Notes on OEIS A359557

The Raise-the-Bar sequence, a sequence of Scott R. Shannon.

Michael Thomas De Vlieger · St. Louis, Missouri · 10 January 2023

Abstract.

Scott Shannon wrote a sequence that requires all prime divisors of the sum of 2 adjacent terms to divide the following term. This leads to a sequence that exhibits phases wherein all terms are divisible by a "sticky" prime, raising the bar for numbers that endeavor to enter the sequence. The phases show in scatterplot as a series of escalating scallops. The sticky primes conglomerate into a sticky composite factor that continues to grow as n increases. This sticky factor yields interesting implications for subsequent terms in the sequence.

Introduction.

Shannon defines A359557 = a to be the following sequence:

$a(1) = 1$, $a(2) = 2$; for $n > 2$, $a(n)$ is the smallest positive number which has not appeared such that all the distinct prime factors of $a(n-2) + a(n-1)$ are factors of $a(n)$.

For convenience, we set the following:

$$i = a(n-2),\ j = a(n-1),\ s = i + j, \text{ and } \varkappa = \mathrm{RAD}(s) = \mathrm{A7947}(s).$$

We consider the candidate k for $a(n)$, and define function $c(x)$ to be TRUE if $a(j) = x$, $j < n$ else FALSE. We summarize the sequence definition as follows:

$$a(n) = n,\ n \le 2; \text{ for } n > 2, \qquad \text{[F1]}$$
$$a(n) = \boxed{\le}\ m(\varkappa)\,\varkappa : \neg c(m(\varkappa)\,\varkappa). \qquad \text{[AXIOM 1]}$$

We can efficiently generate the sequence with a program that stores $c(x)$ and $m(x)$, initializing these globally to FALSE and 1, respectively. Once $a(n) = k = m(\varkappa)\,\varkappa$, we set $c(k)$ to TRUE, and increment $m(\varkappa)$.

$a(3) = 3$ since $s = 3$ and $m(3) = 1$.
$a(4) = 5$ since $s = 5$ and $m(5) = 1$.
$a(5) = 4$ since $s = 8$ thus $\varkappa = 2$, and $m(2) = 2$ (since $a(2) = 2$).
$a(6) = 6$ since $s = 9$ thus $\varkappa = 3$, and $m(3) = 2$.
$a(7) = 10$ since $s = 10$ and $m(10) = 1$.
$a(8) = 8$ since $s = 16$ thus $\varkappa = 2$, and $m(2) = 3$, but $c(6)$ is true, so we move to $m(2) = 4$.
$a(9) = 12$ since $s = 18$ thus $\varkappa = 6$, and $m(6) = 2$, etc.

The sequence begins as follows:

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1, 2, 3, 5, 4, 6, 10, 8, 12, 20, 14, 34, 18, 26, 22, 24,
46, 70, 58, 16, 74, 30, 52, 82, 134, 36, 170, 206, 94,
60, 154, 214, 92, 102, 194, 148, 114, 262, 188, 90, 278,
138, 78, 42, 120, 48, 84, 66, 150, 54, 204, 258, 462,
180, 642, 822, 366, 132, 498, 210, 354, ...
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Some Simply Sticky Theorems.

Lemma 1.1: $2 \mid i \wedge 2 \mid j \Rightarrow 2 \mid k$.
Proof: $i \equiv j \equiv 0 \pmod 2$ implies $s = i + j \equiv 0 \pmod 2$, which in turn implies $2 \mid \varkappa$ as 2 is prime. Since Axiom 1 demands $a(n)$, $n > 2$ is the least novel k that is a multiple $m\varkappa$, $2 \mid k$. ■

Lemma 1.2: $2 \mid i \wedge 2 \mid j \Rightarrow 2 \mid a(n+v)$, $v \ge 0$.
Proof: This is evident through rewriting Lemma 1.1 as $2 \mid a(n-2) \wedge 2 \mid a(n-1) \Rightarrow 2 \mid a(n)$ and induction on n. ■

Theorem 1: Prime $p \mid i \wedge p \mid j \Rightarrow p \mid a(n+v)$, $v \ge 0$. (The "sticky prime" theorem.)
Proof: We prove this theorem through generalization of the congruence argument and application of Axiom 1 in Lemma 1.1 and the induction argument in Lemma 1.2. ■

Theorem 2: $Q \mid i \wedge Q \mid j \Rightarrow Q \mid a(n+v)$, $v \ge 0$. (The "sticky factor" theorem.)
Proof: This is a generalization of Theorem 1. Any factor Q common to i and j divides all subsequent terms of the sequence as a consequence of the sum $i + j$, Axiom 1, and the induction argument of Lemma 1.2. ■

Corollary 2.1: The sequence is not a permutation of natural numbers if we have 2 consecutive even terms, or generally, 2 consecutive terms divisible by a common factor Q.

Corollary 2.2: Q serves as a lower bound for $a(n+v)$, $v \ge 0$.

Summation scrambles the prime factors of s, but whereupon the parity of i and j are the same, we have an even k. This begins at $n = 4$. The multiplier m for the following number $a(5) = 6$ happened to be even, resulting in a sum of evens, which means that subsequent terms are even. It is clear that $2 \mid \varkappa$ hence $2 \mid m(\varkappa)\,\varkappa$ for $n > 4$.

We say the prime divisor p common to i and j is sticky because according to the theorems, this prime divides subsequent terms. It is also true that we might say any factor Q common to i and j, whether or not Q is prime, is sticky, because subsequent terms are also divisible by the factor Q.

The Scalloping Scatterplot.

