	1111	1653	2242	2575	2966	3097	**3287**					
8	**19985**	4305	6431	8746	10079	11648	12172	12954	**13257**				
9	**349670**	16933	25361	34613	39967	46336	48473	51669	52919	**54963**			
10	**1456458**	67721	101652	139166	161021	187172	195979	209242	214431	222948	**232635**		
11	**6107257**	270340	406560	557949	646576	753176	789249	843679	865040	900169	940261	**951295**	
12	**25547835**	1089602	1641539	2257853	2620289	3058147	3206838	3431839	3520364	3666193	3833041	3879081	**3997121**

(FIG. 6.3) Values of $r'(n, k) = r(p_n\#) - s(n, k)$.

		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_10	p_11	p_12
n	r(p_n#)	2	3	5	7	11	13	17	19	23	29	31	37
1	**2**	**2.**											
2	**5**	2.5	**1.66667**										
3	**18**	2.57143	2.	**1.5**									
4	**68**	3.2381	2.19355	1.65854	**1.47826**								
5	**283**	3.67532	2.50442	1.84967	1.62644	**1.42211**							
6	**1161**	4.00345	2.71897	2.00865	1.75643	1.53571	**1.47335**						
7	**4843**	4.35914	2.92982	2.16012	1.88078	1.63284	1.56377	**1.47338**					
8	**19985**	4.64228	3.1076	2.28504	1.98284	1.71575	1.64188	1.54277	**1.50751**				
9	**349670**	4.90604	3.27566	2.40008	2.07856	1.79286	1.71382	1.60781	1.56983	**1.51145**			
10	**1456458**	5.16339	3.43987	2.51261	2.17158	1.86817	1.78422	1.67113	1.63069	1.56839	**1.50308**		
11	**6107257**	5.3875	3.58239	2.61038	2.25257	1.93376	1.84537	1.72632	1.68369	1.61798	1.54899	**1.53103**	
12	**25547835**	5.60503	3.72045	2.7049	2.33076	1.99704	1.90445	1.77959	1.73484	1.66583	1.59332	1.57441	**1.52791**

(FIG. 6.4) Decimal expansions of the ratio $r(p_n\#) / r'(n, k)$.

(CODE 6.2)

```
attributableToPrimek[n_, k_] :=
 Function[P,
   Total@ Map[MoebiusMu[#] Floor[P/#] &,
     Select[Range@ P,
      And[CoprimeQ[#, Times @@ Prime@ Range@ n/Prime@ k],
        Divisible[#, Prime@ k]] &]]][
          Times @@ Prime@ Range@ n]
```

This code is more efficient and constructs $s(n, k)$ in logarithmic time:

(CODE 6.3)

```
attributableToPrimek[n_, k_] :=
 Block[{P = Times @@ Prime@ Range@ n},
  Length@ Function[w,
     ToExpression@
        StringJoin["Module[{n = ", ToString@ P,
         ", k = 0}, Flatten@ Table[k++, ",
         Most@ Flatten@ Map[{#, ", "} &, #], "]]"] &@
      MapIndexed[
       Function[p,
          StringJoin["{", ToString@ Last@ p, ", 0, Log[",
           ToString@ First@ p, ", n/(",
           ToString@ InputForm[
            Times @@ Map[Power @@ # &,
             Take[w, First@ #2 - 1]]],
           ")]}"]]@ w[[First@ #2]] &, w]]@
     Map[{#, ToExpression["p" <> ToString@ PrimePi@ #]} &,
      FactorInteger[P/Prime@ k][[All, 1]]] ]
```

Figure 6.2 is a number triangle $s(n, k)$. Note that the smallest totative, i.e., 2, has the greatest full effect on the value of $r(p_n\#)$, shown at left for reference. It follows that the impact of ignoring the contributions of p_k and its regular multiples varies close to $\log_{p_k} p_n\#$, especially as $p_n\#$ is very much larger than p_k.

Consider a function $r'(n, k) = r(p_n\#) - s(n, k)$. Figure 6.3 shows the values of the regular counting function if we were to ignore the full contribution of p_k to $r(p_n\#)$. This is simply subtracting values in rows of Figure 6.2 from $r(p_n\#)$ at left in the same row.

Finally, Figure 6.4 shows the ratio $r(p_n\#) / r'(n, k)$. We can see that the contribution of smaller ignored primes p_k tends increase markedly as n increases. This seems to have to do mostly with the case of the smallest prime p_k itself.

Determination of whether or not the ratio $r(p_n\#) / r'(n, k)$ converges is an interesting problem beyond the scope of this paper. Some insight into the behavior of the ratio might be gleaned from thinking about the ***

<u>The Tier Jump Ratio</u>. Given the data in Appendix B, we note the ratio of the regular counting function values for primorials $p_{(T+1)}\#$ and the greatest term of tier T, i.e., $(p_{(T+1)} - 1)\, p_T\#$. Curiously the ratio $r(\,(p_{(T+1)}\#) / r(\,(p_{(T+1)} - 1)\, p_T\#)$ continues to hold just over 3/2 for $T = 14$. At first the ratio begins high above the limit at 5/3 and has a low value at Tier 5 below 3/2. The ratios of the regular functions of the primorial $p_T\#$ and the largest level $k\, p_{(T-1)}\#$ less

(FIG. 6.4) The tier jump ratio $r((p_{(T+1)}\#) / r((p_{(T+1)} - 1)\, p_T\#)$.

T	p_T#	r(p_ (T+1)#)	r(k p_T#) with k = p_T+1 - 1	TJR
1	2#	2	1	2
2	3#	5	3	1.66667
3	5#	18	11	1.63636
4	7#	68	44	1.54545
5	11#	283	192	1.47396
6	13#	1161	766	1.51567
7	17#	4843	3223	1.50264
8	19#	19985	13037	1.53294
9	23#	83074	54226	1.53200
10	29#	349670	230293	1.51837
11	31#	1456458	942700	1.54499
12	37#	6107257	3968011	1.53912
13	41#	25547835	16485222	1.54974
14	43#	106115655	67583798	1.57013
15	47#	440396221	279200197	1.57735

(FIG. 6.5) The adjusted jump ratio $r(p_n\#) / r'(n, n - 1)$.

T	p_T#	r(p_n#)	r'(n, n - 1)	TJR
1	2#	2	1	2
2	3#	5	3	1.66667
3	5#	18	12	1.5
4	7#	68	46	1.47826
5	11#	283	199	1.42211
6	13#	1161	788	1.47335
7	17#	4843	3287	1.47338
8	19#	19985	13257	1.50751
9	23#	83074	54963	1.51145
10	29#	349670	232635	1.50308
11	31#	1456458	951295	1.53103
12	37#	6107257	3997121	1.52791

than $p_T\#$ hover around a ratio just above 1.5 thereafter. At tier 13 it is 6107257/3968011 = 1.53912…

Figure 6.4 is a table of values for each tier T (here T pertains to the primorial in the numerator). It seems to follow that the "tier jump" ratio approaches a limit since the primes $p_{(T-1)}$ and p_T are ever more similar in magnitude on average as T increases. Some candidates for the limit, if it can be related to existing constants, regards those slightly greater than 3/2:

1. $\gamma + 1 = 1.5772156649\ldots$

   (γ = Euler-Mascheroni constant).

2. $\pi/2 = 1.5707963267\ldots$

The first candidate is interesting for the following reasons:

1. The relation between γ and the rate of growth of the divisor counting function $\tau(n)$,
2. Divisors are a special case of regular numbers $1 \le m \le n$,
3. The similarity between the tensors of $\tau(n)$ and $r(n)$ as shown by A275055 and A275280, respectively.

The determination of this convergence, if there is a convergence, is beyond the scope of this work.

There are smaller jumps of $r(n)$ for each level k that have to do with increasing the bound on the infinite T-rank tensor that is the product of the prime power ranges of the distinct prime divisors p of A007947(n) (i.e., the squarefree root of n). This is perhaps the easiest "jump" effect to understand. The jumps have a logarithmic relationship to $r(\text{A007947}(n))$.

There are even smaller jumps between turbulent candidates of the same depth that have to do with both a different limit between them and a different configuration of distinct prime divisors.

The "tier jump ratio" can be adjusted to iron out the effects of

The expositions in Sections 2 through 4 set the stage for several theorems.

7. Theorems Regarding A244052.

1. PRIMORIALS IN A244052. In this Section we shall determine two facts. First, the primorials $p_T\#$ must appear in A244052 in strictly increasing order. This implies OEIS A002110 is a subset of A244052. A corollary to this is that primorials $p_T\#$ organize A244052 into "tiers" wherein all terms that succeed $p_T\#$ in the record must have $\omega(n) = \omega(p_T\#) = T$ up to but not including the appearance of the next primorial.

Theorem 5.1. Let primorial $p_T\#$ be the product of the smallest T positive primes p. Let the nonzero positive integer $1 \le m \le n$ whose every prime divisor p divides n be termed a "regular of" or "regular to" n, counted by a "regular counting function" $r(n)$. Primorial $p_T\#$ sets records for the number of regulars $r(n)$ of n. In other words, primorials, which appear in OEIS A002110 by the definition of that sequence, appear in A244052, i.e., A002110 is a subset of A244052.

Corollary 5.1.1. The smallest prime 2 is the only prime in A244052.

Proof. Let's recall the Möbius function examined in Section 4.5 Formula (4.2):

$$\sum_{1 \le t \le n} \mu(t)\ \text{FLOOR}(n/t)$$
where $\text{GCD}(t, n) = 1$.

Let $f(n, t) = \mu(t)\ \text{FLOOR}(n/t)$ and let q_1 be the smallest prime $1 < q_1 < n$ that is coprime to n. We observe $f(n, 1) = n$ and the Möbius function $\mu(p) = -1$ for p prime. Thus $f(n, q_1)$ must have the largest negative value of all t. In Section 4.5 we observed that the totatives t of n are symmetrically arranged about ½ n. We determined that nonzero $|f(n, t)| > 1$ for $t <$ ½ n while nonzero $|f(n, t)| = 1$ for $t <$ ½ n.

***

The presence of a ratio with a constant numerator in Formula (2.2) implies $f(n, q_1)$ generates the largest negative value against $r(n)$. Let S = the sum of $f(n, t)$ across ½ $n < t < n$. $f(n, q_1) \ge S$, in other words, $|f(n, q_1)|$ is at least as large as S. This reinforces the implication that the smallest totative q_1 of n has the most influence on the value of $r(n)$ outside of $t = 1$ for which $f(n, 1) = n$. This suggests that the nonzero integer q_1 increasingly detracts from $r(n)$ as q_1 approaches 0 and implies that odd n or large n with small q_1 have relatively low $r(n)$. Conversely, the pronounced effect of the smallest totatives $1 < t <$ ½ n implies that numbers n that minimize the number and maximize the magnitude of the smallest totatives will have a larger $r(n)$.

This alone does not prove primorials $p_T\#$ are in A244052, however, let's examine some properties of a primorial. Since $p_T\#$ is a product of the T smallest primes, primorials are the smallest number $\omega(n)$ with the highest possible minimum totative q_1.

An exception is $p_1\# = 2\# = 2$ which has no $q_1 < 2$. All primes p have two regular $1 \le m \le n$, these are $\{1, p\}$ which also are divisors of p: both regulars $1 \le m \le p$ divide p. Since 2 is the smallest prime, it appears in A244052 and no other prime appears in the sequence, since odd primes p also have 2 regulars and thus $r(p_{\text{odd}}) = r(2) = 2$ and 2 is already in the sequence. This proves Corollary 5.1.1.

Let's consider the $\omega(n)$-rank regular tensor **R** as discussed in Section 2. We know that this matrix is the tensor product of all the distinct prime power ranges **P** bounded by n of n. The number of terms in the tensor depends on the length δ_p of each range **P** given by Formula (2.2). Specifically, if the distinct prime divisors p of n are more similar to one another, the number of terms in **R** will be greater than that of a similar-magnitude number whose distinct prime factors are not as similar. This suggests that n with a "compact" set of prime divisors, i.e., those that are minimally distinct, will have a higher value of $r(n)$.

Returning to the properties of primorials, we understand that the product of a contiguous set of primes by definition signifies that primorials have a minimally distinct set of distinct prime divisors.

The notion that numbers n with a contiguous set of minimally distinct prime divisors have the most efficient configuration of such divisors to produce a high value of $r(n)$ reinforces something we'd seen from the Möbius function. Numbers n that minimize the number and maximize the magnitude of the smallest totatives have a larger $r(n)$.

Primorials $p_T\#$ appear in A244052 for the following reasons:

1. Primorials $p_T\#$ are products of the smallest T primes, therefore primorials are the smallest numbers with T prime factors.
2. The distinct prime divisors of $p_T\#$ are minimally distinct.
3. The smallest prime q_1 coprime to $p_T\#$ is larger than that of any other number having T distinct prime divisors. This $q_1 = q_{pn}$, the smallest totative of numbers pn that are the products of at least one prime p that divides $n = p_T\#$, and no primes q coprime to n, but $p_T\# < pn$ and thus precedes the latter in A244052. Likewise, products described in the second and third cases of Section 4 are larger than $p_T\#$.

We have thus proved that primorials appear in A244052 thus A002110 is a subset of A244052. Figure 5.1 shows primorials in the column labeled A002110, followed by multiplicity notation and the number of regulars of $p_T\#$. ▪

Lemma 5.1.2. Primorials $p_T\#$ are the smallest terms that have $\omega(n) = T$ and all terms $p_T\# \le n < p_{(T+1)}\#$ in A244052 must have $\omega(n) = T$. Thus we can split A244052 into contiguous "tiers" T, that is, intervals wherein all terms n have $\omega(n) = T$, starting with $p_T\#$.

Proof 5.1.2. We know that $p_T\#$ is the smallest possible number that has T distinct prime divisors by definition of a primorial as the product of the T smallest primes. This implies that terms n with $\omega(n) = T$ cannot precede $p_T\#$ in A244052. There may be terms n that do not have prime divisors p of $p_T\#$ but instead an equal number of prime divisors q coprime to $p_T\#$ as replacements, but these must be greater than $p_T\#$.

Recall that $\omega(n)$ governs the rank of the regular tensor **R**. Some numbers $n_{(T-1)}$ with $\omega(n) = T - 1$ either already appear in tier $(T - 1)$ of A244052—these will be discussed in the next section. Such

(FIG. 5.1) Primorials in A244052.

p#	Position A244052	A002110(n)	A054841(A002110(n))	A010846(A002110(n))
	1	1	0	1
2#	2	2	1	2
3#	4	6	11	5
5#	9	30	111	18
7#	17	210	1111	68
11#	29	2310	11111	283
13#	48	30030	111111	1161
17#	72	510510	1111111	4843
19#	104	9699690	11111111	19985
23#	137	223092870	111111111	83074
29#	174	6469693230	1111111111	349670
31#	232	200560490130	11111111111	1456458
37#	292	7420738134810	111111111111	6107257
41#	369	304250263527210	1111111111111	25547835
43#	470	13082761331670030	11111111111111	106115655
47#	563	614889782588491410	111111111111111	440396221

(FIG. 5.2) "Turbulence" and integer multiples $kp_T\#$ in A244052.

Position in a(n)	p#	k	Position in A244052	A244052(n)	A054841(a(n))	A010846(a(n))
1		1	1	1	0	1
2	2#	1	2	2	1	2
3		2	3	4	2	3
4	3#	1	4	6	11	5
5			5	10	101	6
6		2	6	12	21	8
7		3	7	18	12	10
8		4	8	24	31	11
9	5#	1	9	30	111	18
10			10	42	1101	19
11		2	11	60	211	26
12			12	84	2101	28
13		3	13	90	121	32
14		4	14	120	311	36
15		5	15	150	112	41
16		6	16	180	221	44
17	7#	1	17	210	1111	68
18			18	330	11101	77
19			19	390	111001	80
20		2	20	420	2111	96
21		3	21	630	1211	115
22		4	22	840	3111	131
23		5	23	1050	1121	145
24		6	24	1260	2211	156
25		7	25	1470	1112	166
26		8	26	1680	4111	174
27		9	27	1890	1311	183
28		10	28	2100	2121	192
29	11#	1	29	2310	11111	283
30			30	2730	111101	295
31			31	3570	1111001	313
32			32	3990	11110001	322
34		2	33	4620	21111	382
35			34	5460	211101	395
36		3	35	6930	12111	452
37			36	8190	121101	463
38		4	37	9240	31111	505
39			38	10920	311101	519
40		5	39	11550	11211	551
41			40	13650	112101	567
42		6	41	13860	22111	593
43		7	42	16170	11121	629
44		8	43	18480	41111	660
45		9	44	20790	13111	691
46		10	45	23100	21211	717
47		11	46	25410	11112	743
48		12	47	27720	32111	766

numbers $n_{(T-1)} > p_T\#$ stand at a disadvantage because n with $\omega(n) = T$ have an additional dimension in their regular array. If $n_{(T-1)}$ belongs to Section 3 Case 2 with respect to $p_{(T-1)}\#$, i.e., $qn_{(T-1)}$, this number will have a relatively inefficient configuration of distinct prime divisors, since the smallest totatives of $qn_{(T-1)}$ are smaller than that of $p_{(T-1)}\#$.

The appearance of $p_T\#$ in the record reimposes the efficient configuration of $p_{(T-1)}\#$ at a larger bound p_T times that of $p_{(T-1)}\#$, with an additional dimension in the regular array attributable to p_T, thus incrementing the rank of regular tensor $\mathbf{R}'$. Any larger $n_{(T-1)}$ cannot succeed it. Let's suppose there were a number $n_{(T-1)}$ that would succeed $p_T\#$ in the record. This would happen if and only if $n_{(T-1)}$ had a smallest prime larger than that of $p_T\#$ or the distinct prime divisors of $n_{(T-1)}$ were even more distinct than those of $p_{(T-1)}\#$. Further, $r(n_{(T-1)})$ would have to be larger than $r(p_T\#)$ even though $\mathbf{R}$ of $n_{(T-1)}$ has a lower rank than $\mathbf{R}$ of $p_T\#$.

Finally, if we were to see a primorial $p_T\#$ succeeded by terms n with $\omega(n) = T - 1$, it might happen early in A244052. Recall that the regular tensor $\mathbf{R}$ is most efficient at producing a larger $r(n)$ when the distinct prime divisors p of n are minimally distinct. The ratio $p_{(T+1)}/p_T$ tends to be largest when T is small, meaning that small primes tend to be less distinct than larger primes.

We observe the first 15 primorials $p_T\#$ in A244052 and note $r(p_T\#)$ jumps markedly over that of largest term in tier $T - 1$, i.e., that of $(p_T - 1)\ p_{(T-1)}\#$. In fact, $r(p_T\#) > (r(p_T\#) - \mu(p_T\#) p_{(T-1)}\#)$, that is, the regular counting function of $p_T\#$ minus the number of regulars attributable to its greatest prime divisor p_T. Since the ratio $p_{(T+1)}/p_T$ generally approaches a limit of 1, we can say that there never will be a term with $\omega(n) = T - 1$ that succeeds $p_T\#$ in A244052. ▪

We have shown that A244052 is divided into tiers T that start with $p_T\#$ and all terms n have T distinct prime divisors.

2. Integer Multiples of Primorials in A244052. This section intends to prove that the multiples $kp_T\#$ with

(5.2) $$1 \le k \le (p_{(T+1)} + 1) - 1$$

appear in A244052.

Theorem 5.2. Let primorial $p_T\#$ be the product of the smallest T positive primes p. Let the nonzero positive integer $1 \le m \le n$ whose every prime divisor $p \mid n$ be termed a "regular of" or "regular to" n, counted by a "regular counting function" $r(n)$. Consider integer $1 \le k \le \text{PRIME}(T + 1) - 1$ as a multiplier of $p_T\#$. The numbers $kp_T\#$ set records for the number of regulars $r(n)$ of n. In other words, these numbers $kp_T\#$, which appear in OEIS A060735 by the definition of that sequence, appear in A244052, i.e., A060735 is a subset of A244052.

Lemma 5.2.1. The integer multiples $kp_T\#$ subdivide the tiers T of into "levels" k wherein all numbers

(5.2.1) $$kp_T\# \le n < (k + 1)p_T\#$$

must be divisible by k such that the quotient is a squarefree integer n that appears in level $k = 1$ of the tier and thereby has $\omega(kn) = T$.