Scatterplot lays bare a curious "scalloping" behavior, revealing the sequence to suffer a series of phases that forces terms to be much larger than before, gradually abating until the onset of a new phase. The conjecture is that these are tuned to the emergence of a new sticky prime p such that $p \mid i$ and $p \mid j$.

This has to do with the congruence generalization of summation and parity (Theorem 1). Instead, we may consider three cases as to summation as follows:

$$p \mid i \wedge p \mid j, \qquad \begin{matrix} p \mid i \wedge p \nmid j, \\ p \nmid i \wedge p \mid j, \end{matrix} \qquad p \nmid i \wedge p \nmid j$$

In the first case implies $p \mid k$.
The second case implies $p \nmid k$.
The third case normally yields $p \nmid k$ especially when p is large, however, if $i + j \equiv 0 \pmod p$ then $p \mid k$.

Therefore there are 2 ways for odd p to arise as divisor of k, but only one means by which all subsequent terms in a are divisible by p.

It seems natural to surmise that a common factor Q, the product of sticky primes, would perhaps be a primorial $Q \in$ A2110. Is there something that requires the sticky primes p to emerge in order?

Indeed it is not necessarily true that we would have plenary divisibility of subsequent terms by p in order of the primes. Suppose we have, say, $17 \mid a(j)$, then say $31 \mid i$ and $31 \mid j$ somehow immediately after, we could have $31 \mid a(j+k) : k > 0$ before $19 \mid a(j+k) : k > 0$. We may be able to show that such never happens on account of the greedy approach regarding m.

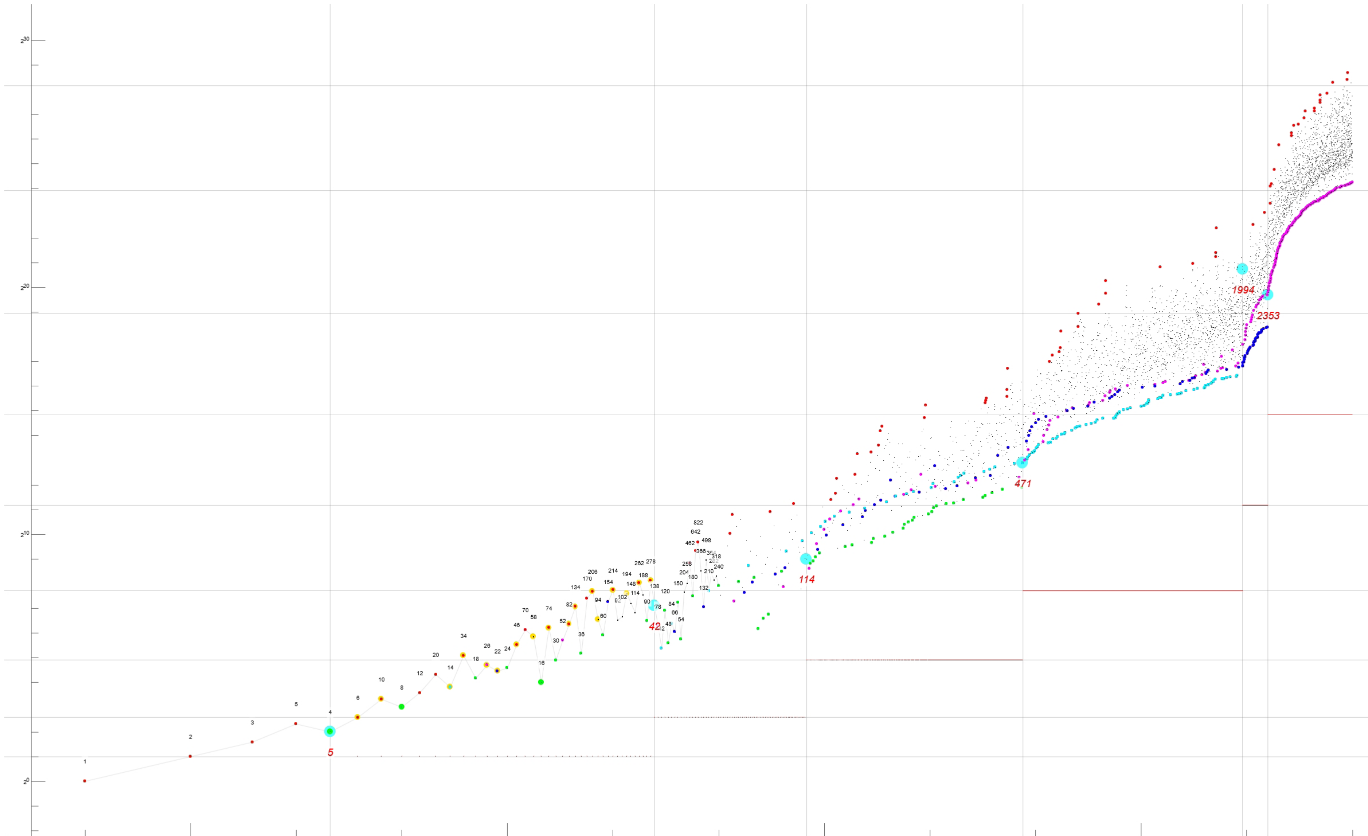

Figure 1: Log-log scatterplot of a(n), n = 1 … 4096 showing records in medium red, highlighting squarefree semiprimes in large gold, prime powers in large green. Phase transitions are highlighted in large light cyan and the transition index is labeled in red. We show 5-smooth numbers in small green, 7-smooth numbers in small cyan, 11-smooth numbers in small blue, and 13-smooth numbers in small magenta. Remaining terms are shown in tiny black dots. The rows of tiny red dots below the sequence indicate the lower bound of Q.

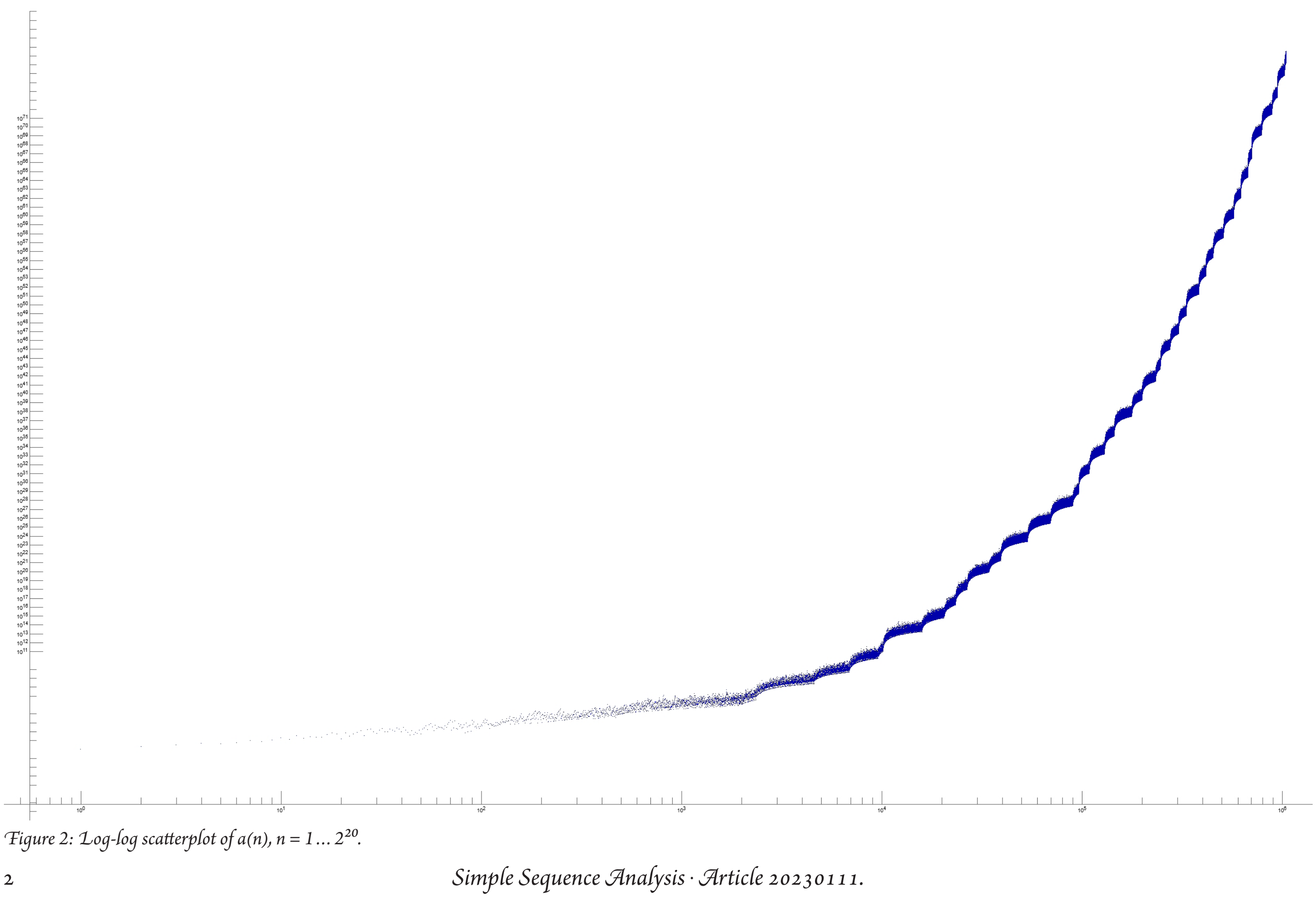

Figure 2: Log-log scatterplot of a(n), n = 1 … 2^{20}.

Phases in the Sequence.

Hence indeed we have phases in the sequence that are governed by divisibility of both i and j by some sticky prime p.

We define a sequence of the positions $n : a(n) = p$, which we call "phase transitions", as follows:

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5, 42, 114, 471, 1994, 2353, 4591, 6904, 9612, 10165,
15922, 20530, 23372, 26742, 34375, 39302, 53538, 69757,
89995, 96260, 108911, 129634, 144879, 177605, 199537,
232930, 246327, 275596, 303470, 331535, 383137, 414878,
452277, 509216, 572672, 621108, 672487, 703926, 790370,
889469, 943629, 1027898, ...
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Note that these indices n pertain to the first emergence of $p \mid a(n)$. They are found by determining when both $a(n)$ and $a(n+1)$ are divisible by p. There doesn't seem to be a way to predict phase transitions except by determining that $Q' : Q' \mid a(n-1) \wedge Q' \mid a(n)$ exceeds $Q : Q \mid a(n-2) \wedge Q \mid a(n-1)$. In other words, we look for the occasion of the following:

$$(a(n-1), a(n)) > (a(n-2), a(n-1)). \qquad \text{[F2]}$$

Therefore, for $n \geq 5$, $2 \mid a(n)$, and for $n \geq 42$, $3 \mid a(n)$, etc. More precisely, for $n \geq 5$, $2 \mid a(n)$, for $n \geq 42$, $6 \mid a(n)$, for $n \geq 114$, $30 \mid a(n)$, etc., since indeed, the previous record primes also divide $a(n)$ as n increases. This suggests that primorials, products $P(\ell)$ = A2110(ℓ) of the first primes, are important in this sequence.