Proof 5.2. We have already proved in Theorem 5.1 that primorials $1 \times p_T\#$ are in A244052. Lemma 5.2.1 shows that primorials delimit A244052 into tiers wherein all terms must have T distinct prime divisors. How about the integer multiples $k > 1$ of primorials such that k does not exceed the next prime?

Note that this is tantamount to Section 3 Case 1. The multiplier k divides n in all cases, since $k < \text{PRIME}(T + 1)$. Therefore, no new distinct prime divisors are introduced, and $kp_T\#$ has the same squarefree kernel as $p_T\#$. The number of distinct prime divisors is conserved among $kp_T\#$, i.e.,

$$\omega(p_T\#) = \omega(kp_T\#) = T.$$

Because of this the rank of the infinite regular array $\mathbf{I}$ common to all $kp_T\#$ is obviously the same. The tensors $\mathbf{R}'$ of $kp_T\#$ have a progressively larger bound as k increases, admitting more regulars from $\mathbf{I}$ such that each one is larger than the last. Thus the regular counting function $r((k + 1)p_T\#) > r(kp_T\#)$.

There may be "turbulent" squarefree terms $t > p_T\#$ with T distinct prime divisors that succeed the primorial in A244052. These are terms that have at least one totative $q < p_{(T+1)}$, the smallest totative of $p_T\#$, and at least one prime divisor $p > p_T$, the greatest prime divisor of $p_T\#$. Therefore the efficiency of these numbers t is less than those numbers $kp_T\#$ at producing regular numbers $1 \le m \le n$. We will return to turbulent numbers in the next topic.

Products $kp_T\#$ represent the reimposition of the most efficient infinite regular array $\mathbf{I}$ pertaining to numbers with T distinct prime divisors with a bound k times larger than that of primorial $p_T\#$. Thus $2p_T\#$ functions as a barrier to any larger squarefree turbulent number t, since these tend toward increasing inefficiency as the smallest totatives of t decrease and greatest prime factors of t increase.

Observing $kp_T\#$ is k times larger than $p_T\#$ we can likewise find turbulent products kt while k is relatively small, that is, k times the level-1 turbulent numbers t in their turn such that

(5.2.2) $$kp_T\# < kt < (k - 1)\ p_T\#/k.$$

Thus the $p_T\# < t < 2p_T\#$ or level-1 turbulent terms are "echoed" until the difference $(t_1 - p_T\#)$ between the smallest turbulent term and the primorial, multiplied by k, exceeds $(k - 1)\ p_T\#/k$. We will show later that the turbulence in tier T always abates within the tier.

Thus k subdivides the tier T into levels in which all terms n must have wherein all numbers $kp_T\# \le n < (k + 1)p_T\#$ must be divisible by k such that the quotient is a squarefree integer n that appears in the first level of the tier and the number of distinct prime divisors is conserved at T. All terms $kp_T\#$ such that $1 \le k \le \text{PRIME}(T + 1) - 1$ appear in A244052. Thus A060735 is a subset of A244052. ▪

3. "Turbulence". Examination of the terms in A244052 shows that indeed there are terms outside of A060735 = $kp_T\#$ with $1 \le k \le \text{PRIME}(T + 1) - 1$.

The terms stand out when we apply multiplicity notation to the sequence. Recall that multiplicity notation writes only the multiplicities of the distinct prime divisors of a number. The position of the multiplicity in the notation denotes the prime, read from left to right. Thus, "2011" signifies $2^2 \times 5^1 \times 7^1 = 140$. This is equivalent to $E(n) = \text{REVERSE}(\text{A054841}(n))$. Let us call the number-like result of multiplicity notation a "word" to better distinguish it from the value n. Figure 5.2 shows the terms in in the fifth column, followed by the corresponding multiplicity notation. The "turbulent" terms are so-named due to the "flaring out" of their multiplicity notation, which corresponds to Section 4 Case 3.

Turbulent terms t have several things in common:

1. $\omega(t) = T$ distinct prime divisors,
2. at least one prime totative $q \le p_T$, greatest prime factor of $p_T\#$,
3. divisible by k if $kp_T\# < t < (k+1)p_T\#$, such that t/k is squarefree with $\omega(t) = T$ distinct prime divisors.

The totatives $q < p_{max}$ of t appear in multiplicity notation as "0," thus turbulent terms are also distinguished by zeros in that notation.

Summary of Theorems. Through Theorems 5.1 and 5.2 and their lemmas, we have a necessary but not sufficient set of conditions for terms in A244052. Let's summarize the conditions:

Theorem 5.3.

Let the integers $T \ge 1$, $1 \le k \le \text{PRIME}(T+1) - 1$, and $p_T\#$ be the "tier", "level", and the primorial, i.e., product of the smallest T primes, respectively. Consider an integer $n \ge 1$. A squarefree term $p_T\# \le n < 2p_T\#$ is in A244052 if and only if $\omega(n) = T$. Further, multiples $kp_T\# \le kn < (k+1)p_T\#$, also may appear in A244052. All n and kn that are in A060735 are also in A244052, but the terms $n = t$ that are not in A060735 require the regular counting function to determine if $r(t)$ sets a record and thus appears in A244052.

This theorem has been proven by Theorems 5.1 and 5.2 and their lemmas. Note that these conditions are necessary but not sufficient, since we are dealing with a regular array bounded by n that is governed by the spacing of primes. The terms of A060735 are certainly in A244052 due to the proofs pertaining to primorials $p_T\#$ and multiples $kp_T\#$. Only the "turbulent" terms t are those that require testing by running a regular counting function $r(t)$.

One might be able to devise a calculus solution for the hypervolume of the T-rank regular tensor **R**. The curved sheet that is formed by the bound n is the floor of a curve produced by the geometric progressions of the distinct prime divisors. This function is beyond the scope of this work, but could further refine the conditions of this paper.

Sequence $a(n)$. Let us define sequence $a(n)$ as the sequence produced by Theorem 5.3. It is fairly easy to write a brute-force algorithm that generates candidate terms for A244052. The following function merely computes the candidates in level $k = 1$ of tier $n = 3$:

(CODE 5.3.1)

```
f[n_] := Select[Range[#, 2 # - 1] &[
  Times @@ Prime@ Range@ n],
  EvenQ@ # && PrimeNu@ # == n &]; f@ 3
```

```
{30, 42}
```

Mapping this function across the first several tiers gives us the primorial followed by the first-level turbulence:

(CODE 5.3.2)

```
f /@ Range@ 8 // Flatten
```

```
{2, 6, 10, 30, 42, 210, 330, 390, 2310, 2730, 3570,
3990, 4290, 30030, 39270, 43890, 46410, 51870,
53130, 510510, 570570, 690690, 746130, 870870,
881790, 903210, 930930, 1009470, 9699690, 11741730,
13123110, 14804790, 15825810, 16546530, 17160990,
17687670, 18888870}
```

This more efficient code produces the same output but uses binary to encode the turbulence. Though it might seem ingenious, there are better ways to do the same:

(CODE 5.3.3)

```
Map[Function[r, Select[#,
  And[r <= # < 2r, PrimeOmega@ # == PrimeOmega@ r] &]],
    Map[Times @@ Prime@ Range@ # &,
      Range@ PrimeOmega@ Max@ #]] &@
   Sort@ Select[#, EvenQ] &@
 Map[Times @@ Prime@ Flatten@ Position[#, 1] &,
  Map[Reverse, IntegerDigits[Range[0, 2^20], 2]]]
```

```
(361 terms)
```

Such a function entails we might leverage multiplicity notation to produce $a(n)$.

Returning to the original algorithm 5.3.1, we can write this function generates all the candidates in all levels in the tier. The segment after the semicolon applies the function to the range 1 through 6 to produce all terms through tier 6:

(CODE 5.3.4)

```
f[n_] := Block[{P = Times @@ Prime@ Range@ n},
  Map[Function[k,
    DeleteCases[
     Select[#, # < (k + 1) P &] &@
      (k Select[Range[#, 2 # - 1] &@ P,
         EvenQ@ # && PrimeNu@ # == n &]), {}]],
   Range[Prime[n + 1] - 1]]];
{1}~Join~Array[f, 6] // Flatten
```

```
{1, 2, 4, 6, 10, 12, 18, 24, 30, 42, 60, 84, 90,
120, 150, 180, 210, 330, 390, 420, 630, 840, 1050,
1260, 1470, 1680, 1890, 2100, 2310, 2730, 3570,
3990, 4290, 4620, 5460, 6930, 8190, 9240, 10920,
11550, 13650, 13860, 16170, 18480, 20790, 23100,
25410, 27720, 30030, 39270, 43890, 46410, 51870,
53130, 60060, 78540, 87780, 90090, 117810, 120120,
150150, 180180, 210210, 240240, 270270, 300300,
330330, 360360, 390390, 420420, 450450, 480480}
```

The brute force approach in codes 5.3.1 and 5.3.4 above sifts through all numbers $p_T\# \le n < 2p_T\#$ to find those that are even and have $\omega(n) = T$. It is effective through about tier 8.

Referring to Figure 5.2, it is evident that the term 4290 generated by this method does not appear in A244052. Appendix B1 shows terms in $a(n)$ but not in A244052 in italics through tier 10 (i.e., 23# in the table). There are generally several more terms in tiers T > 6 that are in $a(n)$ but not A244052 such that by $p_{10}\#$, 174 of the 183 terms in $a(n)$ are in A244052.

6. Aspects of Turbulence.

We can make further observations about "turbulence" in general with notions we have proved. Notably, we can segment A244052 into "tiers" T that start with $p_T\#$ as the smallest term in the tier. Every term n in the tier must have $\omega(n) = T$ distinct prime divisors. Each tier can be further divided into "levels" k that begin with $kp_T\#$ with $1 \le k \le \text{PRIME}(T+1) - 1$ as the smallest term in the level. All terms kn in the level must be divisible by k such that the quotient is a squarefree integer n that appears in level $k = 1$ of the tier and thereby has $\omega(kn) = T$.

Looking at all the "turbulence" (the turbulent terms t) in the tier, aided by multiplicity notation, we observe the following:

1. Let p_T be the maximum prime such that at least one term n in tier T is divisible by p_T.
2. All terms in tier T are divisible by a smaller primorial $p_F\#$, that is the greatest common divisor of all terms in tier T. We will call the "frustum primorial" or simply "frustum".

3. There is a final level k at which we observe turbulence. We will refer to that level as the "reverb" of the tier.

Let's define a few methods of measuring these three aggregate qualities of turbulence.

DISTENSION. Let "distension" i be the number of totatives $q < p_{max}$, i.e., the greatest prime factor of n. In terms of multiplicity notation, i is the number of zeroes in the word $E(n)$.

The greatest distension in the tier, i_T, is the "cardinal distension" of tier T. Cardinal distension i_T is thus descriptive of p_T relative to T. The greatest prime factor of all the terms in the tier is PRIME$(T + i)$. We will prove this after the next definition.

DEPTH. Let "depth," a positive integer j, be the number of largest prime divisors of $p_T\#$ to divide $p_T\#$ by to admit j new prime factors $p \geq (p_T - j + 1)$. In multiplicity notation, depth j refers to the insertion of a zero j places from the right end of the primorial word, thus $j = 1$ is a word ending in "01," $j = 2$ ends in "011," etc. We may insert more zeros in the word after this first zero to produce some turbulent terms.

"Cardinal depth" j_T is the greatest depth in tier T and is descriptive of the least prime totative q_1 of the tier. This also denotes the prime frustum of the tier.

VOLUME. The total number of turbulent candidates per tier, level, depth, or distension in $a(n)$ or turbulent numbers in A244052 is the volume for that compartment. In analysis we will examine the total number of turbulent candidates in level 1 of tier T with depth j. This we will refer to as "volume," with all levels of tier T considered "total volume."

REVERB. Let "reverb" k_t be the greatest value of k such that $kp_T\# < kt_1 < (k - 1)\, p_T\#/k$. This is easy to determine and proves turbulence is always contained within a given tier.

Theorem 6.1. The product $p_{(T-1)}\#\, p_{(T+1)}$ is the smallest squarefree number n with $\omega(n) = T$ distinct prime divisors such that $n > p_T\#$.

Proof 6.1. We already recognize $p_T\#$ is the smallest number with $\omega(n) = T$. In order to distinguish n from $p_T\#$ while conserving the value of $\omega(n) = T$, we need to use Section 4 Case 3, i.e., divide $p_T\#$ by $1 \leq i \leq T$ of its prime divisors p, then multiply by the same number of new primes.

Consider discarding all the prime divisors of $p_T\#$ and producing n by taking the product of the next T smallest primes. Suppose $T = 1$, thus $p_1\# = 2$. Therefore the smallest $n = 3$. In this case there can be no smaller $n > 2$ with $\omega(n) = 1$ (i.e., prime). Now suppose $T = 2$, thus $p_2\# = 6$. Naturally, the smallest product then is that of the next 2 primes. Therefore we take $5 \times 7 = 35$, but $10 = 2 \times 5$ is smaller, as is 14, 15, 21, 22, 26, 33, and 34.

Indeed, we need only to discard one prime divisor p of $p_T\#$ in order to distinguish n. Taking away more than one prime divisor simply means that we'd be adding it back when considering the smallest factor through which we can produce a valid n. If we are only replacing one prime divisor p of $p_T\#$ with another prime that does not divide $p_T\#$, the smallest of these is the minimum prime totative q_1 of $p_T\#$, i.e., $p_{(T+1)}$. Essentially, we are using a factor q_1/p to produce a valid n. This ratio is minimized when the primes p and q_1 are most similar. Since we know $q_1 = p_{(T+1)}$, the prime that necessarily divides $p_T\#$ that is most similar, i.e., a prime such that the difference $p_{(T+1)} - p$ is minimized, is p_T, the very largest prime divisor of $p_T\#$.

Thus the smallest squarefree $n > p_T\#$ is

$$p_T\#/p_T \times p_{(T+1)} = p_{(T-1)}\#\, p_{(T+1)}. \blacksquare$$

Thus, multiplicity notation words 11101, 111101, etc., represent the smallest squarefree numbers $n > p_T\#$ with $\omega(n) = T$ distinct prime divisors. These are equivalent to 330 > 210, 2730 > 2310, etc. Generally the word associated with such an n consists of $T - 1$ ones with a suffix of "01".

Corollary 6.1.1. There is a minimum depth $j = 1$ that produces a turbulent term t_1, and that term is $p_{(T-1)}\#\, p_{(T+1)}$.

Proof 6.1.1. We know from Proof 6.1 that the smallest turbulent term t_1 is $p_{(T-1)}\#\, p_{(T+1)}$. It follows that it is the product of $T - 1$ smallest primes thus the smallest totative of t_1 is p_T. The equation of $\pi(p_T) = T$ which can be written $T - j + 1$ by the definition of depth. This simplifies to $j = 1$. Since p_T is the largest prime divisor of $p_T\#$ there can be no smaller number j. ▪

This fact gives us a starting point for algorithmically generating tier T turbulence.

Corollary 6.1.2. The product

$$p_T\#\, p_{(T+1)}/p_{(T-j+1)} = p_{(T-j)}\# \prod_{1 \leq x \leq j} p_{(T-x+2)}$$

$$\text{with depth } j = \pi(p_T) - \pi(q_1) + 1 \text{ and}$$

$$\text{and } q_1 = \text{smallest prime such that } \text{GCD}(q_1, n) = 1$$

is the smallest squarefree number n with $\omega(n) = T$ distinct prime divisors and the minimum prime totative $q_1 = p_{(T-j+1)}$ such that $n > p_T\#$.

Proof 6.1.2. Corollary 6.1.2 is implied by the fact that the smallest number $n > p_T\#$ with $\omega(n) = T$ and $q_1 = p_{(T-j+1)}$ (i.e., depth j) is

$$p_T\#\, p_{(T+1)}/p_{(T-j+1)},$$

whose word has length $T + 1$, consists of all ones and a zero at j ones from the right. Let's look at it another way. Consider the number divisible by $p_{(T-j)}\#$ but not by $p_{(T-j+1)}$. If we have to construct a squarefree number produced by T primes, we have used $(T - j)$ primes in $p_{(T-j)}\#$ and thus have j to use. All the primes $p \leq p_{(T-j)}$ are unavailable, since $p_{(T-j)}\#$ by definition is the product of the $(T - j)$ smallest primes.The smallest number we can make whose smallest prime factor $p > p_{(T-j+1)}$ is the product of the j primes

$$\text{PRIME}(T - x + 2)$$

$$\text{with } 1 \leq x \leq j.$$

These are the smallest primes greater than $p_{(T-j+1)}$ and include $j - 1$ primes that also divide $p_T\#$. This is therefore equivalent to

$$t_j = p_T\#\, p_{(T+1)}/p_{(T-j+1)}. \quad (6.1.2)$$

We have proved Corollary 6.1.2. ▪

Therefore by Corollaries 6.1.1 and 6.1.2, generally the number t_j is the smallest turbulent term of depth j. This implies that we can check for the presence of depth j in a given tier simply by generating for t_j.

The multiplicity notation words 11111101, 11111011, and 11110111 pertaining to tier T = 7 are of the form

T	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1												
3	1												
4	2												
5	3	1											
6	3	2											
7	4	2	1										
8	4	3	1										
9	5	3	2										
10	6	4	2	1									
11	7	5	3	1									
12	9	6	4	2									
13	9	8	5	3	1								
14	9	8	7	4	2								
15	9	8	7	6	3	1							
16	11	8	7	6	5	2							
17	13	10	7	6	5	4	1						
18	12	12	9	6	5	4	3						
19	13	11	11	8	5	4	3	2					
20	14	12	10	10	7	4	3	2	1				
21	13	13	11	9	9	6	3	2	1				
22	15	12	12	10	8	8	5	2	1				
23	15	14	11	11	9	7	7	4	1				
24	16	14	13	10	10	8	6	6	3				
25	19	15	13	12	9	9	7	5	5	2			
26	20	18	14	12	11	8	8	6	4	4	1		
27	19	19	17	13	11	10	7	7	5	3	3		
28	19	18	18	16	12	10	9	6	6	4	2	2	
29	18	18	17	17	15	11	9	8	5	5	3	1	1
30	18	17	17	16	16	14	10	8	7	4	4	2	
31	23	17	16	16	15	15	13	9	7	6	3	3	1
32	23	22	16	15	15	14	14	12	8	6	5	2	2

(FIG. 6.2.1) Distension i at depth j (horizontal axis) in tier T (vertical axis).

	2	3	5	7	11	13	17	19	23	29	31	37	41
2#	0												
3#		1											
5#			1										
7#				2									
11#				1	3								
13#					2	3							
17#					1	2	4						
19#						1	3	4					
23#							2	3	5				
29#							1	2	4	6			
31#								1	3	5	7		
37#									2	4	6	9	
41#									1	3	5	8	9
43#										2	4	7	8
47#										1	3	6	7
53#											2	5	6
59#											1	4	5
61#												3	4
67#												2	3
71#												1	2
73#													1

(FIG. 6.2.2) Distension i for prime p (horizontal axis) in tier T (vertical axis), a transform of Figure 6.2.1.

.Candidacy

1	0		
2	1		
3	1		
4	2		
5	3	1	
6	3	2	
7	4	2	1
8	4	3	1
9	5	3	2

.Presence in A244052

1	0	
2	1	
3	1	
4	2	
5	3	
6	3	2
7	4	1
8	4	3
9	5	1

(FIG. 6.2.3) The chart on the left is the first 9 rows of Figure 6.2.1, while the chart at right shows the maximum distension at (T, j) observed in A244052. Note that no depth $j > 3$ appears in tier $T \le 9$, while all depth $j = 1$ distension terms appear in the same range.

$$p_T\# \, p_{(T+1)}/p_{(T-j+1)}.$$

These equate to 570,570, 746,130, and 881,790, respectively, with 510,510 = $p_7\#$. See Appendix B1 to observe that these are the first occurring words with a zero in place $T - j + 1$.

Corollary 6.1.3. The terms t for $j = k = 1$ are simply those produced by primorial $p_{(T-1)}\#$ and one of the primes $p \ge p_T$ such that $t < 2p_T\#$.