Let us then define the δ-th phase as that which begins with the δ-th sticky prime $p_\delta \mid a(n+v)$, $v \geq 0$. and ends with the term that precedes the $(\delta+1)$-th sticky prime $p_{(\delta+1)}$ as divisor of all terms that follow the first such divisible by $p_{(\delta+1)}$. Hence, phase 0 begins with $a(1) = 1$ and ends with $a(4) = 5$, followed by phase 1, starting with $a(5) = 4$ and ending with $a(41) = 278$, followed by phase 2, starting with $a(42) = 138$, etc. All the terms in phase 0 are divisible by 1 (i.e., they are unrestrained), as are all subsequent terms. All the terms in phase 1 are even since $p_1 = 2$, as are all subsequent terms. All the terms in phase 2 are trine, since $p_2 = 3$, as are all subsequent terms, etc. It is clear that sticky factor $Q_1 = 2$, $Q_2 = 6$, $Q_3 = 30$, etc., therefore, all terms in phase 2 are congruent to 0 (mod 6), all in phase 3 congruent to 0 (mod 30), etc.

Table 1 shows the first 7 phase transitions and sticky factors.

δ	n	Q_δ
1	5	2
2	42	6
3	114	30
4	471	210
5	1994	2310
6	2353	30030
7	4591	510510

Given 5000 terms of the sequence it would seem indeed that the sticky factors are primorials.

Primorials in the Sequence.

Do primorials define the phases in this sequence?

We study the occasion of primorials $P(\ell)$ in a.

Table 2 lists positions of the smallest primorials in the sequence.

ℓ	n	P(ℓ)
0	1	1
1	2	2
2	6	6
3	22	30
4	60	210
5	178	2310
6	508	30030
7	1477	510510
8	2687	9699690
9	4807	223092870

In brief, $a(m) = P(\ell)$ such that $m < n_\delta$ and $\delta = \ell$. In other words, primorial $P(\ell)$ appears well before the corresponding phase $\delta = \ell$. Now we do not see $P(\ell) \mid a(n)$ for some time as n increases, since $i+P(\ell)$ scrambles the prime divisors of s. It is only whereupon p_ℓ divides both i and j that we have divisibility of by a larger primorial $P(\ell)$ for $n \leq 6903$.

Theorem 3. Let $j < k < \ell$ be indices of primes $p_j < p_k < p_\ell$, respectively. Suppose $p_j \mid a(n_j)$ and $p_j \mid a(n_j+1)$. Then $p_\ell \mid a(n_\ell)$ and $p_\ell \mid a(n_\ell+1)$, $n_j < n_\ell$ while $p_k \nmid a(n_\ell)$ and $p_k \nmid a(n_\ell+1)$ implies primorials $P(1 \dots k)$ cannot enter the sequence unless they already appear.

Proof. Suppose $a(n_j+1)$ is divisible by all primes smaller than p_j in addition to p_j. In this case, $a(n_j \dots n_\ell)$ may harbor primorials P at least as large as $P(j)$, since $P(j)= Q$, the sticky factor, and $Q \mid P$. It is clear that smaller primorials are missing prime factors that Q has, therefore they cannot enter the sequence after $a(n_j+1)$. Also clear is the fact that while we have successive sticky primes p in order of the primes, we have successive sticky factors Q that are successive primorials. Whereupon we have a break and have sticky prime p_ℓ ahead of p_k, primorial $P(k)$ can never enter, since it doesn't have the necessary factor p_ℓ. Primorial $P(k)$ may not enter even when the next sticky prime is p_k, since it still lacks the necessary factor p_ℓ. Primorials $P(M)$, $M > \ell$, may indeed appear until the emergence of p_N, $N > M$. ■

Corollary 3.1. Primorials $P(\ell)$, $\ell \geq \delta$ can enter the sequence only when the sticky prime factor Q_δ is itself a primorial.

Corollary 3.2. Primes $q > 2$ may enter the sequence iff $\delta = 0$. In other words, once we have a sticky prime p, all other primes cannot enter the sequence. For $n \geq 7$, $p_1 = 2$, hence $a(n)$ for $n \geq 5$ is even.

Corollary 3.3. Composite prime powers $q^\varepsilon : \varepsilon > 1$ may enter the sequence iff $\delta \leq 1$. When the sequence is unrestrained, any number may enter, however the greedy approach to Axiom 1 limits the number of composite prime powers in this phase. Once we have $p_1 = 2$, only composite prime powers of $2^\varepsilon : \varepsilon > 1$ may appear. When $p_2 = 3$ appears at $n = 44$, no prime power may appear, since such would either have to be a power of 2 or of 3, and thereby lack the other necessary prime divisor.

Corollary 3.4. Squarefree semiprimes $q_1 q_2 : q_1 < q_2$ may enter the sequence iff $\delta \leq 1$. When the sequence is unrestrained, any number may enter, however the greedy approach to Axiom 1 limits the number of squarefree semiprimes in this phase. Once we have $p_1 = 2$, only even squarefree semiprimes $2q$ may appear. When $p_2 = 3$ appears at $n = 44$, only 6 can appear, however $a(6) = 6$. Once we have 3 sticky primes, semiprimes cannot appear because they lack 1 necessary prime factor, and the number of sticky primes is nondecreasing as n increases.