Proof 6.1.3. We know that terms t must have T distinct prime divisors, thus conserving $T - 1$ of them via $p_{(T-1)}\#$ and adding one via a single prime larger than p_T would honor the constraint. Theorem 5.3, or more specifically, Lemma 5.2.1 is the source of the constraint that confines the terms t in question to those less than twice the primorial on T. The smallest turbulent term $t_1 = p_{(T-1)}\# \, p_{(T+1)}$ as shown by Corollary 6.1.1. ▪

Further, through Formula 5.2.2, we know that terms

$$kp_T\# < kt < (k-1)\, p_T\#/k$$

occur in level k of tier T.

Theorem 6.2. Let integer $i_T \ge 1$ be called the "cardinal distension" of tier T. There is a largest term t_i for $j = k = 1$ such that $t_i < 2p_T\#$. This is the product

$$p_{(T-1)}\# \, p_{(T+i)}.$$

The term t_i has the largest value of i in the tier.

Proof 6.2. Let's look at the problem using multiplicity notation. This problem starts with a constant array of $(T - 1)$ ones, i zeros, and a final one, e.g., for $T = 4$, we start with 11101. We merely need to add zeros in a monotonically increasing fashion until we have a word whose decimal value exceeds $2p_T\#$. Once we have this number, we need to subtract 1 to get i_T. Algebraically, we would multiply $p_{(T-1)}\#$ by primes $p_{(T+i)}$, incrementing i until the product exceeds $2p_T\#$. Similarly we would need to subtract one from the result to get i_T.

The number t_i as described has the largest possible value of i for $k = 1$. Suppose $t_i = ab$. The ratio b/a is largest when a is small and b large. Since we must conserve the number T of distinct prime divisors, necessarily, $a = p_{(T-1)}\#$ and prime $b = p_{(T+i)}$.

Depths $j > 1$ require a totative $p_{(T-j+1)}$. Let's recall Formula 6.1.2, which describes the smallest term t for a given j:

$$t_j = p_T\# \, p_{(T+1)}/p_{(T-j+1)}$$

We determine distension by dividing t_j by the largest prime divisor $p_{(T+1)}$ and multiplying by $p_{(T+i)}$, incrementing i until the product exceeds $2p_T\#$. Hence we start with the number $p_T\#/p_{(T-j+1)}$. This number must be larger than $p_{(T-1)}\#$ since both have the same number of prime divisors and the latter is the product of the smallest contiguous set of primes. Thus, the distension i with respect to $j > 1$ must be smaller than that of $j = 1$.

Finally, recall Formula 5.2.2:

$$kp_T\# < kt_1 < (k-1)\, p_T\#/k.$$

This formula implies that the largest t in level 1 is disqualified as it is by definition at the top of a range that is progressively shortened toward the bottom of the range. The distension seen in level 1 is the largest possible distension. Therefore, t_i has the largest possible value of i in the tier. ▪

For this reason, we call the distension pertaining to $j = 1$ the "cardinal distension" i_T of tier T.

We can approach cardinal distension using several methods. The first two methods involve an iterative test. We'll look at one that uses multiplicity notation later. This program assembles the word described at the start of Proof 6.2 using arrays dependent on i:

(CODE 6.2)

```
Map[With[{n = #}, i = 1;
  While[Times @@ Prime@ Flatten@ Position[#, 1] &@
    Join[ConstantArray[1, n - 1],
      ConstantArray[0, i], {1}] <
      2 (Times @@ Prime@ Range@ n), i++]; i - 1] &,
  Range@ 30]
```

```
{1, 1, 1, 2, 3, 3, 4, 4, 5, 6, 7, 9, 9, 9, 9, 11, 13,
12, 13, 14, 13, 15, 15, 16, 19, 20, 19, 19, 18, 18}
```

Thus with this program it is clear that in tier 12, the cardinal distension is 9, which is to say $p_{12}\#\, p_{21}$, a number with multiplicity notation "111111111110000000001" or decimal 14640915779490.

We'll return to algorithms after Proof 6.2.1.

Corollary 6.2.1. Let the j-distension i be the distension that pertains to the maximum prime divisor for turbulent terms t of depth j. The j-distension i decreases as j increases.

Proof 6.2.1. Recall the discussion of Formula 6.1.2 in Proof 6.2. We mentioned the smallest prime totative $p_{(T-j+1)}$ and deriving $p_T\#/p_{(T-j+1)}$ from t_j. We had determined that $p_T\#/p_{(T-j+1)} > p_{(T-1)}\#$. Compare the smallest prime totatives of depths j and $j + 1$, i.e., $p_{(T-j+1)}$ and $p_{(T-j+2)}$ respectively. It is clear that $p_{(T-j+1)} > p_{(T-j+2)}$, thus, since the number T of distinct prime divisors must be conserved, it is clear that the largest prime divisor that does not exceed $2p_T\#$ must be smaller for terms t with larger values of j. ▪

We can approach programming j-distension in an algebraic or multiplicity notation method. These methods integrate the concept of 1-distension applied in code 6.2. This function uses an algebraic approach:

(CODE 6.2.1)

```
iDistension[n_, j_] := -1 + SelectFirst[Range@ n,
  Function[i, # Prime[n + i]/Prime[n - j + 1] > 2 #] &[
    Times @@ Prime@ Range@ n]]
      /. k_ /; MissingQ@ k -> 1
iDistension[5, #] & /@ Range@ 5
```

```
{3, 1, 0, 0, 0}
```

This shows distension for all possible depths in tier 5. Thus, for $j = 2$, the distension is 1, i.e., in multiplicity notation we have "111011" = decimal 4290. Figure 6.2.1 shows the j-distensions in tiers $1 \le T \le 32$.

Corollary 6.2.2. There is a maximum or "cardinal depth" j_T for tier T. This implies that there is a "frustum primorial" $p_{(T-j)}\#$ that divides all terms n of tier T.

Proof 6.2.2. Figure 6.2.1 suggests this is true. We can at least say that j_T cannot exceed T, leaving 1. We studied similar scenarios in Proofs 5.1 (odd numbers) and 6.1 (discarding all prime divisors of primorial). Such numbers do not qualify for the reasons stated in those proofs.

Figure 6.2.1 suggests DISTENSION$(T + 1, j + 1)$ = DISTENSION$(T, j) - 1$ in all visible cases. If this holds true for all T, all we would need to know in order to produce the same chart is cardinal

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1												
3	1												
4	2												
5	3	1											
6	3	2											
7	4	3	1										
8	4	3	1										
9	5	3	2										
10	6	7	2	1									
11	7	9	3	1									
12	9	11	8	2									
13	9	16	11	6	1								
14	9	16	15	6	3								
15	9	15	14	9	3	1							
16	11	20	16	13	6	2							
17	13	28	24	16	12	6	1						
18	12	33	32	18	10	7	3						
19	13	36	48	33	13	7	4	2					
20	14	40	50	49	27	10	5	3	1				
21	13	38	54	43	34	15	5	2	1				
22	15	40	57	56	32	24	9	3	1				
23	15	43	56	59	44	21	15	5	1				
24	16	49	70	58	52	35	14	10	3				
25	19	64	91	96	66	54	33	15	12	2			
26	20	78	130	124	107	61	50	29	14	10	1		
27	19	76	150	153	117	89	44	33	18	10	6		
28	19	72	140	186	137	89	63	29	23	10	5	3	
29	18	65	113	153	149	86	48	31	15	10	4	1	1
30	18	59	95	106	114	93	46	22	12	4	4	2	
31	23	78	123	141	129	124	93	40	20	11	5	4	1
32	23	113	162	165	162	130	116	82	34	17	9	3	2

(FIG. 6.3) Total volume or population of distinct turbulent candidates in depth j (horizontal axis) of tier T (vertical axis) of $a(n)$.

distension. The consideration of depth leaves us with a primorial $p_{(T-j)}\#$ that divides all the terms generated by any j-distension i. The depth or j parameter is actually a relational one tied to T itself. In actuality, the coordinates $(T,\ j)$ and $(T + 1, j + 1)$ refer to the same prime, p = PRIME$(T - j + 1)$. We could then write the table a new objective way as Figure 6.2.2, solely in terms of q.

The prime p has a distension i that predicts the "staying power" or "tolerance" a larger primorial has if p "goes missing."

Instead of looking at distension i with respect to prime p, we can simply use position in the list to signify the n-th prime and note $\pi(q)$ of the smallest prime q for which p no longer can be tolerated as a gap in the prime decomposition of a turbulent candidate. We arrive at this with the following code:

(CODE 6.2.2)

```
Table[n + iDistension[n, 1] /. {} -> {0}, {n, 70}]
```

This yields:

```
{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 18, ...}
```

With the exception of $a(1)$, the resultant data matches OEIS A020900: "Greatest k such that prime $p_k < 2\, p_n$". Thus, cardinal distension i is merely an artifact of $p_k < 2\, p_n$. Harvey P. Dale wrote an elementary script that generates the sequence:

(CODE 6.2.2.1)

```
Map[PrimePi@ NextPrime[#, -1] &,2 Prime@ Range@ 70]
```

This is essentially $\pi(2\, p_T)$.

Now we can express distension and depth solely in terms of A020900$(T) = \pi(2\, p_T)$.

Cardinal distension i_T:

Tier T	Tier GPF=prime(T+i)	Tier GCD=p_(T-j+1)#	Cardinal Distension i	Cardinal Depth j	∑(i,j)	Primary Volume	Primary Volume A244052	Total Volume	Total Volume A244052	Position of p_T#	Position of p_T# A244052	Population	Population A244052
0	0	0	0	0	0	0	**0**	0	**0**	1	**1**	1	**1**
1	1	1	0	0	0	0	**0**	0	**0**	2	**2**	2	**2**
2	3	1	1	1	1	1	**1**	1	**1**	4	**4**	5	**5**
3	4	2	1	1	1	1	**1**	2	**2**	9	**9**	8	**8**
4	6	3	2	1	2	2	**2**	2	**2**	17	**17**	12	**12**
5	8	3	3	2	4	4	**4**	8	**7**	29	**29**	20	**19**
6	9	4	3	2	5	5	**5**	8	**8**	49	**48**	24	**24**
7	11	4	4	3	7	8	**5**	17	**14**	73	**72**	25	**22**
8	12	5	4	3	8	8	**7**	12	**11**	108	**104**	34	**33**
9	14	6	5	3	10	10	**6**	13	**9**	142	**137**	41	**37**
10	16	6	6	4	13	16	**10**	34	**28**	183	**174**	64	**58**
11	18	7	7	4	16	20	**13**	31	**24**	247	**232**	67	**60**
12	21	8	9	4	21	30	**18**	51	**38**	314	**292**	91	**77**
13	22	8	9	5	26	43	**21**	83	**60**	405	**369**	125	**101**
14	23	9	9	5	30	49	**26**	76	**52**	530	**470**	122	**93**
15	24	9	9	6	34	51		72		652	**563**	124	
16	27	10	11	6	39	68		95		776		153	
17	30	10	13	7	46	100		162		929		222	
18	30	11	12	7	51	115		163		1151		229	
19	32	11	13	8	57	156		228		1380		298	
20	34	11	14	9	63	199		301		1678		373	
21	34	12	13	9	67	205		274		2051		352	
22	37	13	15	9	73	237		316		2403		398	
23	38	14	15	9	79	259		331		2801		419	
24	40	15	16	9	86	307		388		3220		484	
25	44	15	19	10	96	452		599		3704		699	
26	46	15	20	11	106	624		865		4403		967	
27	46	16	19	11	114	715		951		5370		1057	
28	47	16	19	12	122	776		1037		6427		1145	
29	47	16	18	13	128	694		865		7572		977	
30	48	18	18	12	133	575		669		8549		795	
31	54	18	23	13	144	792		959		9344		1089	
32	55	19	23	13	154	1018		1225		10433		1361	
33	58	19	25	14	166	1344				11794			
34	59	21	25	13	177	1498							
35	62	21	27	14	191	2088							
36	62	21	26	15	203	2465							

(FIG. 6.5) Summary of various aggregate aspects of each tier T of $a(n)$. The total of all j-distensions (i.e., the sum of i values for each depth j) appears in the column marked "Σ(*i*,*j*)". The term "volume" means the number of turbulent terms, while "population" means all terms. Primary volume refers to turbulence for level $k = 1$, while "total volume" refers to all turbulence in the tier. Figures that pertain to A244052 appear in bold.

$$\text{(6.2.2.2)} \quad \begin{aligned} i_T &= \text{A020900}(T) - T \\ &= \pi(2\,p_T) - T, \\ f(1) &= 1, \\ f(x) &:= \pi(2\,p_x) - x \end{aligned}$$

Thus:

$$\text{(6.2.2.3)} \quad \begin{aligned} \text{DISTENSION}(T, j) &= \text{A020900}(T - j + 1) - T - j + 1 \\ &= f(T - j + 1). \end{aligned}$$

Therefore we could produce the tables using Formula 6.2.2.3 and need not probe primorials. This code is vastly more efficient than codes 6.2 and 6.2.1. Figures 6.2.1, 6.2.2, and Appendix A4 derive from $\pi(2\,p_x) - x$.

This code produces Figure 6.2.1:

(CODE 6.2.2.4)

```
Table[TakeWhile[
  Map[
    If[# == 1, 1, PrimePi[2 Prime@ #]] - n &[
    n - # + 1] &, Range@ n], # > 0 &],
  {n, 12}] // TableForm
```

This association of j-distension with A020900 shows that there is a cardinal depth j_T for tier T, hence a frustum primorial $p_{(T-j)}\#$ that divides all terms n of tier T. Also, there is a totative $p_{(T-j+1)}$ common to all terms n in tier T. Because j-distension boils down to a far more elementary problem, it becomes easy to determine these parameters for large T. ▪

Now we can define an algorithm for cardinal depth j_T:

(CODE 6.2.2.5)

```
jDepth[n_] := -1 + SelectFirst[Range@ n,
  Function[j, # Prime[n + 1]/Prime[n - j + 1] > 2 #] &[
    Times @@ Prime@ Range@ n]]
      /. j_ /; MissingQ@ j -> 1
```

Appendix A4 shows j-distension for tiers $1 \le T \le 60$.

The definition and affirmation of the aggregate parameters of turbulence in tier T sets the stage for an efficient generation of turbulent candidates for A244052.

Corollary 6.2.3. We can determine the largest level k of tier T with $1 \le k < (p_{(T+1)} + 1)$, for which turbulent candidates t appear. This is effectively the number of levels k of tier T for which we have turbulence.

Proof 6.2.3. We can write a program that looks for the "reverberation" of the smallest turbulent term t_1 in tier T. We proved in Proof 6.1 that $t_1 = p_{(T-1)}\#\, p_{(T+1)}$. Proof 5.2 shows that terms $kp_T\# < kt < (k - 1)\, p_T\#/k$ relate to level-1 turbulent terms t. The formula implies that the ratio in Formula 6.2.3 must be true if kt is in $a(n)$.

$$\text{(6.2.3)} \quad kt < (k + 1)\, p_T\#$$

Which simplifies to:

$$\text{(6.2.3.1)} \quad kp_{(T+1)} < (k + 1)\, p_T$$

The sequence generated by these formulas matches OEIS A102551 for all $n > 1$. Thus we can use the following formula as well:

$$\text{(6.2.3.2)} \quad \begin{aligned} f(1) &= 1, \\ f(x) &= \text{FLOOR}(p_x/(p_{(x+1)} - p_x)) \end{aligned}$$

The following code computes the "reverb" or the maximum level k for which turbulence occurs in tier T:

(CODE 6.2.3)

```
Table[Floor[Prime[n]/(Prime[n + 1] - Prime[n])] -
  Boole[n == 1], {n, 30}]
```

```
{1, 1, 2, 1, 5, 3, 8, 4, 3, 14, 5, 9, 20, 10, 7, 8,
29, 10, 16, 35, 12, 19, 13, 11, 24, 50, 25, 53, 27, 8}
```

Thus we have proved that we can determine the largest level k for which we can find a term n in $a(n)$ such that n is not in A060735.

Using figures at [https://primes.utm.edu/notes/gaps.html] $f(n)$ never exceeds p_n, since the difference $p_{(n+1)} - p_n$ remains small compared to p_n for $n \le 486570087$. It is beyond the scope of this paper to prove $\text{PRIME}(p_{(n+1)} - p_n) < p_n$ for all n, i.e., that we could find "turbulence" even in level $k = p_T - 1$ for some tier T. If this occurs, it only can possibly occur for $T > 486570087$.

Theorem 6.3. The population or "volume" of turbulent terms $t_{(i,}$

$_{j)}$ for $j = 1$ equals i_T, but for $j > 1$, there may be many more terms.

Proof 6.3. Terms with depth j involve retaining the first $(T - 1)$ prime divisors of $p_T\#$ as $p_{(T-1)}\#$ and multiplying by a prime p_i with $1 \le i \le i_T$. Because there are no further combinations of prime factors other than the single range pertaining to p_i, we have i_T possible combinations and none further for terms with $j = 1$.

Let's look at an example in mutliplicity notation. For $T = 5$, we have 111101, 1111001, and 11110001 representing the terms pertaining to depth of 1 and cardinal distension of 3. These are the only possible combinations for depth of 1.

When depth j exceeds 1, there may be several variants owing to the possibility of more than one prime distended from the smallest configuration. It may be easier to visualize this through example in multiplicity notation.

For $T = 11$ and $j = 2$, we have the following combinations:

```
11111111011
111111110101
111111110011
1111111101001
11111111010001
1111111100101
11111111001001
```

The 2-distension at tier 11 is 4, but there are 7 possible combinations. In smaller tiers, terms with $j > 1$ may be constrained to i terms, but the potential exists to have alternative configurations with more than one prime smaller than the greatest prime factor distended from the "home" position, i.e., compact and leaving only the minimum prime totative q_1 less than j distinct prime factors. ▪

Let's examine the j-distensions of tiers 1–12. Figure 6.4 below at left shows the first 12 rows of Figure 6.2.1, while the right side shows the first 12 rows of Figure 6.3. The chart on the left shows the j-distension for $k = 1$. The tier T (vertical axis) is compared against the depth j (horizontal axis). The chart on the right instead shows the number of turbulent candidates with depth j.

(FIG. 6.4)

```
     j-distension                          Volume
 1|   0                             1|   0
 2|   1                             2|   1
 3|   1                             3|   1
 4|   2                             4|   2
 5|   3    1                        5|   3    1
 6|   3    2                        6|   3    2
 7|   4    2    1                   7|   4   _3    1
 8|   4    3    1                   8|   4    3    1
 9|   5    3    2                   9|   5    3    2
10|   6    4    2    1             10|   6   _7    2    1
11|   7    5    3    1             11|   7   _9    3    1
12|   9    6    4    2             12|   9  _11   _8    2
```

It's clear that there are more than one term with distension i and depth j for values of $1 < j < j_T$.

Summary of Aggregate Aspects of Turbulence.

Having arrived at definitions, proofs, and reviewed data regarding turbulent candidates in $a(n)$ and turbulent terms in A244052, we can summarize what we know about terms in these sequences that are not in A060735.

Figure 6.5 charts various aspects pertaining to turbulence for each tier $1 \le T \le 36$ when known.

Let's recap the various metrics in the chart by column.

1. Cardinal distension i_T. This is a relative measure of the largest prime factor PRIME$(i_T + T)$ in the tier. With respect to multiplicity notation, this is the number of digits between the last digit representing the greatest prime factor of primorial $p_{(T-j)}\#$ and that of $p_{(T-j)}\#\ p_i\#$. In all cases we have proven this must be a product of $p_{(T-1)}\#$ with level $k = 1$ and depth $j = 1$.

2. Cardinal depth j_T. This measures the maximum number of largest prime divisors of $p_T\#$ we can divide $p_T\#$ by to arrive at the "frustum primorial" $p_{(T-j)}\#$ with $j = j_T$. This primorial is the greatest prime divisor of all numbers in tier T.