Corollary 3.2 implies the only primes in the sequence are 2, 3, abd 5. As a consequence of Corollary 3.3, the only composite prime powers in the sequence are 4, 8, and 16. Corollary 3.4 implies the number of squarefree semiprimes is finite; they are all even:

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6, 10, 14, 34, 26, 22, 46, 58, 74, 82,
   134, 206, 94, 214, 194, 262, 278
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Therefore, it is possible that not all primorials are in the sequence. Indeed, we cannot find $P(10)$ in the smallest 2^{26} terms. It is clear that this primorial will never appear because for $\delta = 8$ we have the following:

$$Q_8 = 15825810 = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 31.$$

We will return to the question of phases after introducing another aspect of this sequence.

Constitutive Analysis.

There are 3 classes of constitutive relations between 2 integers k and n [2]. These may be examined in a set-theoretic manner, as they pertain to the set of prime divisors of k and n. These classes are useful in analyzing the behavior of sequences with axioms that constrain output multiplicatively.

Let the squarefree kernel $\kappa = \operatorname{rad}(k)$, product of the set $\mathcal{K}$ of distinct prime divisors of k and let $\nu = \operatorname{rad}(n)$, of the set $\mathcal{N}$ of distinct prime divisors of n.

The first and easiest to understand on account of its familiarity is the relationship of coprimality. If $(k, n) = 1$, i.e., $\mathcal{K} \cap \mathcal{N} = \varnothing$, then we say k is coprime to n, or n-coprime. Borrowing Knuth's notation, we write $k \perp n$. Coprimality is symmetric, that is, $k \perp n$ implies $n \perp k$.

The second is more obscure, but includes divisors $d \mid n$. We say k is regular to n (or k is n-regular) iff k is indivisible by primes q that do not divide n. We write it this way so as to include the empty product $k = 1$, since $1 \mid n$, but we may think of k instead as a product limited to the primes $p \mid n$. Set theoretically, we have $\mathcal{K} \subseteq \mathcal{N}$. We can also write $k \mid n^{\varepsilon} : \varepsilon \geq 0$. Written this way, we see that this class includes divisors $d \mid n^{\varepsilon} : 0 \leq \varepsilon \leq 1$. Then it is clear that there are nondivisor n-regular k, which we call "semidivisors" and write $k ¦ n$. Generally we write n-regular k as $k \parallel n$ in resonance with Knuth's notation for coprimality. It is clear that $k \mid n$ implies $n \mid k$ iff $k = n$. Similarly, $k \parallel n$ implies $n \parallel k$ iff $\mathcal{K} = \mathcal{N}$. Thus, $k ¦ n$ implies $n ¦ k$ iff $\mathcal{K} = \mathcal{N}$ and neither term divides the other.

Finally, the last class is semicoprimality, written $k \between n$, or k is semicoprime to n. This includes k that is a product of at least 1 prime $p \mid n$ and at least 1 prime q coprime to n, implying composite k. The n-semicoprime k is such that $(k, n) > 1$ and not equal to either k or n, $\mathcal{K} \cap \mathcal{N} \neq \varnothing$. Symmetric semicoprimality ($k \between n$ and $n \between k$) has $\mathcal{K}$ include at least one prime that does not divide n, and $\mathcal{N}$ include at least one prime that does not divide k. An example of this is $6 \between 10$ and $10 \between 6$. Oftentimes we have $\mathcal{K} \supset \mathcal{N}$, e.g., $2p \between p$, $p > 2$.

Though there are 3 major classes of constitutive relation, we may regard divisibility and semidivisibility as distinct, therefore we have coprimality ($\perp$) which is always symmetric, semicoprimality ($\between$), divisibility ($\mid$), and semidivisibility ($¦$), the last 3 only conditionally symmetric. Therefore, for example, we may more accurately write $2p \between p$, $p > 2$ as $2p \between\!\mid p$, $p > 2$, because $p \mid 2p$. We may express $6 \mid 12$ more accurately by $6 \mid¦ 12$, since 12 semidivides 6 (i.e., $12 \mid 6^2$).

We may assign symbols to the various non-coprime binary relations. If $k \perp n$, then the state is ⓪. For noncoprime (open) relations, we have the following:

	$k \between n$	$k \mid n$	$k ¦ n$
$n \between k$	①	④	⑦
$n \mid k$	②	⑤	⑧
$n ¦ k$	③	⑥	⑨

Therefore, k ⑤ n means that $k \mid n$ and $n \mid k$, implying $k = n$, while k ⑧ n means that $k ¦ n$ and $n \mid k$, which is the inverse of k ⑥ n. We remark that the "neutral" states ①③⑦⑨ apply to k and n both composite, since primes must either divide or be coprime to other numbers, and these states do not include either relation.

For our purposes we are interested in the set $\mathcal{S}$ of prime divisors of s (which is equal to that of its squarefree kernel $\varkappa = \operatorname{rad}(n)$) and set $\mathcal{K}$ of prime divisors of k.

Theorem 4: Axiom 1 implies the sequence is completely $\varkappa$-regular.
Proof: $a(n) = m\,\varkappa$ for some integer multiple m, therefore $\varkappa \mid a(n)$, i.e., $\mathcal{S} \subseteq \mathcal{K}$. ■

Theorem 4.1: Squarefree $\varkappa$ prohibits $\varkappa ¦ k$.
Proof: Semidivisibility requires multiplicity of at least one prime power divisor exceeding 1. Since $\varkappa$ is squarefree by definition, we cannot have $\varkappa ¦ k$. Furthermore, $\varkappa ¦ k$ is ruled out by Axiom 1. ■

Theorems 4 and 4.1 show that we are limited to $\varkappa \mid k$ (④⑤⑥).