3. This is the sum of all distensions at all depths j in level 1 of tier T, i.e., the row sums of Figure 6.2.1.

4. This is the sum of all turbulent candidates at all depths j in level 1 of tier T, i.e., the row sums of Figure 6.3.

5. The sum of all turbulent candidates at all depths j in all levels of tier T.

6. The penultimate column t gives the number of level 1 turbulent terms of tiers in A244052.

7. The last column t_k gives the total number of turbulent terms in all levels of tiers in A244052.

7. Generation of Turbulent Candidates.

We can now marshal the aggregate parameters of distension i and depth j to generate all potential turbulent terms t for each level k of tier T of A244052. Through Proof 5.2.2 we know that the turbulent candidates of level 1 "reverberate" in level $k > 1$ until extinction per Corollary 6.2.3. It is sufficient that we only generate turbulent candidates $p_T\# \le n < 2p_T\#$, then find $kp_T\# \le kn < (k - 1)\ p_T\#/k$, having found all terms t for $k = 1$.

Code 7.1 shows the "SWEEP" function, which integrates many of the concepts discussed in this paper to produce the squarefree numbers $p_T\# \le n < 2p_T\#$.

(CODE 7.1)

```
sweep[n_] := Block[{primorialP, jDepth, jFrustum,
  encode, decode},
  primorialP[x_] := Times @@ Prime@ Range@ x;
    jDepth[x_] := -1 + SelectFirst[Range@ x,
      Function[j, # Prime[x + 1]/Prime[x - j + 1] >
        2 #] &[Times @@ Prime@ Range@ x]]
          /. j_ /; MissingQ@ j -> 1;
  jFrustum[x_] := primorialP[x - # + 1] &@
    (SelectFirst[Range@ x, Function[k,
       # Prime[x + 1]/Prime[x - k + 1] > 2 #] &@
         primorialP@ x] /. k_ /; MissingQ@ k -> 1);
  encode[x_] := If[x == 1, {0}, Function[f,
   ReplacePart[Table[0, {PrimePi[f[[-1, 1]]]}], #] &@
    Map[PrimePi@ First@ # -> Last@ # &, f]]@
      FactorInteger@ x];
  decode[x_] := Times @@ Flatten@
    MapIndexed[Prime[#2]^#1 &, x];
  Map[Select[#,
    Function[k, # <= k <= 2 # &@ primorialP@ n]] &,
    {{primorialP[n]}}~
    Join~Map[Function[j, Function[i, Map[decode@
      Join[encode@ jFrustum@ n,
        ConstantArray[1, jDepth@ n - j], {0},
        Reverse@ IntegerDigits@ FromDigits@ #] &,
        Reverse@ Rest@ Permutations[
        ConstantArray[1, j]~Join~
          ConstantArray[0, i], {i + j}]]][
        If[# == 1, 1, PrimePi[2 Prime@ #]] - n]
          &[n - j + 1]], Range@ jDepth@ n] ] ]
```

The core of the program assembles the multiplicity notation word encoding a candidate t this way:

	1	2	3	4	5
5	**01** **001** **0001**	*011*			
6	**01** **001** **0001**	**011** **0101**			
7	**01** **001** **0001** **00001**	**011** *0101* *0011*	*0111*		
8	**01** **001** **0001** **00001**	**011** **0101** **01001**	*0111*		
9	**01** **001** **0001** **00001** **000001**	**011** *0101* *01001*	*0111* *01101*		
10	**01** **001** **0001** **00001** **000001** **0000001**	**011** **0101** *0011* **01001** *00101* *010001* **001001**	*0111* *01101*	*01111*	
11	**01** **001** **0001** **00001** **000001** **0000001** **00000001**	**011** **0101** *0011* **01001** *00101* **00011** **010001** *001001* **0100001**	*0111* *01101* *011001*	*01111*	
12	**01** **001** **0001** **00001** **000001** **0000001** **00000001** **000000001** **0000000001**	**011** **0101** **0011** **01001** *00101* **00011** *010001* *001001* *000101* **0100001** **01000001**	**0111** *01101* *01011* *00111* *011001* **010101** *010011* *0110001*	*01111* *011101*	
13	**01** **001** **0001** **00001** **000001** **0000001** **00000001** **000000001** **0000000001**	**011** **0101** **0011** **01001** *00101* *00011* **010001** *001001* *000101* *0100001* **0010001** **0001001** *01000001* **00100001** **010000001** *0100000001*	**0111** *01101* **01011** *00111* *011001* *010101* *001101* *010011* *0110001* **0101001** *01100001*	*01111* *011101* *011011* *010111* *0111001* *0110101*	*011111*
14	**01** **001** **0001** **00001** **000001** **0000001** **00000001** **000000001** **0000000001**	**011** **0101** **0011** **01001** *00101* *00011* **010001** **001001** *000101* **0100001** *0010001* **01000001** *00100001* **010000001** *001000001* **0100000001**	**0111** *01101* *01011* *00111* *011001* *010101* *001101* *010011* *0110001* *0101001* *0011001* *01100001* *01010001* **011000001** *0110000001*	*01111* *011101* *011011* *010111* *0111001* *01110001*	*011111* *0111101* *0111011*
	1	2	3	4	5

(FIG. 8.1) Abbreviations of multiplicity notations for turbulent terms t in the first level of tier T in $a(n)$ in reverse lexicographic order. The chart is arranged in tiers (vertical axis) and depths (horizontal axis). The data through tier 14 is validated. Terms shown to be in appear in bold, while those that are not are in italic. See Appendix B1. Below tier 5, the cardinal depth is 1 and all candidates are validated to exist in A244052.

1. Compute and encode the frustum primorial $p_{(T-j)}\#$ using the cardinal depth J as an array of $(T - J)$ ones.
2. Encode depth j as an array of j ones.
3. Encode the totative $p_{(T-j+1)}$ as a single zero.
4. Find the permutations outside of the empty permutation of the set consisting of j ones and i zeros.
5. Concatenate all the above arrays into the permutations of valid multiplicity notations with T ones, depth $j \le J$ and j-distension i.
6. Convert the multiplicity notation to decimal and check if the number $t < 2p_T\#$.
7. Output.

We then can place the SWEEP routine into Code 7.2 and generate all tier T candidates, including those that belong to A060735 proven to be in A244052, and the "turbulent candidates" t that require testing through the regular counting function $r(t)$ to determine if they set records and are part of A244052.

(CODE 7.2)

```
Flatten[Table[{#, FromDigits@ encode@ #} & /@
  Sort@ Flatten@ Map[Function[k, DeleteCases[
  Map[Select[#, # < (k + 1) primorialP@ n &] &,
  (k sweep[n])], {}]], Range[Prime[n + 1] - 1]],
  {n, 0, 16}], 1]
```

The functions generate candidates and numbers in A060735 up to $T = 40$, i.e., $a(n) < p_{40}\# \approx 1.665 \times 10^{68}$.

8. Analysis and Extension of A244052.

Extension.

Given the code in this paper, we can generate the OEIS sequences A244052 and A244053.

I computed the first 54 terms of both sequences 18 June 2014. David Corneth extended the sequences to 149 terms 10 February 2015. In November 2016, I used a congruence test based on regular $m \mid n^m$ (see Code 2.2) to extend the sequence to 162 terms before encountering memory problems. It is possible to divide the range of n by a small primorial in order to segment the testing process. This took hours per term but I reached term 188 in late November.

On 10 January I wrote a program (Code 2.6) that wrote an expression similar to the most efficient Hamming number generator. This code generated all terms of both sequences up to and including tier 11 (term 313) in a single day. Tier 12 was computed the following day. Tier 13 was processed over the next several days and across the weekend.

On 15 January I wrote the level-range program (Code 2.6.2). This enabled processing all m that share a common squarefree root at the same time. The next day the terms of tier 14 that have GPF = p_{14} were processed. I followed this with processing all the reverberated turbulent candidates, then returned to Code 2.6 to wrap up all turbulent candidates of tier 14 on 24 January 2017. This brought the number of terms in each of A244052 and A244053 to 563.

At the time or writing, Code 2.6 and 2.6.2 were expected to require between 8:15 and 36 hours to bore through tier 15. I have decided that generation of these terms is beyond my current capacity, that either a more efficient method might exist, or that someone with a faster system or compiled programs might easily outdo this sort of slog. Thus the dataset stands at 563 confirmed terms.

Appendix B1 contains the full dataset of 563 terms of A244052 and A244053.

Remarks.

The following remarks pertain to the data in Appendix B1.

Firstly, 1 and 2 are in the sequences $a(n)$ and A244052 because they are the empty product and the smallest primorial, respectively. The number 2 sets up the first tier, $T = 1$.

Generally, the terms of $a(n)$ appear in the columns A244052(n) and A054841(A244052(n)) were generated by constructing the terms of A002110 and A060735 and the turbulent candidates $p_T\# < m < 2\,p_t\#$ such that $\omega(m) = T$, and their reverberations in levels k. The algorithm takes advantage of the concepts of distension and depth and the relationship of cardinal distension and depth with A020900, as well as the "multiplicity notation" of A054841. Thus, $a(n)$ can be rather efficiently constructed and all that remains is to validate the turbulent candidates and their echoes.

A. The number 4 is the smallest term of the form $k\,p_T\#$ with $1 < k < p_{(T+1)}$. All tiers $T \geq 1$ have "levels" organized by $k\,p_T\#$.

B. The primorial 6 sets the greatest-seen value for the "tier jump ratio" which is the regular counting function values for 6 and that of the largest term in tier 1, 4, thus $5/3 = 1.6666\ldots$

C. The number 10 is the smallest "turbulent" term, having depth $j = 1$, distension $i = 1$, i.e., GPF $= p_3$ and LPT $= p_2$. The number 10 is of the abbreviated notation class 01, which represents the smallest possible turbulent term of tier T. The turbulent form 01 appears in all tiers $T \geq 2$. The term "turbulence" refers to the appearance of the A054841 "multiplicity notation" of m in tier T.

D. The number 24 is the smallest term of A244052 that is not in A275881, the sequence of numbers that have at least $n/2$ regulars. This is to say that 24 has less than $24/2 = 12$ regulars. OEIS A275881 = {1, 2, 3, 4, 6, 8, 10, 12, 18, 30}. Of these, {3, 4, 8} are neither in $a(n)$ nor A244052, since they are odd primes or perfect prime powers.

E. The term 30 is the largest term in A275881, with 18 regulars. It is the largest number to have at least half of its range $1 \leq k \leq n$ regular. Additionally, 30 is the largest number in A020490 = {1, 2, 3, 4, 6, 8, 10, 12, 18, 24, 30}: numbers that have at least as many divisors as totatives. The number 30 is also the largest number without composite totatives.

F. The term 84 is the smallest "echo turbulent" term, appearing in the "reverb" segment of tier $T = 3$ level $k = 2$. This is to say that $2\,p_3\# < 84 < 3\,p_3\#$. All tiers $T \geq 3$ have "reverb" segments that come after the "primary turbulence" and precede the "tail" wherein all terms are of the form $k\,p_T\#$. The number 84 is the "echo" of 42: both are of the turbulent form 01.

G. The primorial 2310 sets the lowest-seen value for the "tier jump ratio" which is the regular counting function values for 2310 and that of the largest term in tier 4, 2100, thus $283/192 = 1.47396\ldots$, below 3/2.

H. The number 4290 is the first turbulent candidate in $a(n)$ with a negative value of $r(a(n)) - r(a(n-1))$, i.e., its regular function value 315 is lower than that of the preceding candidate, 3990, with $r(3990) = 322$. Further, 4290 is of the turbulent form 011, the first candidate with depth $j = 2$. From this, it might seem to follow that all terms of $a(n)$ with depth $j = 2$ do not appear in A244052. Tier $T = 5$ thus first instance of $p_T\# < m < 2p_T\#$ with $\omega(a(n)) = T$ that does not appear in A244052. Though 4290 in tier $T = 5$ disqualifies, no term of $a(n)$ disqualifies in tier $T = 6$.

<u>Conjecture</u>. Tier $T = 6$ is the greatest T having all of $p_T\# < a(n) < 2p_T\#$ with $\omega(a(n)) = T$ appear in A244052.

The term 17160990 of turbulent form 0111 in tier $T = 8$, is the only turbulent candidate of that tier that does not appear in A244052.

I. The number 46410 in tier $T = 6$ is the smallest term in A244052 with depth $j = 2$, of turbulent form 011. With $r(46410) = 1257$, it bests the preceding term 43890 (turbulent form 001) by just 4 regulars.

All tiers T>=6 have depth-2 turbulent terms, and tier T=6 is the smallest tier with

qualified depth j=2 turbulence.

J. The term 881790 is the smallest depth $j = 3$ candidate; $r(881790)$ is insufficient to qualify it. The number is of the turbulent form 0111. The number 17160990 in tier $T = 8$ and 380570190 in tier $T = 9$ are also of this class and depth; the former the only candidate disqualified in tier 8. These observations may lead us to conjecture that no term with $j > 2$ can qualify…

K. The number 11797675890 of tier $T = 10$ is the smallest depth $j = 4$ candidate. No candidate of this depth appears in tiers 10 through 14 of A244052.

L. Tier 11 is the smallest tier that has more depth $j = 2$ candidates than $j = 1$.

M. The term 10491388397490 is the smallest depth $j = 3$ term in A244052, it is of the turbulent form 0111 in tier $T = 12$. This tier also includes a second depth 3 term 13326898775190 of the form 010101. Thus tier $T = 12$ is the smallest tier with qualified depth j=3 turbulence.

N. The number 22006326882540 is the smallest echo-turbulent (secondary-turbulent) number that appears in $a(n)$ but not in A244052. This number is 2×11003163441270, the latter disqualified in the primary turbulence of tier $T = 12$.

<u>Conjecture</u>. This supports the conjecture that a candidate that fails to qualify in primary turbulence will not qualify in higher levels $k > 1$. In all smaller tiers T, the smallest disqualifying candidate in primary turbulence, t, was sufficiently large such that $2t > 3\,p_T\#$ and thereby did not appear in secondary turbulence. This is the smallest instance that $2t < 3\,p_T\#$.

Thinking about how $r(n)$ relates to $r(2\,n)$, I do not think that it is necessarily always true that qualification of $2t$ as part of A244052 is totally dependent on whether t is in A244052. It is beyond the aim of this study to examine this aspect.

O. The number 568815710072610 in tier $T = 13$ is the smallest depth $j = 5$ candidate, but is disqualified in tier 13. Tier 13 has 44 turbulent candidates but disqualifies 23; it is the first tier to disqualify more than half of the turbulent candidates predicted by the necessary-but-insufficient condition described above. Tier $T = 12$ has 31 but disqualifies 13. We conjecture that tier 13 is the smallest T where more turbulent terms of $a(n)$ are not such in A244052, and all larger T disqualify more than half such candidates. An exception may be one or more smaller $T > 13$. It is also interesting to observe the regular function, i.e., the values of $r(c) = $ A010846$(a(n))$ holding narrowly in the channel $r(p_{13}\#) = 25547835 < r(c) < 27862209$. This supports the conjecture that $r(p_T\#)$ necessarily sets the floor for $r(c)$, and that all c in tier T of $a(n)$ have $r(c) > r(p_T\#)$.

P. The term 902259402184140 in tier $T = 13$ is the echo-disqualification of the depth $j = 4$, class -01111 candidate in level $k = 2$.

Do echo-disqualifications happen in every tier $T > 12$? The answer seems to be yes.

Is the apparent "heritability" of qualification/disqualification attributable to the size of numbers as the tiers increase, since $r(n)$ is not multiplicative? E.g., $r(6) = 5$ but $r(2 \times 6) = 8$. Indeed $r(451129701092070) = 26608037$ but $r(902259402184140) = 31986440$.

Analysis.

Let's move on to explore the aggregate qualities of turbulence in $a(n)$ and in A244052 with the understanding that terms in both sequences that are in A060735 (i.e., primorials and their integer multiples) are well understood.

<u>Turbulent term abbreviation</u>. There are several ways to abbreviate turbulent terms using multiplicity notation. One method is to eliminate the first run of 1s, that is, extract the "suffix" The number of 1s in the first run = $T - j$. We are left with i zeros and j ones in the suffix. This is useful for analyzing level $k = 1$ turbulence. If we need to look at higher levels, the level might be made to precede the suffix, e.g., for tier 5 level 2 we would have 2:01 = 211101 = decimal 5460. Another method is to write run lengths in multiplicity notation. The first method would have 903210 = "111110101" as 51111 versus "0101." Because the first method seems clearer, we'll apply this to make Figure 8.1 more concise. We can use these abbreviations to recognize classes of turbulence.

Figure 8.1 examines the appearance of first-level turbulence in A244052, given turbulent candidates in $a(n)$. Those that are shown not to be in A244052 appear in italic. Verification at the time of this paper's writing ran through tier 14 (i.e., primorial 41) and primorial $p_{15}\#$. A fuller table appears in Appendix B2.

The chart in Figure 8.2 below summarizes the number of primary turbulent distensions in $a(n)$ versus those in A244052. Looking at Figure 8.1, what we are counting is the number of cells to the right of each tier designation that have at least one term written in boldface. The decimal equivalents of such terms appear in A244052.

(FIGURE 8.2)

T	a(n)					A244052(n)		
1	**0**					**0**		
2	**1**					**1**		
3	**1**					**1**		
4	**2**					**2**		
5	**3**	**1**				**3**		
6	**3**	**2**				**3**	**2**	
7	**4**	**2**	**1**			**4**	**1**	
8	**4**	**3**	**1**			**4**	**3**	
9	**5**	**3**	**2**			**5**	**1**	
10	**6**	**4**	**2**	**1**		**6**	**4**	
11	**7**	**5**	**3**	**1**		**7**	**5**	
12	**9**	**6**	**4**	**2**		**9**	**5**	**2**
13	**9**	**8**	**5**	**3**	**1**	**9**	**7**	**3**
14	**9**	**8**	**7**	**4**	**2**	**9**	**8**	**2**

The chart in Figure 8.3 below summarizes the volume of first-level turbulent terms in $a(n)$ versus those in A244052 as shown in Figure 8.1. Totals in the right chart are in italic as they represent prevalidated figures:

(FIGURE 8.3)

T	a(n)					A244052(n)		
1	**0**					**0**		
2	**1**					**1**		
3	**1**					**1**		
4	**2**					**2**		
5	**3**	**1**				**3**		
6	**3**	**2**				**3**	**2**	
7	**4**	**3**	**1**			**4**	**1**	
8	**4**	**3**	**1**			**4**	**3**	
9	**5**	**3**	**2**			**5**	**1**	
10	**6**	**7**	**2**	**1**		**6**	**4**	
11	**7**	**9**	**3**	**1**		**7**	**6**	
12	**9**	**11**	**8**	**2**		**9**	**7**	**2**
13	**9**	**16**	**11**	**6**	**1**	**9**	**9**	**3**
14	**9**	**16**	**15**	**6**	**3**	**9**	**10**	**2**

The charts reveal the following:

1. Full distension at depth $j = 1$ in tiers $T \leq 11$.
2. Note that no depth $j > 3$ appears in tier $T \leq 11$, while all depth $j = 1$ distension terms appear in the same range.
3. Prevalidated terms in tier 12 show that depth 3 may exist

This suggests that all numbers $a(n)$ of the form $p_{(T-1)}\#\, p_{(T+x)}$ with $1 \leq x \leq i$ appear in A244052. Looking at terms 146, 149, and 151 in , i.e., tier 9 depth 1 terms that have "beaten out" those with $j > 1$ suggests that terms in depth 1 are more "efficient" somehow than those with $j > 1$. This would imply that all depth $j = 1$ terms are in A244052.

Let's take a look at some other analyses before conjecturing.

One

Figure 8.3 tabulates the following in order by row. The total of all j-distensions (i.e., the sum of i values for each depth j) is first, followed by the total volume of distinct level 1 turbulent candidates in $a(n)$. Finally, the total volume of level 1 turbulence in tiers of A244052 includes figures in bold for confirmed totals, and italic for pre-confirmed totals.