Theorem 4.2: $\varkappa$ ⑤ k implies novel $\varkappa$.
Proof: Axiom 1 demands $k = m\,\varkappa$ where m is the smallest that yields a product $m\,\varkappa$ unprecedented in the sequence. If $\varkappa = k$, then $m = 1$ and $a(n) = 1\varkappa$, and it is clear that $a(n)$ is the first appearance of a number $k : \varkappa \mid k$. ■

State ⑤ also signifies $\varkappa = k$.

Theorem 4.3: $\varkappa$ ④ k implies $m \nparallel \varkappa$.
Proof: Again we reflect on Axiom 1. For each occasion of $\varkappa$, m increments. These m are either $\varkappa$-regular or $\varkappa$-nonregular. If is even, then $m = 1$ and even m are $\varkappa$-regular and, through $k = m\,\varkappa$, $\mathcal{S} = \mathcal{K}$. For m that is $\varkappa$-nonregular, we introduce some prime factor $q \perp \varkappa$ to k, thus $\mathcal{S} \subset \mathcal{K}$. Theorem 4 shows that we are restricted to $\varkappa \parallel k$, particularly, $\varkappa \mid k$. Among the 3 variants associated with $\varkappa \mid k$, we have $\varkappa \mid\!\between k = \varkappa$ ④ k. ■

Corollary 4.4: $\varkappa$ ⑥ k implies $m \parallel \varkappa$, $m > 1$.

Generally, terms that derive from state ⑤ seem to arrive earlier than the other two states. Terms from state ④ appear along trajectories divisible by primorials or "infill" primes that are missing from the sticky factor P. Looking at a dataset records seen in the first 2^{20} terms, 900 records derive from ⑥, 225 from ⑤, and only 21 from ④. The latter state more rarely sets records as n increases.

Theorem 4.5: Let $a(n_\delta) = \mu p_\delta$, where p_δ is the δ-th sticky prime. We observe both $\varkappa_\delta$ ④ $a(n_\delta)$ and $\mu = \varkappa$ for all $\delta \leq 162$, $n \leq 2^{26}$, as demonstrated below.

δ	n	a(n−1)	rad(s(n))		a(n)	m(n)
1	5	4	3	(4)	6	2
2	42	138	26	(4)	78	3
3	114	510	114	(4)	570	5
4	471	7560	2130	(4)	14910	7
5	1994	1741740	14070	(4)	154770	11
6	2353	840840	168630	(4)	2192190	13
7	4591	203182980	3273270	(4)	55645590	17
8	6904	807116310	30120090	(4)	933722790	31
9	9612	42397344990	933722790	(4)	35481466020	38
10	10165	172896974250	8720021310	(4)	200560490130	23
11	15922	31225193549550	283551037770	(4)	36578083872330	129
12	20530	6108540006679110	33604255915230	(4)	1377774492524430	41

Proof: The equality $\mu = \varkappa$ implies the following:

$$a(n_\delta) = \mu p_\delta = \varkappa p_\delta.$$

and that though $p_\delta \mid a(n_\delta - 1)$, $s_\delta = (a(n_\delta - 2) + a(n_\delta - 1))$ is not divisible by p_δ. In order to have a new phase, we need a new sticky prime, which means we need 2 consecutive terms divisible by a new sticky prime p.

We know that prime p either divides s or it does not. Suppose $p \mid a(n-1)$ and $p \mid s$. Then by Theorem 1, subsequent terms are indeed likewise divisible by p. However, a prime p that divides an addend and a sum implies $p \mid a(n-2)$, therefore p cannot be newly sticky. By Corollary 4.4, if $a(n) = p\varkappa$, then we must have $p \mid \varkappa$, which implies $\varkappa$ ⑥ k. Since Theorem 4.2 rules out state ⑤, we are left with $a(n) = \varkappa p$, $\varkappa$ ④ $\varkappa p$.

We can have $\varkappa$ ④ $\varkappa p$ without p a new sticky prime. This frequently occurs on account of $p \perp a(n-1)$; indeed p either divides $a(n-1)$ or it does not. Since we have covered the following cases:

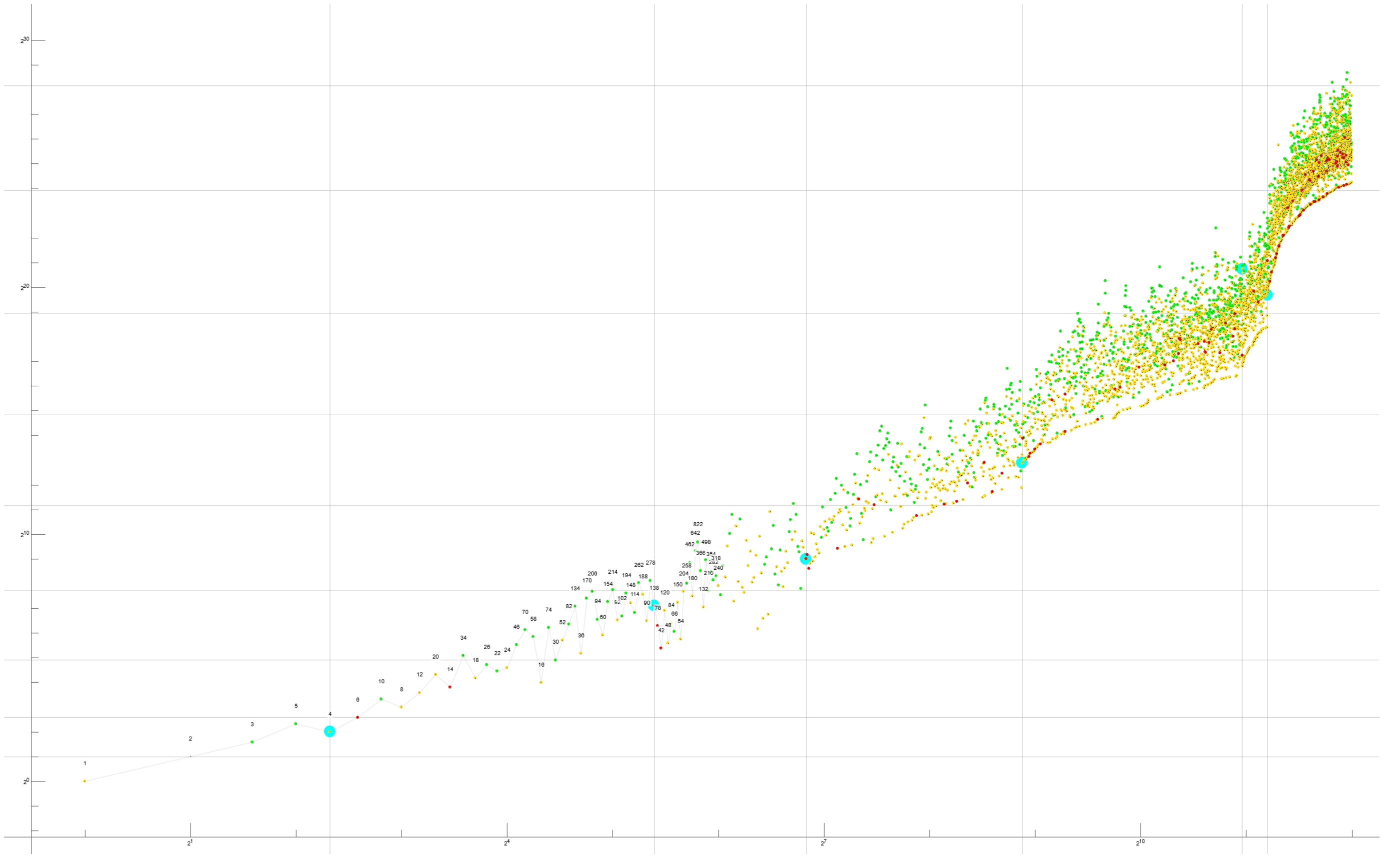

Figure 3: Log-log scatterplot of a(n), n = 1 … 4096 showing the constitutive states of s(n) and a(n). Red represents s(n) |◊ a(n) or state ④, gold represents s(n) |¦ a(n) or state ⑥, and green represents s(n) = a(n) or state ⑤. We highlight phase transitions in large cyan.