(FIG. 8.3)

Tier ->	1	2	3	4	5	6	7	8	9	10	11	12
T(i, j)	0	1	1	2	4	5	7	8	10	13	16	21
Volume	**0**	**1**	**1**	**2**	**4**	**5**	**8**	**8**	**10**	**16**	**20**	**30**
A244052	**0**	**1**	**1**	**2**	**4**	**5**	**7**	**6**	**7**	**10**	**14**	*23*

```
0
1
1
2
3   1
3   2
4   2   1
4   3   1
5   3   2
6   4   2   1
7   5   3   1
```

Regular counting functions $r(n) = \text{A010846}(n)$:

(CODE C 1.1)

```
r[n_] := Function[d,
  Length@ Prepend[
    Select[Range[2, n],
      SubsetQ[d, FactorInteger[#][[All, 1]]] &], 1]]
  [FactorInteger[n][[All, 1]]]
   (* prime divisor subset PDS approach *)
```

(CODE C 1.2)

```
r[n_] := Select[Range@ n,
  First@ NestWhile[Function[s, {#1/s, s}]@ GCD[#1, #2] &
    @@ # &, {#, n},
    And[First@ # != 1, ! CoprimeQ @@ #] &] == 1 &]
   (* Recursive GCD approach *)
```

(CODE C 1.3)

```
r[n_] := Count[Range@ n,
  k_ /; First@
   NestWhile[Function[s, {#1/s, s}]@ GCD[#1, #2] & @@ # &,
     {k, n}, And[First@ # != 1, ! CoprimeQ @@ #] &] == 1]
   (* Recursive GCD approach *)
```

(CODE C 1.4)

```
r[n_] := Count[Range@ n, k_ /; PowerMod[n, k, k] == 0]
   (* Congruency test: k | n^k *)
```

(CODE C 1.5)

```
r[n_] := With[{m = n^Floor@ Log2@ n},
  Count[Range@ n, k_ /; Divisible[m, k]]]
   (* Congruency test: k | n^floor(log_2 n) *)
```

(CODE C 1.6)

```
r[n_] := Total@
  Map[MoebiusMu[#] Floor[n/#] &,
   Select[Range[n - 1], CoprimeQ[#, n] &]]
   (* Moebius Function across totatives of n *)
```

(CODE C 1.7)

```
Most efficient r(n): O(log n) time:

r[n_] := Length@
  Function[w,
    ToExpression@
       StringJoin["Module[{n = ", ToString@ n,
        ", k = 0}, Flatten@ Table[k++, ",
        Most@ Flatten@ Map[{#, ", "} &, #], "]]"] &@
     MapIndexed[
      Function[p,
         StringJoin["{", ToString@ Last@ p, ", 0, Log[",
          ToString@ First@ p, ", n/(",
          ToString@
           InputForm[
            Times @@ Map[Power @@ # &,
             Take[w, First@ #2 - 1]]],
          ")]}"]]@ w[[First@ #2]] &, w]
   ]@ Map[{#, ToExpression["p" <> ToString@ PrimePi@ #]} &,
    FactorInteger[n][[All, 1]]]
   (* Regular Count
      Logarithmic Constructor Approach | 201701101605
   (p_8# in 1s,
    p_9# in 4.625s,
    p_10# in 20s,
    p_11# in 83.5s,
    p_12# in 355s,
    p_13# in 1530s (25.5m),
    p_14# in (110m),
    p_15# in (7:52:30) h,
    p_16# in (33:51) h) *)
```

(CODE C 2.1)

```
r[n_] := Function[w,
   ToExpression@
       StringJoin["Module[{k = 0, n = ", ToString@ n, "}, ",
         StringDrop[#, -1], "; k]"] &@
      StringJoin@
       Fold[{StringJoin["Do[", First@ #1, ToString@ #2,
           "],"],
           StringJoin[Last@ #1]} &, {"k++, ", ""}, #] &@
    Reverse@ MapIndexed[
      Function[p,
         StringJoin["{", ToString@ Last@ p, ", 0, Log[",
          ToString@ First@ p, ", n/(",
          ToString@
           InputForm[
            Times @@ Map[Power @@ # &,
              Take[w, First@ #2 - 1]]],
          ")]}"]]@ w[[First@ #2]] &, w]]@
     Map[{#, ToExpression["p" <> ToString@ PrimePi@ #]} &,
   FactorInteger[n][[All, 1]]]
   (* Regular Count
      Logarithm Approach | Do Count | 201701131240
   (p_8# in 1.33 s,
    p_9# in 5.55 s,
    p_10# in 23.5s *)
```

(CODE C 2.2)

Compute $r(n)$ for all n with the same squarefree root—very efficient:

```
levelwise[T_, k1_: 1, k2_: 1, m_: 1] :=
 Module[{P = Times @@ Prime@ Range@ T,
   n = If[k2 == 1, NextPrime@ Prime@ T - 1,
     Min[k2, NextPrime@ Prime@ T - 1]], x, r},
       x = If[m == 1, P, m];
  r[x_] := Function[w,
     ToExpression@
       StringJoin["With[{n=", ToString@ x,
         "}, Flatten@ Table[",
         ToString@ InputForm[Times @@
           Map[Power @@ # &, w]], ", ",
         Most@ Flatten@ Map[{#, ", "} &, #], "]]"] &@
     MapIndexed[
       Function[p,
          StringJoin["{", ToString@ Last@ p, ", 0, Log[",
           ToString@ First@ p, ", n/(",
           ToString@
            InputForm[
             Times @@ Map[Power @@ # &,
               Take[w, First@ #2 - 1]]],
           ")]}"]]@ w[[First@ #2]] &, w]]@
     Map[{#, ToExpression["p" <> ToString@ PrimePi@ #]} &,
     FactorInteger[x][[All, 1]]];
  Function[w,
    Map[Function[k, {k x, Length@ TakeWhile[w, # <= k x &]}],
      Range[k1, n]]]@ Sort@ r[x n]]
     (* Segmented Levelwise Regular Count kx | 201701151118
       (k p_8# in 3.63s,
        k p_9# in 16.5s,
        k p_10# in 73s,
        k p_11# in 325.5s (5:25.5m),
        k p_12# in 1486s (25m),
        k p_13# in (6220s 1:53h),
        k p_14# in 8:44h) *)
```

(CODE C 3.1)

```
primorialP[n_] := Times @@ Prime@ Range@ n; (* A002110(n) *)
```

(CODE C 3.2)

```
encode[n_] :=
  If[n == 1, {0},
   Function[f,
     ReplacePart[Table[0, {PrimePi[f[[-1, 1]]]}], #] &@
      Map[PrimePi@ First@ # -> Last@ # &, f]]@
    FactorInteger@
      n] (* Reverse@ IntegerDigits@ A054841(n) *);
```

(CODE C 3.3)

```
decode[n_] := Times @@ Flatten@
  MapIndexed[Prime[#2]^#1 &, n];
```

(CODE C 4.1)

```
sweep[n_] := Block[{primorialP, jDepth, tierGCD,
   encode, decode},
   primorialP[x_] := Times @@ Prime@ Range@ x;
   jDepth[x_] := -1 +
      SelectFirst[Range@ x,
       Function[j, # Prime[x + 1]/Prime[x - j + 1] > 2 #] &[
        Times @@ Prime@ Range@ x]] /. j_ /; MissingQ@ j -> 1;
   tierGCD[x_] :=
    primorialP[x - # + 1] &@
     (SelectFirst[Range@ x,
       Function[k, # Prime[x + 1]/Prime[x - k + 1] > 2 #] &@
         primorialP@ x] /. k_ /; MissingQ@ k -> 1);
   encode[x_] :=
    If[x == 1, {0},
     Function[f,
       ReplacePart[Table[0, {PrimePi[f[[-1, 1]]]}], #] &@
        Map[PrimePi@ First@ # -> Last@ # &, f]]@
         FactorInteger@ x];
   decode[x_] := Times @@ Flatten@
    MapIndexed[Prime[#2]^#1 &, x];
   Map[Select[#,
      Function[k, # <= k <= 2 # &@ primorialP@ n]] &,
       {{primorialP[n]}}~Join~
    Map[
      Function[j,
       Function[i,
           Map[decode@
              Join[encode@ tierGCD@ n,
               ConstantArray[1, jDepth@ n - j], {0},
               Reverse@ IntegerDigits@ FromDigits@ #] &,
             Reverse@
              Rest@ Permutations[
                ConstantArray[1, j]~Join~
                 ConstantArray[0, i], {i + j}]]]][
          If[# == 1, 1, PrimePi[2 Prime@ #]] - n] &[
           n - j + 1]],
      Range@ jDepth@ n]]]
   (* sweep | 201609142100
      output all level-1 candidates for A244052
      including A060735 terms. *);
```

(CODE C 4.2)

```
tier[n_] :=
  Sort@ Flatten@
    Map[Function[k,
      DeleteCases[
       Map[Select[#, # < (k + 1) primorialP@ n &] &,
         (k sweep[n])], {}]],
     Range[Prime[n + 1] - 1]]
    (* generate all candidates in tier n *);
```

(CODE C 4.3)

```
jDepth[n_] := -1 +
    SelectFirst[Range@ n,
     Function[k, # Prime[n + 1]/Prime[n - k + 1] > 2 #] &[
      Times @@ Prime@ Range@ n]] /. k_ /; MissingQ@ k -> 1
    (* algebraic j-depth of primorial(omega) *);
```

(CODE C 4.4)

```
jNumber[n_] := Block[{primorialP, j},
   primorialP[x_] := Times @@ Prime@ Range@ x;
   j = (SelectFirst[Range@ n,
       Function[k, # Prime[n + 1]/Prime[n - k + 1] > 2 #] &@
         primorialP@ n] /. k_ /; MissingQ@ k -> 1) - 1;
   Floor[primorialP[n + 1]/Prime[n - j + 1]]]
   (* algebraic deepest tier-T turbulent number m = number
      with least prime totative lpt of tier *);
```

(CODE C 4.5)

```
tierGCD[n_] := Block[{primorialP, j},
   primorialP[x_] := Times @@ Prime@ Range@ x;
   j = (SelectFirst[Range@ n,
        Function[k, # Prime[n + 1]/Prime[n - k + 1] > 2 #] &@
          primorialP@ n] /. k_ /; MissingQ@ k -> 1) - 1;
   primorialP[n - j]]
    (* algebraic j-depth of primorial(T) = GCD of tier,
       formerly "jFrustum" *);
```

(CODE C 4.6)

```
primeFrustum[n_] :=
 n + 1 - SelectFirst[Range@ n,
    Function[k, # Prime[n + 1]/Prime[n - k + 1] > 2 #] &[
     Times @@ Prime@ Range@ n]] /. k_ /; MissingQ@ k -> 1
   (* primorial frustum at depth of
      primorial(omega) = pi(gpf(GCD of tier T))*)
```

(CODE C 5.1)

```
iDistension[n_, j_] := -1 +
    SelectFirst[Range@ n,
     Function[i, # Prime[n + i]/Prime[n - j + 1] > 2 #] &[
      Times @@ Prime@ Range@ n]] /. k_ /; MissingQ@ k -> 1
    (* Distension of primorial(n) at depth j, i.e., greatest
       pi(gpf) for numbers m with lpt=prime(T-j+1) *);
```

(CODE C 5.2)

```
iNumber[n_, j_] := Block[{primorialP, i},
  primorialP[x_] := Times @@ Prime@ Range@ x;
  i = (SelectFirst[Range@ n,
      Function[k, # Prime[n + k]/Prime[n - j + 1] > 2 #] &@
        primorialP@ n] /. k_ /; MissingQ@ k -> 1) - 1;
  If[i == 0, 1,
   primorialP[n - j] *
   (Times @@ Prime@ Range[n - j + 2, n]) * Prime[n + i]]]
    (* algebraic most distended omega-turulent candidate
       at depth j, i.e., m with omega(m)=T that has
       greatest gpf with lpt at prime(T-j+1) *)
```

(CODE C 5.3)

```
f[T_] := {{T, primorialP@ T},
  Map[Function[w,
   {StringReverse@ ToString@ FromDigits@ Reverse@
   Take[#, -(Length@ # - Length@ TakeWhile[#, # == 1 &])] &@
   encode@ w, #, Lookup[t, #]} &@ w] /@
   Select[#, Function[k, # <= k <= 2 # &@ primorialP[T]]] &,
     Map[Function[j, Function[i,
       Map[Times @@ Flatten@ MapIndexed[Prime[#2]^#1 &, #]&@
        Join[encode@ tierGCD@ T,
         ConstantArray[1, jDepth@ T - j], {0},
         Reverse@ IntegerDigits@ FromDigits@ #] &,
         Reverse@ Rest@ Permutations[
           ConstantArray[1, j]~Join~
           ConstantArray[0, i], {i + j}]]]][
      If[# == 1, 1, PrimePi[2 Prime@ #]] - T] &[T - j + 1]],
    Range@ jDepth@ T]]}
    (* Analyze Tier T turbulent terms by depth class:
       Format: abbreviated notation -
         n - r(n) - Presence in A244052/3 *)
```

(CODE C 6.1)

```
attributableToGPF[n_] :=
 Function[P,
   Total@ Map[MoebiusMu[#] Floor[P/#] &,
     Select[Range@ P,
      And[CoprimeQ[#, Times @@ Prime@ Range[n - 1]],
        Divisible[#, Prime@ n]] &]]][
          Times @@ Prime@ Range@ n]
    (* portion of r(p_n#) attributable to p_n
       201702071500, based on Mertens function *)
```

(CODE C 6.2)

```
attributableToPrimek[n_, k_] :=
 Function[P,
   Total@ Map[MoebiusMu[#] Floor[P/#] &,
     Select[Range@ P,
      And[CoprimeQ[#, Times @@ Prime@ Range@ n/Prime@ k],
        Divisible[#, Prime@ k]] &]]][
          Times @@ Prime@ Range@ n]
    (* portion of r(p_n#) attributable to p_k
       with 1<=k<=n 201702072030 *)
```

(CODE C 6.3)

```
attributableToPrimek[n_, k_] :=
 Block[{P = Times @@ Prime@ Range@ n},
  Length@ Function[w,
     ToExpression@
        StringJoin["Module[{n = ", ToString@ P,
         ", k = 0}, Flatten@ Table[k++, ",
         Most@ Flatten@ Map[{#, ", "} &, #], "]]"] &@
      MapIndexed[
       Function[p,
          StringJoin["{", ToString@ Last@ p, ", 0, Log[",
           ToString@ First@ p, ", n/(",
           ToString@ InputForm[
            Times @@ Map[Power @@ # &,
             Take[w, First@ #2 - 1]]],
           ")]}"]]@ w[[First@ #2]] &, w]]@
      Map[{#, ToExpression["p" <> ToString@ PrimePi@ #]} &,
       FactorInteger[P/Prime@ k][[All, 1]]] ]
```

Some interesting runs:

1. Generate candidates and multiplicity notations for the candidates in tier T:

```
{#, FromDigits@ encode@ #} & /@ tier@ 5 // TableForm
```

2. Generate deepest candidate m (with maximum j) in tier T:

```
{primorialP@ #, jDepth@ #, FromDigits@ encode@ jNumber@ #,
    2 primorialP@ #} & /@ Range@ 8 // TableForm

{primorialP@ #, jNumber@ #, FromDigits@ encode@ jNumber@ #,
tierGCD@ #,
    primeFrustum@ #,
    FromDigits@ encode[Times @@ Prime@ Range@ primeFrustum@
#],
    2 primorialP@ #} & /@ Range@ 12 // TableForm
```

3. Compare cardinal distension to tier GCD (cardinal depth) for tier T:

```
{#, # + iDistension[#, 1], primeFrustum@ #} & /@ Range@ 36
// TableForm
```

4. Generate Table 1:

```
Table[Map[iDistension[n, #] &, Range@ jDepth@ n] /. {} ->
{0}, {n, 40}] //
  TableForm  (* T(n,k) with n = omega, k = distension at
depth = position(k) *)

With[{n =
   12}, {#,
    FromDigits@ encode@ # & /@
       Map[iNumber[n, #] &, Range@ jDepth @n] /. {} -> {0},
2 #} &@ primorialP @ n]
   (* deepest j, most distended i for j *)
```

5. Generate multiplicity notations present in tier T sorted by depth:

```
Table[{primorialP@ n,
   FromDigits[encode@#] & /@
     Select[#, Function[k, # <= k <= 2 # &@ primorialP[n]]]
& /@
    Map[Function[j,
      Function[i,
          Map[Times @@ Flatten@ MapIndexed[Prime[#2]^#1 &,
#](* decode *)&@
            Join[encode@ tierGCD@ n,
             ConstantArray[1, jDepth@ n - j], {0},
             Reverse@ IntegerDigits@ FromDigits@ #] &,
           Reverse@
            Rest@ Permutations[
              ConstantArray[1, j]~Join~
               ConstantArray[0, i], {i + j}]]]][
         If[# == 1, 1, PrimePi[2 Prime@ #]] - n] &[n - j +
1]],
     Range@ jDepth@ n]}, {n, 1, 15}] // TableForm
    (* signatures per class 20161023 *)
```

6. Generate Table 3:

```
Table[Length@ Select[#, Function[k, # <= k <= 2 # &@ pri-
morialP[n]]] & /@
    Map[Function[j,
      Function[i,
           Map[Times @@ Flatten@ MapIndexed[Prime[#2]^#1 &,
#](* decode *)&@
            Join[encode@ tierGCD@ n,
             ConstantArray[1, jDepth@ n - j], {0},
             Reverse@ IntegerDigits@ FromDigits@ #] &,
          Reverse@ Rest@
            Permutations[
             ConstantArray[1, j]~Join~ConstantArray[0, i],
{i + j}]]]][
        If[# == 1, 1, PrimePi[2 Prime@ #]] - n] &[n - j + 1]],
    Range@ jDepth@ n], {n, 2, 32}] // TableForm
    (* Sweep Program | Enumerator 20161032 *)
```

7. Generate Table 6:

```
Table[(-1) attributableToPrimek[n, k], {n, 8}, {k, n}] //
TableForm
```

8. Generate Table 7 (This may be more efficient via memoization in a Do loop):

```
Table[r[Times @@ Prime@ Range@ n] + attributableToPrimek[n,
k], {n, 8}, {k, n}]
  // TableForm
```

Appendix A1: Values of regular function $r(n)$ for $1 \le n \le 540$ = OEIS A010846.

1	1	79	2	157	2	235	6	313	2	391	5	469	6
2	2	80	14	158	10	236	11	314	11	392	20	470	29
3	2	81	5	159	7	237	7	315	25	393	8	471	8
4	3	82	9	160	18	238	23	316	12	394	11	472	13
5	2	83	2	161	5	239	2	317	2	395	6	473	5
6	5	84	28	162	24	240	49	318	34	396	48	474	37
7	2	85	5	163	2	241	2	319	5	397	2	475	8
8	4	86	9	164	11	242	15	320	22	398	11	476	30
9	3	87	7	165	18	243	6	321	8	399	19	477	9
10	6	88	11	166	10	244	11	322	24	400	23	478	11
11	2	89	2	167	2	245	9	323	5	401	2	479	2
12	8	90	32	168	38	246	32	324	30	402	36	480	65
13	2	91	5	169	3	247	5	325	8	403	5	481	5
14	6	92	10	170	24	248	12	326	11	404	12	482	11
15	5	93	7	171	8	249	8	327	8	405	16	483	19
16	5	94	9	172	11	250	20	328	13	406	26	484	18
17	2	95	5	173	2	251	2	329	5	407	5	485	6
18	10	96	20	174	29	252	45	330	77	408	43	486	33
19	2	97	2	175	8	253	5	331	2	409	2	487	2
20	8	98	13	176	14	254	10	332	12	410	29	488	13
21	5	99	8	177	7	255	19	333	9	411	8	489	8
22	7	100	15	178	10	256	9	334	11	412	12	490	43
23	2	101	2	179	2	257	2	335	6	413	6	491	2
24	11	102	25	180	44	258	33	336	50	414	41	492	41
25	3	103	2	181	2	259	5	337	2	415	6	493	5
26	7	104	11	182	22	260	30	338	16	416	17	494	25
27	4	105	16	183	7	261	9	339	8	417	8	495	26
28	8	106	9	184	12	262	11	340	31	418	25	496	14
29	2	107	2	185	6	263	2	341	5	419	2	497	6
30	18	108	21	186	29	264	41	342	40	420	96	498	38
31	2	109	2	187	5	265	6	343	4	421	2	499	2
32	6	110	21	188	11	266	24	344	13	422	11	500	24
33	6	111	7	189	11	267	8	345	19	423	9	501	8
34	8	112	14	190	24	268	12	346	11	424	13	502	11
35	5	113	2	191	2	269	2	347	2	425	8	503	2
36	14	114	26	192	25	270	53	348	38	426	36	504	59
37	2	115	5	193	2	271	2	349	2	427	6	505	6
38	8	116	10	194	10	272	14	350	39	428	12	506	25
39	6	117	8	195	18	273	18	351	11	429	19	507	12
40	11	118	9	196	16	274	11	352	17	430	29	508	12
41	2	119	5	197	2	275	8	353	2	431	2	509	2
42	19	120	36	198	36	276	36	354	35	432	32	510	86
43	2	121	3	199	2	277	2	355	6	433	2	511	6
44	9	122	9	200	19	278	11	356	12	434	26	512	10
45	8	123	7	201	7	279	9	357	19	435	21	513	11
46	8	124	10	202	10	280	36	358	11	436	12	514	12
47	2	125	4	203	5	281	2	359	2	437	5	515	6
48	15	126	33	204	33	282	33	360	58	438	37	516	42
49	3	127	2	205	6	283	2	361	3	439	2	517	5
50	12	128	8	206	10	284	12	362	11	440	36	518	28
51	6	129	7	207	8	285	19	363	12	441	14	519	8
52	9	130	23	208	14	286	24	364	29	442	25	520	38
53	2	131	2	209	5	287	5	365	6	443	2	521	2
54	16	132	31	210	68	288	29	366	35	444	40	522	44
55	5	133	5	211	2	289	3	367	2	445	6	523	2
56	11	134	10	212	11	290	27	368	14	446	11	524	13
57	6	135	12	213	7	291	8	369	9	447	8	525	30
58	8	136	12	214	10	292	12	370	28	448	21	526	12
59	2	137	2	215	6	293	2	371	6	449	2	527	5
60	26	138	27	216	26	294	48	372	38	450	64	528	53
61	2	139	2	217	5	295	6	373	2	451	5	529	3
62	8	140	27	218	10	296	13	374	25	452	12	530	31
63	8	141	7	219	7	297	11	375	15	453	8	531	9
64	7	142	10	220	28	298	11	376	13	454	11	532	31
65	5	143	5	221	5	299	5	377	5	455	16	533	5
66	22	144	23	222	31	300	55	378	52	456	44	534	39
67	2	145	6	223	2	301	5	379	2	457	2	535	6
68	10	146	10	224	17	302	11	380	31	458	11	536	14
69	6	147	10	225	13	303	8	381	8	459	11	537	8
70	20	148	11	226	10	304	14	382	11	460	31	538	12
71	2	149	2	227	2	305	6	383	2	461	2	539	8
72	18	150	41	228	34	306	40	384	31	462	79	540	69
73	2	151	2	229	2	307	2	385	16	463	2		
74	9	152	12	230	25	308	28	386	11	464	14		
75	9	153	8	231	17	309	8	387	9	465	21		
76	10	154	22	232	12	310	27	388	12	466	11		
77	5	155	6	233	2	311	2	389	2	467	2		
78	23	156	31	234	37	312	41	390	80	468	48		

4

	2^δ		
	1	2	4

6

	2^δ		
	1	2	4
3^δ	3	6	

8

	2^δ			
	1	2	4	8

9

	3^δ		
	1	3	9

10

	2^δ			
	1	2	4	8
5^δ	5	10		

12

	2^δ			
	1	2	4	8
3^δ	3	6	12	
	9			

14

	2^δ			
	1	2	4	8
7^δ	7	14		

15

	3^δ		
	1	3	9
5^δ	5	15	

16

	2^δ				
	1	2	4	8	16

18

	2^δ				
	1	2	4	8	16
3^δ	3	6	12		
	9	18			

20

	2^δ				
	1	2	4	8	16
5^δ	5	10	20		

21

	3^δ		
	1	3	9
7^δ	7	21	

22

	2^δ				
	1	2	4	8	16
11^δ	11	22			

24

	2^δ				
	1	2	4	8	16
3^δ	3	6	12	24	
	9	18			

25

	5^δ		
	1	5	25

26

	2^δ				
	1	2	4	8	16
13^δ	13	26			

27

	3^δ			
	1	3	9	27

28

	2^δ				
	1	2	4	8	16
7^δ	7	14	28		

30

5^0

	2^δ				
	1	2	4	8	16
3^δ	3	6	12	24	
	9	18			
	27				

5^1

	2^δ		
	5	10	20
3^δ	15	30	

5^2

	2^δ
3^δ	25

32

	2^δ					
	1	2	4	8	16	32

33

	3^δ			
	1	3	9	27
11^δ	11	33		

34

	2^δ					
	1	2	4	8	16	32
17^δ	17	34				

35

	5^δ		
	1	5	25
7^δ	7	35	

36

	2^δ					
	1	2	4	8	16	32
3^δ	3	6	12	24		
	9	18	0			
	27					

38

	2^δ					
	1	2	4	8	16	32
19^δ	19	38				

39

	3^δ			
	1	3	9	27
13^δ	13	39		

40

	2^δ					
	1	2	4	8	16	32
5^δ	5	10	20	40		
	25					

42

7^0

	2^δ					
	1	2	4	8	16	32
3^δ	3	6	12	24		
	9	18	36			
	27					

7^1

	2^δ		
	7	14	28
3^δ	21	42	

44

	2^δ					
	1	2	4	8	16	32
11^δ	11	22	44			

45

	3^δ			
	1	3	9	27
5^δ	5	15	45	
	25			

46

	2^δ					
	1	2	4	8	16	32
23^δ	23	46				

48

	2^δ					
	1	2	4	8	16	32
3^δ	3	6	12	24	48	
	9	18	36			
	27					

49

	7^δ		
	1	7	49

50

	2^δ					
	1	2	4	8	16	32
5^δ	5	10	20	40		
	25	50				

51

	3^δ			
	1	3	9	27
17^δ	17	51		

52

	2^δ					
	1	2	4	8	16	32
13^δ	13	26	52			

54

	2^δ					
	1	2	4	8	16	32
3^δ	3	6	12	24	48	
	9	18	36			
	27	54				

55

	5^δ		
	1	5	25
11^δ	11	55	

56

	2^δ					
	1	2	4	8	16	32
7^δ	7	14	28	56		
	49					

57

	3^δ			
	1	3	9	27
19^δ	19	57		

58

	2^δ					
	1	2	4	8	16	32
29^δ	29	58				

60

5^0

	2^δ					
	1	2	4	8	16	32
3^δ	3	6	12	24	48	
	9	18	36			
	27	54				

5^1

	2^δ			
	5	10	20	40
3^δ	15	30	60	
	45			

5^2

	2^δ	
3^δ	25	50

62

	2^δ					
	1	2	4	8	16	32
31^δ	31	62				

63

	3^δ			
	1	3	9	27
7^δ	7	21	63	
	49			

64

	2^δ						
	1	2	4	8	16	32	64

65

	5^δ		
	1	5	25
13^δ	13	65	

66

11^0

	2^δ						
	1	2	4	8	16	32	64
3^δ	3	6	12	24	48		
	9	18	36				
	27	54					

11^1

	2^δ		
	11	22	44
3^δ	33	66	

Appendix A3: Regular numbers $1 \le m \le n$ for $n = 2520$ = OEIS A275280(2520).

2520 7^0:

5^0 2^δ 3^δ

1	2	4	8	16	32	64	128	256	512	1024	2048
3	6	12	24	48	96	192	384	768	1536		
9	18	36	72	144	288	576	1152	2304			
27	54	108	216	432	864	1728					
81	162	324	648	1296							
243	486	972	1944								
729	1458										
2187											

5^1 2^δ 3^δ

5	10	20	40	80	160	320	640	1280
15	30	60	120	240	480	960	1920	
45	90	180	360	720	1440			
135	270	540	1080	2160				
405	810	1620						
1215	2430							

5^2 2^δ 3^δ

25	50	100	200	400	800	1600
75	150	300	600	1200	2400	
225	450	900	1800			
675	1350					

5^3 2^δ 3^δ

125	250	500	1000	2000
375	750	1500		
1125	2250			

5^4 2^δ 3^δ

625	1250	2500
1875	750	

7^1:

5^0 2^δ 3^δ

7	14	28	56	112	224	448	896	1792
21	42	84	168	336	672	1344		
63	126	252	504	1008	2016			
189	378	756	1512					
567	1134	2268						
1701								

5^1 2^δ 3^δ

35	70	140	280	560	1120	2240
105	210	420	840	1680		
315	630	1260	2520			
945	1890					

5^2 2^δ 3^δ

175	350	700	1400
525	1050	2100	
1575			

5^3 2^δ 3^δ

875	1750

7^2:

5^0 2^δ 3^δ

49	98	196	392	784	1568
147	294	588	1176	2352	
441	882	1764			
1323					

5^1 2^δ 3^δ

245	490	980	1960
735	1470		
2205			

5^2 2^δ 3^δ

1225	2450

7^3:

5^0 2^δ 3^δ

343	686	1372
1029	2058	

5^1 2^δ 3^δ

1715

7^4:

5^0 2^δ 3^δ

2401

APPENDIX A4: j-Distension for Tiers 1 through 60

Tier T \ Depth j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0																									
2	1																									
3	1																									
4	2																									
5	3	1																								
6	3	2																								
7	4	2	1																							
8	4	3	1																							
9	5	3	2																							
10	6	4	2	1																						
11	7	5	3	1																						
12	9	6	4	2																						
13	9	8	5	3	1																					
14	9	8	7	4	2																					
15	9	8	7	6	3	1																				
16	11	8	7	6	5	2																				
17	13	10	7	6	5	4	1																			
18	12	12	9	6	5	4	3																			
19	13	11	11	8	5	4	3	2																		
20	14	12	10	10	7	4	3	2	1																	
21	13	13	11	9	9	6	3	2	1																	
22	15	12	12	10	8	8	5	2	1																	
23	15	14	11	11	9	7	7	4	1																	
24	16	14	13	10	10	8	6	6	3																	
25	19	15	13	12	9	9	7	5	5	2																
26	20	18	14	12	11	8	8	6	4	4	1															
27	19	19	17	13	11	10	7	7	5	3	3															
28	19	18	18	16	12	10	9	6	6	4	2	2														
29	18	18	17	17	15	11	9	8	5	5	3	1	1													
30	18	17	17	16	16	14	10	8	7	4	4	2														
31	23	17	16	16	15	15	13	9	7	6	3	3	1													
32	23	22	16	15	15	14	14	12	8	6	5	2	2													
33	25	22	21	15	14	14	13	13	11	7	5	4	1	1												
34	25	24	21	20	14	13	13	12	12	10	6	4	3													
35	27	24	23	20	19	13	12	12	11	11	9	5	3	2												
36	26	26	23	22	19	18	12	11	11	10	10	8	4	2	1											
37	28	25	25	22	21	18	17	11	10	10	9	9	7	3	1											
38	28	27	24	24	21	20	17	16	10	9	9	8	8	6	2											
39	28	27	26	23	23	20	19	16	15	9	8	8	7	7	5	1										
40	28	27	26	25	22	22	19	18	15	14	8	7	7	6	6	4										
41	30	27	26	25	24	21	21	18	17	14	13	7	6	6	5	5	3									
42	30	29	26	25	24	23	20	20	17	16	13	12	6	5	5	4	4	2								
43	32	29	28	25	24	23	22	19	19	16	15	12	11	5	4	4	3	3	1							
44	32	31	28	27	24	23	22	21	18	18	15	14	11	10	4	3	3	2	2							
45	32	31	30	27	26	23	22	21	20	17	17	14	13	10	9	3	2	2	1	1						
46	32	31	30	29	26	25	22	21	20	19	16	16	13	12	9	8	2	1	1							
47	35	31	30	29	28	25	24	21	20	19	18	15	15	12	11	8	7	1								
48	38	34	30	29	28	27	24	23	20	19	18	17	14	14	11	10	7	6								
49	38	37	33	29	28	27	26	23	22	19	18	17	16	13	13	10	9	6	5							
50	38	37	36	32	28	27	26	25	22	21	18	17	16	15	12	12	9	8	5	4						
51	39	37	36	35	31	27	26	25	24	21	20	17	16	15	14	11	11	8	7	4	3					
52	39	38	36	35	34	30	26	25	24	23	20	19	16	15	14	13	10	10	7	6	3	2				
53	39	38	37	35	34	33	29	25	24	23	22	19	18	15	14	13	12	9	9	6	5	2	1			
54	41	38	37	36	34	33	32	28	24	23	22	21	18	17	14	13	12	11	8	8	5	4	1			
55	42	40	37	36	35	33	32	31	27	23	22	21	20	17	16	13	12	11	10	7	7	4	3			
56	43	41	39	36	35	34	32	31	30	26	22	21	20	19	16	15	12	11	10	9	6	6	3	2		
57	42	42	40	38	35	34	33	31	30	29	25	21	20	19	18	15	14	11	10	9	8	5	5	2	1	
58	42	41	41	39	37	34	33	32	30	29	28	24	20	19	18	17	14	13	10	9	8	7	4	4	1	
59	42	41	40	40	38	36	33	32	31	29	28	27	23	19	18	17	16	13	12	9	8	7	6	3	3	
60	42	41	40	39	39	37	35	32	31	30	28	27	26	22	18	17	16	15	12	11	8	7	6	5	2	2

Depth j

1	**0**														
2	**1**														
3	**1**														
4	**2**														
5	**3**	**1**													
6	**3**	**2**													
7	**4**	**3**	**1**												
8	**4**	**3**	**1**												
9	**5**	**3**	**2**												
10	**6**	**7**	**2**	**1**											
11	**7**	**9**	**3**	**1**											
12	**9**	**11**	**8**	**2**											
13	**9**	**16**	**11**	**6**	**1**										
14	**9**	**16**	**15**	**6**	**3**										
15	**9**	**15**	**14**	**9**	**3**	**1**									
16	**11**	**20**	**16**	**13**	**6**	**2**									
17	**13**	**28**	**24**	**16**	**12**	**6**	**1**								
18	**12**	**33**	**32**	**18**	**10**	**7**	**3**								
19	**13**	**36**	**48**	**33**	**13**	**7**	**4**	**2**							
20	**14**	**40**	**50**	**49**	**27**	**10**	**5**	**3**	**1**						
21	**13**	**38**	**54**	**43**	**34**	**15**	**5**	**2**	**1**						
22	**15**	**40**	**57**	**56**	**32**	**24**	**9**	**3**	**1**						
23	**15**	**43**	**56**	**59**	**44**	**21**	**15**	**5**	**1**						
24	**16**	**49**	**70**	**58**	**52**	**35**	**14**	**10**	**3**						
25	**19**	**64**	**91**	**96**	**66**	**54**	**33**	**15**	**12**	**2**					
26	**20**	**78**	**130**	**124**	**107**	**61**	**50**	**29**	**14**	**10**	**1**				
27	**19**	**76**	**150**	**153**	**117**	**89**	**44**	**33**	**18**	**10**	**6**				
28	**19**	**72**	**140**	**186**	**137**	**89**	**63**	**29**	**23**	**10**	**5**	**3**			
29	**18**	**65**	**113**	**153**	**149**	**86**	**48**	**31**	**15**	**10**	**4**	**1**	**1**		
30	**18**	**59**	**95**	**106**	**114**	**93**	**46**	**22**	**12**	**4**	**4**	**2**			
31	**23**	**78**	**123**	**141**	**129**	**124**	**93**	**40**	**20**	**11**	**5**	**4**	**1**		
32	**23**	**113**	**162**	**165**	**162**	**130**	**116**	**82**	**34**	**17**	**9**	**3**	**2**		
33	**25**	**118**	**264**	**220**	**190**	**170**	**126**	**108**	**73**	**28**	**14**	**6**	**1**	**1**	
34	**25**	**121**	**258**	**363**	**204**	**158**	**132**	**89**	**74**	**48**	**17**	**6**	**3**		
35	**27**	**131**	**310**	**431**	**471**	**213**	**156**	**126**	**85**	**72**	**45**	**14**	**5**	**2**	
36	**26**	**148**	**314**	**485**	**495**	**463**	**169**	**118**	**94**	**62**	**50**	**30**	**8**	**2**	**1**
37	**28**	**148**	**383**	**491**	**590**	**506**	**420**	**134**	**92**	**73**	**45**	**37**	**21**	**4**	**1**
38	**28**	**162**	**390**	**665**	**614**	**643**	**485**	**380**	**110**	**73**	**58**	**36**	**27**	**16**	**3**
--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_10	p_11	p_12
n	r(p_n#)	2	3	5	7	11	13	17	19	23	29	31	37
1	**2**	1											
2	**5**	3	2										
3	**18**	11	9	6									
4	**68**	47	37	27	22								
5	**283**	206	170	130	109	84							
6	**1161**	871	734	583	500	405	373						
7	**4843**	3732	3190	2601	2268	1877	1746	1556					
8	**19985**	15680	13554	11239	9906	8337	7813	7031	6728				
9	**349670**	66141	57713	48461	43107	36738	34601	31405	30155	28111			
10	**1456458**	281949	248018	210504	188649	162498	153691	140428	135239	126722	117035		
11	**6107257**	1186118	1049898	898509	809882	703282	667209	612779	591418	556289	516197	505163	
12	**25547835**	5017655	4465718	3849404	3486968	3049110	2900419	2675418	2586893	2441064	2274216	2228176	2110136

		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_10	p_11	p_12
n	r(p_n#)	2	3	5	7	11	13	17	19	23	29	31	37
1	**2**	**1**											
2	**5**	2	**3**										
3	**18**	7	9	**12**									
4	**68**	21	31	41	**46**								
5	**283**	77	113	153	174	**199**							
6	**1161**	290	427	578	661	756	**788**						
7	**4843**	1111	1653	2242	2575	2966	3097	**3287**					
8	**19985**	4305	6431	8746	10079	11648	12172	12954	**13257**				
9	**349670**	16933	25361	34613	39967	46336	48473	51669	52919	**54963**			
10	**1456458**	67721	101652	139166	161021	187172	195979	209242	214431	222948	**232635**		
11	**6107257**	270340	406560	557949	646576	753176	789249	843679	865040	900169	940261	**951295**	
12	**25547835**	1089602	1641539	2257853	2620289	3058147	3206838	3431839	3520364	3666193	3833041	3879081	**3997121**

		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_10	p_11	p_12
n	r(p_n#)	2	3	5	7	11	13	17	19	23	29	31	37
1	**2**	**2.**											
2	**5**	2.5	**1.66667**										
3	**18**	2.57143	2.	**1.5**									
4	**68**	3.2381	2.19355	1.65854	**1.47826**								
5	**283**	3.67532	2.50442	1.84967	1.62644	**1.42211**							
6	**1161**	4.00345	2.71897	2.00865	1.75643	1.53571	**1.47335**						
7	**4843**	4.35914	2.92982	2.16012	1.88078	1.63284	1.56377	**1.47338**					
8	**19985**	4.64228	3.1076	2.28504	1.98284	1.71575	1.64188	1.54277	**1.50751**				
9	**349670**	4.90604	3.27566	2.40008	2.07856	1.79286	1.71382	1.60781	1.56983	**1.51145**			
10	**1456458**	5.16339	3.43987	2.51261	2.17158	1.86817	1.78422	1.67113	1.63069	1.56839	**1.50308**		
11	**6107257**	5.3875	3.58239	2.61038	2.25257	1.93376	1.84537	1.72632	1.68369	1.61798	1.54899	**1.53103**	
12	**25547835**	5.60503	3.72045	2.7049	2.33076	1.99704	1.90445	1.77959	1.73484	1.66583	1.59332	1.57441	**1.52791**

Appendix B1: Analytical Review of $a(n)$ vs. OEIS A244052 for Tiers 1 through 15.

This table regards the terms of the necessary-but-not-sufficient condition for terms of OEIS A244052, here referred to as $a(n)$.

The first column lists n, with $a(n)$ in the fifth column. The second and third columns list primorial $p_T\#$ and level $1 \le k < p_{(T+1)}$ for the first term in that tier and level. The fourth column lists the position of $a(n)$ in A244052, if that term appears in that sequence. If the term has been confirmed not appear in A244052, we place "." in that column. If a term is implicitly a part of A244052 and no potentially unconfirmed terms appear before it, the number appears in italics. "Notation" used in this paper, i.e., A054841(n) reversed, appears in the sixth column. The value of $r(a(n))$ = A010846($a(n)$), when known, appears in the last column. If the term pertains to the corresponding term in A244052, i.e., if it is in A244053, it is shown in bold. Finally, the "deficit" of regulars of turbulent candidates in $a(n)$ that fail to set records and appear in A244052 is shown in the last column.

n	p#	k	A244052	a(n)	A054841(a(n))		A010846(a(n))	Deficit
1		1	1	1	0		1	.
2	2#	1	2	2	1		2	.
3		2	3	4	2	(A)	3	.
4	3#	1	4	6	11	(B)	5	.
5			5	10	101	(C)	6	.
6		2	6	12	21		8	.
7		3	7	18	12		10	.
8		4	8	24	31	(D)	11	.
9	5#	1	9	30	111	(E)	18	.
10			10	42	1101		19	.
11		2	11	60	211		26	.
12			12	84	2101	(F)	28	.
13		3	13	90	121		32	.
14		4	14	120	311		36	.
15		5	15	150	112		41	.
16		6	16	180	221		44	.
17	7#	1	17	210	1111		68	.
18			18	330	11101		77	.
19			19	390	111001		80	.
20		2	20	420	2111		96	.
21		3	21	630	1211		115	.
22		4	22	840	3111		131	.
23		5	23	1050	1121		145	.
24		6	24	1260	2211		156	.
25		7	25	1470	1112		166	.
26		8	26	1680	4111		174	.
27		9	27	1890	1311		183	.
28		10	28	2100	2121	(G)	192	.
29	11#	1	29	2310	11111		283	.
30			30	2730	111101		295	.
31			31	3570	1111001		313	.
32			32	3990	11110001		322	.
33			.	*4290*	111011	(H)	*315*	7
34		2	33	4620	21111		382	.
35			34	5460	211101		395	.
36		3	35	6930	12111		452	.
37			36	8190	121101		463	.
38		4	37	9240	31111		505	.
39			38	10920	311101		519	.
40		5	39	11550	11211		551	.
41			40	13650	112101		567	.
42		6	41	13860	22111		593	.
43		7	42	16170	11121		629	.
44		8	43	18480	41111		660	.
45		9	44	20790	13111		691	.
46		10	45	23100	21211		717	.
47		11	46	25410	11112		743	.
48		12	47	27720	32111		766	.

n	p#	k	A244052		a(n)	A054841(a(n))		A010846(a(n))	Deficit
49	13#	1	48		30030	111111		1161	.
50			49		39270	1111101		1224	.
51			50		43890	11111001		1253	.
52			51		46410	1111011	(I)	1257	.
53			52		51870	11110101		1285	.
54			53		53130	111110001		1306	.
55		2	**54**	(a)	60060	211111		1526	.
56			55		78540	2111101		1597	.
57			56		87780	21111001		1631	.
58		3	57		90090	121111		1779	.
59			58		117810	1211101		1856	.
60		4	59		120120	311111		1977	.
61		5	60		150150	112111		2144	.
62		6	61		180180	221111		2294	.
63		7	62		210210	111211		2420	.
64		8	63		240240	411111		2538	.
65		9	64		270270	131111		2645	.
66		10	65		300300	212111		2743	.
67		11	66		330330	111121		2836	.
68		12	67		360360	321111		2921	.
69		13	68		390390	111112		3001	.
70		14	69		420420	211211		3080	.
71		15	70		450450	122111		3153	.
72		16	71		480480	511111		3223	.
73	17#	1	72		510510	1111111		4843	.
74			73		570570	11111101		4939	.
75			74		690690	111111001		5119	.
76			75		746130	11111011		5138	.
77			76		870870	1111110001		5364	.
78			.		*881790*	11110111	(J)	*5235*	129
79			.		*903210*	111110101		*5317*	47
80			77		930930	11111100001		5436	.
81			.		*1009470*	111110011		*5412*	24
82		2	78		1021020	2111111		6225	.
83			79		1141140	21111101		6337	.
84			80		1381380	211111001		6546	.
85			81		1492260	21111011		6560	.
86		3	82		1531530	1211111		7178	.
87			83		1711710	12111101		7299	.
88		4	84		2042040	3111111		7928	.
89			85		2282280	31111101		8055	.
90		5	86		2552550	1121111		8553	.
91			87		2852850	11211101		8685	.
92		6	88		3063060	2211111		9099	.
93			89		3423420	22111101		9236	.
94		7	90		3573570	1112111		9580	.
95			91		3993990	11121101		9719	.
96		8	92		4084080	4111111		10010	.
97			93		4564560	41111101		10155	.
98		9	94		4594590	1311111		10414	.
99		10	95		5105100	2121111		10777	.
100		11	96		5615610	1111211		11120	.
101		12	97		6126120	3211111		11441	.
102		13	98		6636630	1111121		11740	.
103		14	99		7147140	2112111		12027	.
104		15	100		7657650	1221111		12293	.
105		16	101		8168160	5111111		12549	.
106		17	102		8678670	1111112		12799	.
107		18	103		9189180	2311111		13037	.

n	p#	k	A244052	a(n)	A054841(a(n))		
108	19#	1	104 (b)	9699690	11111111	19985	.
109			105	11741730	111111101	20605	.
110			106	13123110	111111011	20929	.
111			107	14804790	1111111001	21453	.
112			108	15825810	11111110001	21713	.
113			109	16546530	1111110101	21769	.
114			.	*17160990*	111110111	*21559*	210
115			110	17687670	11111101001	22028	.
116			111	18888870	111111100001	22443	.
117		2	112	19399380	21111111	25289	.
118			113	23483460	211111101	26005	.
119			114	26246220	211111011	26370	.
120		3	115	29099070	12111111	28924	.
121			116	35225190	121111101	29701	.
122		4	117	38798760	31111111	31776	.
123			118	46966920	311111101	32594	.
124		5	119	48498450	11211111	34150	.
125		6	120	58198140	22111111	36204	.
126		7	121	67897830	11121111	38028	.
127		8	122	77597520	41111111	39660	.
128		9	123	87297210	13111111	41161	.
129		10	124	96996900	21211111	42543	.
130		11	125	106696590	11112111	43827	.
131		12	126	116396280	32111111	45029	.
132		13	127	126095970	11111211	46156	.
133		14	128	135795660	21121111	47233	.
134		15	129	145495350	12211111	48240	.
135		16	130	155195040	51111111	49202	.
136		17	131	164894730	11111121	50130	.
137		18	132	174594420	23111111	51014	.
138		19	133	184294110	11111112	51861	.
139		20	134	193993800	31211111	52680	.
140		21	135	203693490	12121111	53468	.
141		22	136	213393180	21112111	54226	.
142	23#	1	137	223092870	111111111	83074	.
143			138	281291010	1111111101	86054	.
144			139	300690390	11111111001	86978	.
145			140	340510170	1111111011	88168	.
146			141	358888530	111111110001	89598	.
147			.	*363993630*	11111110101	*89097*	501
148			.	*380570190*	1111110111	*89235*	363
149			142	397687290	1111111100001	91214	.
150			.	*406816410*	11111101101	*90163*	1051
151			143	417086670	11111111000001	91993	.
152			.	*434444010*	111111101001	*91713*	280
153		2	144	446185740	211111111	103747	.
154			145	562582020	2111111101	107188	.
155			146	601380780	21111111001	108267	.
156		3	147	669278610	121111111	117837	.
157			148	843873030	1211111101	121572	.
158		4	149 (c)	892371480	311111111	128844	.
159		5	150	1115464350	112111111	137989	.
160		6	151	1338557220	221111111	145890	.
161		7	152	1561650090	111211111	152876	.
162		8	153	1784742960	411111111	159160	.
163		9	154	2007835830	131111111	164894	.
164		10	155	2230928700	212111111	170187	.
165		11	156	2454021570	111121111	175094	.
166		12	157	2677114440	321111111	179692	.
167		13	158	2900207310	111112111	184000	.
168		14	159	3123300180	211211111	188085	.
169		15	160	3346393050	122111111	191949	.
170		16	161	3569485920	511111111	195627	.
171		17	162 (d)	3792578790	111111211	199150	.
172		18	163	4015671660	231111111	202514	.
173		19	164	4238764530	111111121	205745	.
174		20	165	4461857400	312111111	208859	.
175		21	166	4684950270	121211111	211853	.
176		22	167	4908043140	211121111	214739	.
177		23	168	5131136010	111111112	217534	.
178		24	169	5354228880	421111111	220238	.
179		25	170	5577321750	113111111	222860	.
180		26	171	5800414620	211112111	225410	.
181		27	172	6023507490	141111111	227879	.
182		28	173	6246600360	311211111	230293	.

n	p#	k	A244052	a(n)	A054841(a(n))		Deficit
183		1	174	6469693230	1111111111	349670	.
184			175	6915878970	11111111101	352998	.
185			176	8254436190	111111111001	362468	.
186			177	8720021310	11111111011	363560	.
187			178	9146807670	1111111110001	368348	.
188			179	9592993410	11111111100001	371174	.
189			180	10407767370	111111110101	373037	.
190			181	10485364890	111111111000001	376626	.
191				10555815270	11111110111	*370805*	5821
192				11125544430	111111110011	*376388*	238
193			182	11532931410	1111111101001	378944	.
194				11797675890	11111101111 (K)	*374310*	4634
195			183	11823922110	1111111110000001	384339	.
196				12095513430	11111111010001	*381784*	2555
197				12328305990	1111111100101	*382299*	2040
198				12598876290	111111101101	*380250*	4089
199			184	12929686770	11111111001001	385136	.
200		2	185	12939386460	2111111111	431624	.
201			186	13831757940	21111111101	435480	.
202			187	16508872380	211111111001	446460	.
203			**188**	17440042620	21111111011	447583	.
204			189	18293615340	2111111110001	453304	.
205			190	19185986820	21111111100001	456596	.
206		3	191	19409079690	1211111111	487176	.
207			192	20747636910	12111111101	491365	.
208			193	24763308570	121111111001	503307	.
209		4	194	25878772920	3111111111	530392	.
210			195	27663515880	31111111101	534821	.
211		5	196	32348466150	1121111111	566239	.
212			197	34579394850	11211111101	570877	.
213		6	198	38818159380	2211111111	597138	.
214			199	41495273820	22111111101	601928	.
215		7	200	45287852610	1112111111	624417	.
216			201	48411152790	11121111101	629355	.
217		8	202	51757545840	4111111111	648948	.
218			203	55327031760	41111111101	654009	.
219		9	204	58227239070	1311111111	671297	.
220			205	62242910730	13111111101	676486	.
221		10	206	64696932300	2121111111	691878	.
222			207	69158789700	21211111101	697158	.
223		11	208	71166625530	1111211111	710992	.
224			209	76074668670	11112111101	716352	.
225		12	210	77636318760	3211111111	728833	.
226			211	82990547640	32111111101	734288	.
227		13	212	84106011990	1111121111	745602	.
228			213	89906426610	11111211101	751131	.
229		14	214	90575705220	2112111111	761438	.
230			215	96822305580	21121111101	767049	.
231		15	216	97045398450	1221111111	776445	.
232		16	217	103515091680	5111111111	790734	.
233		17	218	109984784910	1111112111	804368	.
234		18	219	116454478140	2311111111	817412	.
235		19	220	122924171370	1111111211	829930	.
236		20	221	129393864600	3121111111	841952	.
237		21	222	135863557830	1212111111	853533	.
238		22	223	142333251060	2111211111	864733	.
239		23	224	148802944290	1111111121	875544	.
240		24	225	155272637520	4211111111	885995	.
241		25	226	161742330750	1131111111	896134	.
242		26	227	168212023980	2111121111	905970	.
243		27	228	174681717210	1411111111	915521	.
244		28	229	181151410440	3112111111	924832	.
245		29	230	187621103670	1111111112	933876	.
246		30	231	194090796900	2221111111	942700	.

n	p#	k	A244052	a(n)	A054841(a(n))		
247	31#	1	232	200560490130	11111111111 (L)	1456458	.
248			233	239378649510	111111111101	1490913	.
249			234	255887521890	111111111011	1502991	.
250			235	265257422430	1111111111001	1512524	.
251			236	278196808890	11111111110001	1522919	.
252			237 (e)	283551037770	1111111110101	1524612	.
253			238	297382795710	11111111101001	1535022	.
254			239	304075581810	111111111100001	1543031	.
255				322640788470	111111110111	*1540418*	2613
256			240	325046311590	111111111010001	1555179	.
257			241	338431883790	1111111110011	1559290	.
258			242	342893741190	1111111111000001	1571546	.
259				354940756170	11111111100101	*1569756*	1790
260				357520873710	1111111101101	*1562052*	9494
261			243	366541585410	1111111110100001	1583729	.
262				374960916330	11111111011001	*1572472*	11257
263			244	381711900570	11111111110000001	1598293	.
264				387958500930	111111111001001	*1589991*	8302
265				390565164990	111111101111	*1565005*	33288
266				393312729810	11111111100011	*1591583*	6710
267			245	394651287030	111111111100000001	1606850	.
268		2	246	401120980260	21111111111	1781418	.
269			247	478757299020	211111111101	1821171	.
270			248	511775043780	211111111011	1835018	.
271			249	530514844860	2111111111001	1846173	.
272			250	556393617780	21111111110001	1858221	.
273			251	567102075540	2111111110101	1860041	.
274			252	594765591420	21111111101001	1872102	.
275		3	253	601681470390	12111111111	2000647	.
276			254	718135948530	121111111101	2043784	.
277			255	767662565670	121111111011	2058749	.
278			256	795772267290	1211111111001	2070946	.
279		4	257	802241960520	31111111111	2170709	.
280			258	957514598040	311111111101	2216328	.
281		5	259	1002802450650	11211111111	2311529	.
282			260	1196893247550	112111111101	2359167	.
283		6	261	1203362940780	22111111111	2432654	.
284		7	262	1403923430910	11121111111	2539521	.
285		8	263	1604483921040	41111111111	2635501	.
286		9	264	1805044411170	13111111111	2722895	.
287		10	265	2005604901300	21211111111	2803271	.
288		11	266	2206165391430	11112111111	2877848	.
289		12	267	2406725881560	32111111111	2947459	.
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291		14	269	2807846861820	21121111111	3074583	.
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295		18	273	3610088822340	23111111111	3292514	.
296		19	274	3810649312470	11111112111	3341183	.
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306		29	284	5816254213770	11111111121	3744918	.
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311		34	289	6819056664420	21111121111	3907885	.
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383		19	347	**140994024561390**	**111111121111**		**13583096**	.
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398		34	362	252305096583540	211111211111	15807228	.
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408			372	353588144099190	11111111111011	25986958	.
409			373	386480064480510	111111111110101	26265842	.
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467			407	1215998251658190	121111111110011		35346324	.
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523		36	463	10953009486979560	3311111111111		65020786	.
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556				22802323804350090	11111111111010101	*112771135*	476448
557				23204648147550870	11111111111100011	*113178162*	69421
558			487	23293697005168590	111111111111010000001	113662782	.
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562				23909874157879710	1111111110111101	*112558938*	1103844
563				23914632282611070	11111111111001101	*113312242*	350540
564				23991246389840730	111111111111000101	*113639249*	23533
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571			490	25208247443949570	1111111111110100000001	114871071	.
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575				25713258758096910	11111111111010011	*114286845*	813148
576				25743498104253930	111111111101110001	*113877652*	1222341
577				25811934519240870	111111111110110000001	*114776005*	323988
578				25894164320194170	1111111111101010001	*114566414*	533579
579				26134048498147590	1111111110111011	*113608109*	1491884
580		2	492	26165522663340060	21111111111111	127090969	.
581			493	28599524771557740	211111111111101	128285120	.
582			494	29994623540902020	211111111111011	128897969	.
583			495	32250527933884260	2111111111111001	130002703	.
584			496	33237285545323860	211111111110111	130151298	.
585			497	33823724418463980	2111111111110101	130617066	.
586			498	35901531096210780	21111111111110001	131635114	.
587			499	36970117387623420	2111111111110011	131817247	.
588			500	37118532150319620	211111111111100001	132161350	.
589				37480343274514140	2111111111101101	*131872459*	288891
590			501	37652825296025940	21111111111101001	132251188	.
591			502	38929192255213260	211111111111010001	132777964	.
592		3	503	39248283995010090	12111111111111	141076766	.
593			504	42899287157336610	121111111111101	142368151	.
594			505	44991935311353030	121111111111011	143029979	.
595			506	48375791900826390	1211111111111001	144227906	.
596			507	49855928317985790	121111111110111	144380748	.
597			508	50735586627695970	1211111111110101	144891000	.

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598	43	4	509	52331045326680120	31111111111111	151849368
599	.		510	57199049543115480	311111111111101	153213665
600	.		511	59989247081804040	311111111111011	153912067
601	.		512	64501055867768520	3111111111111001	155180143
602		5	513	65413806658350150	11211111111111	160724367
603			514	71498811928894350	112111111111101	162147368
604			515	74986558852255050	112111111111011	162875241
605		6	516	78496567990020180	22111111111111	168329480
606			517	85798574314673220	221111111111101	169802111
607			518	89983870622706060	221111111111011	170554836
608		7	519	91579329321690210	11121111111111	175018121
609			520	100098336700452090	111211111111101	176533741
610		8	521	104662090653360240	41111111111111	181009970
611			522	114398099086230960	411111111111101	182563621
612		9	523	117744851985030270	13111111111111	186452254
613			524	128697861472009830	131111111111101	188040080
614		10	525	130827613316700300	21211111111111	191448498
615			526	142997623857788700	212111111111101	193067448
616		11	527	143910374648370330	11112111111111	196074638
617		12	528	156993135980040360	32111111111111	200388101
618		13	529	170075897311710390	11111211111111	204433583
619		14	530	183158658643380420	21121111111111	208246367
620		15	531	196241419975050450	12211111111111	211855056
621		16	532	209324181306720480	51111111111111	215283086
622		17	533	222406942638390510	11111121111111	218549836
623		18	534	235489703970060540	23111111111111	221671944
624		19	535	248572465301730570	11111112111111	224663116
625		20	536	261655226633400600	31211111111111	227535271
626		21	537	274737987965070630	12121111111111	230298900
627		22	538	287820749296740660	21112111111111	232962695
628		23	539	300903510628410690	11111111211111	235534832
629		24	540	313986271960080720	42111111111111	238022005
630		25	541	327069033291750750	11311111111111	240430516
631		26	542	340151794623420780	21111211111111	242765608
632		27	543	353234555955090810	14111111111111	245032569
633		28	544	366317317286760840	31121111111111	247235492
634		29	545	379400078618430870	11111111121111	249378636
635		30	546	392482839950100900	22211111111111	251465288
636		31	547	405565601281770930	11111111112111	253499018
637		32	548	418648362613440960	61111111111111	255482472
638		33	549	431731123945110990	12112111111111	257418712
639		34	550	444813885276781020	21111121111111	259310145
640		35	551	457896646608451050	11221111111111	261159104
641		36	552	470979407940121080	33111111111111	262967531
642		37	553	484062169271791110	11111111111211	264737575
643		38	554	497144930603461140	21111112111111	266470911
644		39	555	510227691935131170	12111211111111	268169500
645		40	556	523310453266801200	41211111111111	269834638
646		41	557	536393214598471230	11111111111121	271467914
647		42	558	549475975930141260	22121111111111	273070560
648		43	559	562558737261811290	11111111111112	274643977
649		44	560	575641498593481320	31112111111111	276189318
650		45	561	588724259925151350	13211111111111	277707756
651		46	562	601807021256821380	21111111211111	279200197
652	47#	1	563 (f)	614889782588491410	111111111111111	440396221
653			*564*	693386350578511590	1111111111111101	
654			*565*	757887406446280110	1111111111111011	
655			*566*	771882918568531770	11111111111111001	
656			*567*	794857523833903530	1111111111110111	
657			*568*	798048441231871830	111111111111110001	
658			*569*	843685980760953330	11111111111110101	
659			*570*	872285505532511070	111111111111101001	
660			*571*	876545009221892010	1111111111111100001	
661				880788066951082290	1111111111101111	
662				884841394456609590	11111111111101101	
663				914836017997511610	111111111111011001	
664				928876054548572130	11111111111111000001	
665				951390574049585670	11111111111110011	
666				955041577211912190	111111111111110000001	
667				958084079847184290	1111111111111010001	
668				980499923587053870	11111111111011101	

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669	47		983641101983469930	111111111111100101
670	.		997799870344687410	11111111111101011
671	.		1004819888620217670	1111111111110110001
672	.		1013737209132377730	111111111110111001
673			1015283129390299770	11111111111110100001
674			1031623594763151390	111111111111010101
675			1033538145201932370	1111111111111100000001
676			1043882654161857510	111111111111101000001
677			1051263176683549830	1111111111011111
678			1064809135702021710	11111111111101100001
679			1080392685785122710	1111111111111001001
680			1085869190528612490	11111111111111000000001
681			1090618462934890890	1111111111100111
682			1094803759242923730	111111111111011000001
683			1094996698434428790	111111111111100011
684			1105670126598167130	11111111111011011
685			1113449065768349310	1111111111101110001
686			1123764085420346370	1111111110111111
687			1127588580322514310	111111111111001101
688			1129681228476530730	1111111111111010000001
689			1133094768018543330	1111111111110101001
690			1143150469872681270	111111111110110101
691			1144893741652891230	11111111111110010001
692			1148411171528791170	111111111111010011
693			1164365758518632670	111111111111110000000001
694			1170274102345838490	11111111110111101
695			1177144269586775490	111111111111100100001
696			1179923636858997030	11111111111011100001
697			1184787629865629790	1111111111110110000001
698			1186880278019646210	11111111111110100000001
699			1200742216855471290	11111111111101010001
700			1202701291723061130	1111111111111000101
701			1208523161630554770	11111111111010111
702			1209944410899934710	111111111101111001
703			1213160922404320890	111111111110111000001
704		2	1229779565176982820	211111111111111
705			1386772701157023180	2111111111111101
706			1515774812892560220	2111111111111011
707			1543765837137063540	21111111111111001
708			1589715047667807060	2111111111110111
709			1596096882463743660	211111111111110001
710			1687371961521906660	21111111111110101
711			1744571011065022140	211111111111101001
712			1753090018443784020	2111111111111100001
713			1761576133902164580	2111111111101111
714			1769682788913219180	21111111111101101
715			1829672035995023220	211111111111011001
716		3	1844669347765474230	121111111111111
717			2080159051735534770	1211111111111101
718			2273662219338840330	1211111111111011
719			2315648755705595310	12111111111111001
720			2384572571501710590	1211111111110111
721			2394145323695615490	121111111111110001
722		4	2459559130353965640	311111111111111
723			2773545402314046360	3111111111111101
724			3031549625785120440	3111111111111011
725		5	3074448912942457050	112111111111111
726			3466931752892557950	1121111111111101
727		6	3689338695530948460	221111111111111
728			4160318103471069540	2211111111111101
729		7	4304228478119439870	111211111111111
730			4853704454049581130	1112111111111101
731		8	4919118260707931280	411111111111111
732		9	5534008043296422690	131111111111111
733		10	6148897825884914100	212111111111111
734		11	6763787608473405510	111121111111111
735		12	7378677391061896920	321111111111111
736		13	7993567173650388330	111112111111111
737		14	8608456956238879740	211211111111111
738		15	9223346738827371150	122111111111111
739		16	9838236521415862560	511111111111111

n	p#	k	a(n)	A054841(a(n))
740	47	17	10453126304004353970	111111211111111
741	.	18	11068016086592845380	231111111111111
742	.	19	11682905869181336790	111111121111111
743	.	20	12297795651769828200	312111111111111
744		21	12912685434358319610	121211111111111
745		22	13527575216946811020	211121111111111
746		23	14142464999535302430	111111112111111
747		24	14757354782123793840	421111111111111
748		25	15372244564712285250	113111111111111
749		26	15987134347300776660	211112111111111
750		27	16602024129889268070	141111111111111
751		28	17216913912477759480	311211111111111
752		29	17831803695066250890	111111111211111
753		30	18446693477654742300	222111111111111
754		31	19061583260243233710	111111111121111
755		32	19676473042831725120	611111111111111
756		33	20291362825420216530	121121111111111
757		34	20906252608008707940	211111211111111
758		35	21521142390597199350	112211111111111
759		36	22136032173185690760	331111111111111
760		37	22750921955774182170	111111111112111
761		38	23365811738362673580	211111121111111
762		39	23980701520951164990	121112111111111
763		40	24595591303539656400	412111111111111
764		41	25210481086128147810	111111111111211
765		42	25825370868716639220	221211111111111
766		43	26440260651305130630	111111111111121
767		44	27055150433893622040	311121111111111
768		45	27670040216482113450	132111111111111
769		46	28284929999070604860	211111112111111
770		47	28899819781659096270	111111111111112
771		48	29514709564247587680	521111111111111
772		49	30129599346836079090	111311111111111
773		50	30744489129424570500	213111111111111
774		51	31359378912013061910	121111211111111
775		52	31974268694601553320	311112111111111
776	53#	1	32589158477190044730	1111111111111111
777			36278497172720993190	11111111111111101
778			37508276737897976010	111111111111111001
779			40909794684132183810	11111111111111011
780			41197615433428924470	1111111111111110001
781			42296567385289206990	111111111111110101
782			43657174563782890110	11111111111111100001
783			44715356980330526490	11111111111110111
784			44886954128959872930	111111111111111000001
785			46231131793223086710	111111111111101101
786			46456885488760276530	1111111111111101001
787			46896593906200308270	11111111111101111
788			47084858032680437970	111111111111110011
789			48486308953868115330	111111111111011101
790			48576292824490821390	1111111111111110000001
791			49230430891074322890	11111111111111010001
792			50617203592231346070	111111111111110100001
793			50778456231900767370	1111111111111011001
794			51035851954844787030	11111111111111100000001
795			51464844826418153130	111111111111101011
796			51716155544091628590	1111111111111100101
797			51966495950113855110	11111111111011111
798			53255454096871536510	1111111111110111001
799			53469245562535412610	1111111111111100011
800			53728072084016019690	111111111110111101
801			53810005857685887810	11111111111110110001
802			53975325061853184990	111111111111011011
803			54725190650375735490	111111111111111000000001
804			54777521695702415610	1111111111111101000001
805			54803687218365755670	11111111111111001001
806			55325780670578448030	111111111111101100001
807			56347453055502819210	111111111111110010001
808			56434884192207150630	11111111111101110001
809			56526960710983873110	1111111111111010101
810			56661439327462899930	11111111111111000101

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811	53		57551067098016461970	11111111111111010000001
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814	.		58257536209926643590	111111111111110001001
815			58443128870678241690	1111111111111010011
816			59012800485722513430	1111111111101111001
817			59284373428592842530	1111111111110110101
818			59644308911083666770	1111111111111110000000001
819			59810495338810286070	111111111110111011
820			59873105109256128690	1111111111111011000001
821			59901704634027686430	11111111111110101001
822			60865792091025932010	111111111111010111
823			60978750566914009830	1111111111111100100001
824			61294013205833277870	1111111111110110011
825			61589076595549593090	111111111111101010001
826			61711385201487531510	111111111111110100000001
827			61932270892808285970	11111111111110100101
828			62024527424329439970	11111111110111111
829			62103868041437632410	11111111111111100000000001
830			62234695654754332710	11111111111111000011
831			62535952753526842590	11111111111011110001
832			62793744382878378870	1111111111110111000001
833			62823739006419280890	11111111111101101001
834			62904654735041249130	11111111111110110000001
835			63045826857317874570	1111111111111100010001
836			63333647606614615230	111111111111111000000000001
837			63676841903873308110	111111111111101001001
838			63743168461322239890	1111111111111001101
839			63987785673198116730	111111111111110000101
840			64066282241188136910	11111111111111001000001
841			64127053777696539630	111111111101111101
842			64297528887429007170	111111111110111100001
843			64593421795332500070	111111111111011010001
844			64953357277823324310	11111111111101100101
845		2	65178316954380089460	2111111111111111
846			72556994345441986380	21111111111111101
847			75016553475795952020	211111111111111001
848			81819589368264367620	21111111111111011
849			82395230866857848940	2111111111111110001
850			84593134770578413980	211111111111110101
851			87314349127565780220	21111111111111100001
852			89430713960661052980	21111111111110111
853			89773908257919745860	211111111111111000001
854			92462263586446173420	211111111111101101
855			92913770977520553060	2111111111111101001
856			93793187812400616540	21111111111101111
857			94169716065360875940	211111111111110011
858			96972617907736230660	211111111111011101
859			97152585648981642780	2111111111111110000001
860		3	97767475431570134190	1211111111111111
861			108835491518162979570	12111111111111101
862			112524830213693928030	121111111111111001
863			122729384052396551430	12111111111111011
864			123592846300286773410	1211111111111110001
865			126889702155867620970	121111111111110101
866		4	130356633908760178920	3111111111111111
867			145113988690883972760	31111111111111101
868			150033106951591904040	311111111111111001
869		5	162945792385950223650	1121111111111111
870			181392485863604965950	11211111111111101
871			187541383689489880050	112111111111111001
872		6	195534950863140268380	2211111111111111
873			217670983036325959140	22111111111111101
874			225049660427387856060	221111111111111001
875		7	228124109340330313110	1112111111111111
876			253949480209046952330	11121111111111101
877		8	260713267817520357840	4111111111111111
878			290227977381767945520	41111111111111101
879		9	293302426294710402570	1311111111111111
880		10	325891584771900447300	2121111111111111
881		11	358480743249090492030	1111211111111111
882	53	12	391069901726280536760	3211111111111111
883	.	13	423659060203470581490	1111121111111111
884	.	14	456248218680660626220	2112111111111111
885	.	15	488837377157850670950	1221111111111111
886		16	521426535635040715680	5111111111111111
887		17	554015694112230760410	1111112111111111
888		18	586604852589420805140	2311111111111111
889		19	619194011066610849870	1111111211111111
890		20	651783169543800894600	3121111111111111
891		21	684372328020990939330	1212111111111111
892		22	716961486498180984060	2111211111111111
893		23	749550644975371028790	1111111121111111
894		24	782139803452561073520	4211111111111111
895		25	814728961929751118250	1131111111111111
896		26	847318120406941162980	2111121111111111
897		27	879907278884131207710	1411111111111111
898		28	912496437361321252440	3112111111111111
899		29	945085595838511297170	1111111112111111
900		30	977674754315701341900	2221111111111111
901		31	1010263912792891386630	1111111111211111
902		32	1042853071270081431360	6111111111111111
903		33	1075442229747271476090	1211211111111111
904		34	1108031388224461520820	2111112111111111
905		35	1140620546701651565550	1122111111111111
906		36	1173209705178841610280	3311111111111111
907		37	1205798863656031655010	1111111111121111
908		38	1238388022133221699740	2111111211111111
909		39	1270977180610411744470	1211121111111111
910		40	1303566339087601789200	4121111111111111
911		41	1336155497564791833930	1111111111112111
912		42	1368744656041981878660	2212111111111111
913		43	1401333814519171923390	1111111111111211
914		44	1433922972996361968120	3111211111111111
915		45	1466512131473552012850	1321111111111111
916		46	1499101289950742057580	2111111121111111
917		47	1531690448427932102310	1111111111111121
918		48	1564279606905122147040	5211111111111111
919		49	1596868765382312191770	1113111111111111
920		50	1629457923859502236500	2131111111111111
921		51	1662047082336692281230	1211112111111111
922		52	1694636240813882325960	3111121111111111
923		53	1727225399291072370690	1111111111111112
924		54	1759814557768262415420	2411111111111111
925		55	1792403716245452460150	1121211111111111
926		56	1824992874722642504880	4112111111111111
927		57	1857582033199832549610	1211111211111111
928		58	1890171191677022594340	2111111112111111
929	59	1	1922760350154212639070	11111111111111111

T	j	Abbreviated Notation	a(n)	r(a(n))	Deficit of r(a(n)) from local record
2	1	01	10	6	.
3	1	01	42	19	.
4	1	01	330	77	.
		001	390	80	.
5	1	01	2730	295	.
		001	3570	313	.
		0001	3990	322	.
	2	*011*	*4290*	315	7
6	1	01	39270	1224	.
		001	43890	1253	.
		0001	53130	1306	.
	2	011	46410	1257	.
		0101	51870	1285	.
7	1	01	570570	4939	.
		001	690690	5119	.
		0001	870870	5364	.
		00001	930930	5436	.
	2	011	746130	5138	.
		0101	*903210*	5317	47
		0011	*1009470*	5412	24
	3	*0111*	*881790*	5235	129
8	1	01	11741730	20605	.
		001	14804790	21453	.
		0001	15825810	21713	.
		00001	18888870	22443	.
	2	011	13123110	20929	.
		0101	16546530	21769	.
		01001	17687670	22028	.
	3	*0111*	*17160990*	21559	210
9	1	01	281291010	86054	.
		001	300690390	86978	.
		0001	358888530	89598	.
		00001	397687290	91214	.
		000001	417086670	91993	.
	2	011	340510170	88168	.
		0101	*363993630*	89097	501
		01001	*434444010*	91713	280
	3	*0111*	*380570190*	89235	363
		01101	*406816410*	90163	1051
10	1	01	6915878970	352998	.
		001	8254436190	362468	.
		0001	9146807670	368348	.
		00001	9592993410	371174	.
		000001	10485364890	376626	.
		0000001	11823922110	384339	.
	2	011	8720021310	363560	.
		0101	10407767370	373037	.
		0011	*11125544430*	376388	238
		01001	11532931410	378944	.
		00101	*12328305990*	382299	2040
		010001	*12095513430*	381784	2555
		001001	12929686770	385136	.
	3	*0111*	*10555815270*	370805	5821
		01101	*12598876290*	380250	4089
	4	*01111*	*11797675890*	374310	4634

T	j	Abbreviated Notation	a(n)	r(a(n))	Deficit of r(a(n)) from local record
11	1	01	239378649510	1490913	.
		001	265257422430	1512524	.
		0001	278196808890	1522919	.
		00001	304075581810	1543031	.
		000001	342893741190	1571546	.
		0000001	381711900570	1598293	.
		00000001	394651287030	1606850	.
	2	011	255887521890	1502991	.
		0101	283551037770	1524612	.
		0011	338431883790	1559290	.
		01001	297382795710	1535022	.
		00101	*354940756170*	1569756	1790
		00011	*393312729810*	1591583	6710
		010001	325046311590	1555179	.
		001001	*387958500930*	1589991	8302
		0100001	366541585410	1583729	.
	3	*0111*	*322640788470*	1540418	2613
		01101	*357520873710*	1562052	9494
		011001	*374960916330*	1572472	11257
	4	*01111*	*390565164990*	1565005	33288
12	1	01	8222980095330	6186873	79616
		001	8624101075590	6225357	38484
		0001	9426343036110	6299923	74566
		00001	10629705976890	6406043	49034
		000001	11833068917670	6505900	73795
		0000001	12234189897930	6537922	62133
		00000001	13437552838710	6630529	79620
		000000001	14239794799230	6689583	55912
		0000000001	14640915779490	6718369	52500
	2	011	9814524629910	6313295	.
		0101	10293281928930	6351940	.
		0011	11406069164490	6432105	.
		01001	11250796526970	6426795	.
		00101	*12467098854210*	6507153	30769
		00011	13075250017830	6546072	.
		010001	*12687068424030*	6533350	4572
		001001	*14058643388790*	6614065	16464
		000101	*14744430871170*	6653154	65215
		0100001	14123340321090	6633671	.
		01000001	*14602097620110*	6665869	23714
	3	0111	10491388397490	6357009	.
		01101	*11003163441270*	6395648	10395
		01011	*12192694624110*	6475789	30111
		00111	*14552571002970*	6602524	87059
		011001	*12026713528830*	6470557	35343
		010101	13326898775190	6550909	.
		010011	*13976991398370*	6589848	40681
		0110001	*13562038660170*	6577198	53331
	4	*01111*	*13228272327270*	6489221	56851
		011101	*13873553904210*	6527793	102736

T	j	Abbreviated Notation	a(n)	r(a(n))	Deficit of r(a(n)) from local record
13	1	01	319091739796830	25691103	.
		001	348774692336070	25969369	.
		0001	393299121144930	26367005	.
		00001	437823549953790	26742668	.
		000001	452665026223410	26863379	.
		0000001	497189455032270	27213034	.
		00000001	526872407571510	27436416	.
		000000001	541713883841130	27545383	.
		0000000001	586238312649990	27862209	.
	2	011	353588144099190	25986958	.
		0101	386480064480510	26265842	.
		0011	405332750552730	26409945	.
		01001	435817945052490	26664539	.
		00101	457077357006270	26809231	54148
		00011	499596180913830	27090540	122494
		010001	485155825624470	27041323	.
		001001	508821963459810	27186671	26363
		000101	556154239130490	27469181	76202
		0100001	501601785815130	27162445	50589
		0010001	526070165610990	27307929	.
		0001001	575006925202710	27590902	.
		01000001	550939666387110	27513287	32096
		00100001	577814772064530	27659372	.
		010000001	583831586768430	27737411	.
		0100000001	600277546959090	27846798	15411
	3	0111	422024559086130	26450355	.
		01101	461282657605770	26729791	133588
		01011	483784250659710	26874055	.
		00111	536085250731030	27171248	265168
		011001	520169805385230	27129502	83532
		010101	545543942233290	27274267	271116
		001101	604521665717970	27572381	289828
		010011	596292215929410	27555986	306223
		0110001	579056953164690	27507433	151939
		0101001	607303633806870	27652803	209406
		01100001	598686002424510	27628947	233262
	4	*01111*	451129701092070	26608037	134631
		011101	493095254682030	26887509	153814
		011011	517148681739690	27031706	181328
		010111	573056647333170	27328648	216735
		0111001	556043585066970	27287265	258118
		0110101	583167662387310	27432006	227366
	5	*011111*	568815710072610	27072943	472440

T	j	Abbreviated Notation	a(n)	r(a(n))	Deficit of r(a(n)) from local record
14	1	01	14299762385778870	107157561	.
		001	16125263966942130	108653223	.
		0001	17950765548105390	110072228	.
		00001	18559266075159810	110529213	.
		000001	20384767656323070	111855268	.
		0000001	21601768710431910	112704111	.
		00000001	22210269237486330	113118627	.
		000000001	24035770818649590	114325008	.
		0000000001	25252771872758430	115099993	.
	2	011	14997311770451010	107693596	.
		0101	16911862209231990	109190977	.
		0011	18485058693811710	110239894	.
		01001	18826412648012970	110611835	.
		00101	20577706847828130	111664732	190536
		00011	23204648147550870	113178162	69421
		010001	19464596127606630	111069449	.
		001001	21275256232500270	112123642	.
		000101	23991246389840730	113639249	23533
		0100001	21379146566387610	112397465	.
		0010001	23367904386516690	113455693	207089
		01000001	22655513525574930	113247583	.
		00100001	24763003155860970	114308560	16448
		010000001	23293697005168590	113662782	.
		001000001	25460552540533110	114725094	374899
		0100000001	25208247443949570	114871071	.
	3	*0111*	16618642772661930	108793184	.
		01101	18740171637257070	110293311	235902
		01011	20483443417467030	111343381	511887
		00111	21482635779294690	111882799	514666
		011001	20861700501852210	111717231	138037
		010101	22802323804350090	112771135	476448
		001101	23914632282611070	113312242	350540
		010011	25713258758096910	114286845	813148
		0110001	21568876790050590	112175914	221551
		0101001	23575283933311110	113231219	431563
		0011001	24725297783716530	113772794	552214
		01100001	23690405654645730	113507268	155514
		01010001	25894164320194170	114566414	533579
		011000001	25104758231042490	114359639	.
		0110000001	25811934519240870	114776005	323988
	4	*01111*	19835154277048110	110490165	579284
		011101	22367301631564890	111991797	1126830
		011011	24447980853105810	113041819	1283189
		010111	25640565284964630	113580618	1519375
		0111001	24899448986081670	113418083	906925
		01110001	25743498104253930	113877652	1222341
	5	*011111*	21203095951327290	111057835	797433
		0111101	23909874157879710	112558938	1103844
		0111011	26134048498147590	113608109	1491884