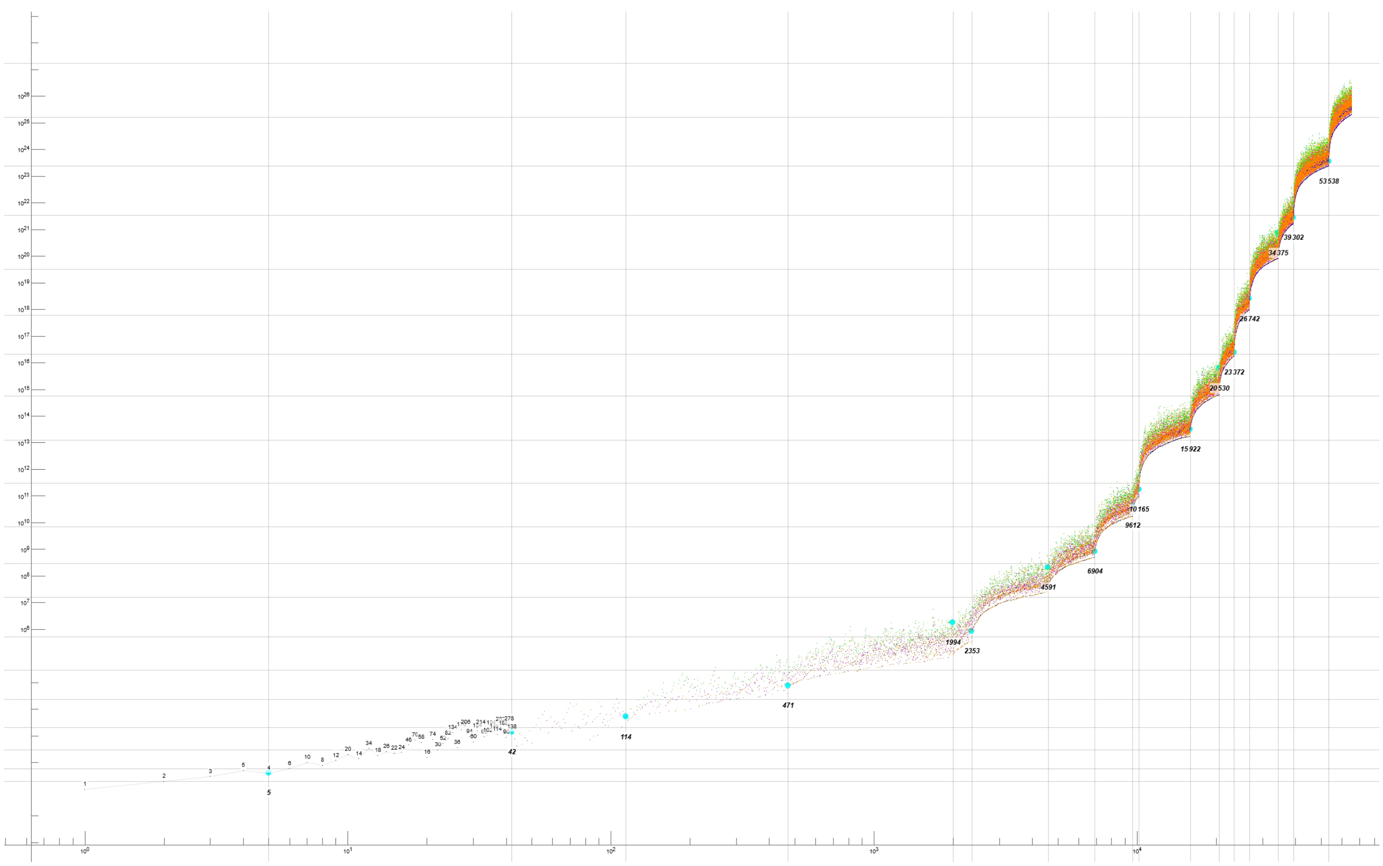

Figure 4: Log-log scatterplot of a(n), n = 1 … 2^{20}, using the same color scheme as Figure 3.

$$p \mid a(n{-}1) \text{ and } p \perp s,$$
$$p \perp a(n{-}1) \text{ and } p \mid s,$$
$$p \mid a(n{-}1) \text{ and } p \mid s,$$

we have covered all possible cases that apply to the emergence of a new sticky prime p. Therefore the proposition is true, $\varkappa_\delta$ ④ $a(n_\delta) = \varkappa_\delta p_\delta$. ■

Extended Study of Phases.

Define a function A287352(n) as the first differences of the indices of prime divisors of n with multiplicity. Hence, if we have a number like $n = 84 = 2 \times 2 \times 3 \times 7$, we have prime indices 1.1.2.4, which would be registered as 1,0,1,2. We abbreviate that as 1.0.1.2. Since we are dealing with squarefree kernels, or the presence or absence of a distinct prime divisor, we dispense with any zeros.

In this sequence it is evident that we are often dealing with a run of numbers divisible by a primorial $P(\ell)$. As regards A287352(n), we would have a series of ℓ 1s.

We might compactify this primorial in such notation by writing "1^ℓ". For instance, the number below can be compactified as shown after the number:

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4940321356659258570
  =  2 ×  3 ×  5 ×  7 × 11 × 13 × 17 × 19 × 23 ×
    31 × 37 × 41 × 43 × 47 × 233
  = 1^9.2.1^4.36
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Therefore we might compactify the unwieldy decimal expansion of RAD(k) via A287352(RAD(k)). An extended list of $a(n_\delta) = Q_\delta$ appears in Appendix Table A. Following is an abridged list of the first 16 sticky factors, Q:

δ	n	A287352(P(δ))	Decimal P(δ)
1	5	1	2
2	42	1^2	6
3	114	1^3	30
4	471	1^4	210
5	1994	1^5	2310
6	2353	1^6	30030
7	4591	1^7	510510
8	6904	1^7.4	15825810
9	9612	1^8.3	300690390
10	10165	1^9.2	6915878970
11	15922	1^9.2.3	297382795710
12	20530	1^9.2^2.1	12192694624110
13	23372	1^9.2^2.1^2	573056647333170
14	26742	1^9.2.1^4	21203095951327290
15	34375	1^15	614889782588491410
16	39302	1^16	32589158477190044730

From the extended Table A, it seems likely that $P(16)$ is the last primorial such that P is the greatest common factor of adjacent terms.

We can find the following primorials $P(\ell)$ in the sequence:

ℓ	n
1	2
2	6
3	22
4	60
5	178
6	508
7	1477
8	2687
9	4807
10	-
11	10166
12	11322
13	-
14	20994
15	23567
16	27026
17	35431
18	43358

Per Theorem 3 and Corollary 3.1, in the light of Table A, these are likely the only primorials in the sequence.

Open questions.

The rate of emergence of sticky primes abates as n increases. Do all primes become sticky? If so, what are the implications for the sequence when we have an infinite number of sticky primes? How are there any terms that follow the sequence? Is the sequence infinite?

The "gaps" in the sticky factor are filled in for $\delta = 15 \ldots 16$. Does this happen again, or are there too many gaps as n increases? This is the same question as "will there ever be another chance for primorials to enter the sequence?"

Conclusion.

We have shown that 2 consecutive terms divisible by a "sticky" prime p imply subsequent terms must also be divisible by p. Consequently, the number of primes, composite prime powers, and squarefree semiprimes in the sequence are finite. These sticky primes do not necessarily arise in order, therefore not all primorials appear in the sequence. There is a constitutive pattern in the sequence that accompanies the sticky primes; since $p \mid a(n{-}1)$ but not the sum $s = a(n{-}2){+}a(n{-}1)$, we have RAD$(s) \mid a(n) = p$ RAD(s).

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved January 2023.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

Code:

[C1] Generate terms of the sequence:

```
nn = 2^6; c[_] = False; q[_] = 1; t = 2;
f[n_] := Times @@ FactorInteger[n][[All, 1]];
Array[Set[{a[#], c[#]}, {#, True}] &, t];
Set[{i, j, k}, {a[t - 1], a[t], f[a[t - 1] + a[t]]}];
Monitor[
  Do[m = q[k];
    While[c[k m], m++];
    m *= k; While[c[k q[k]], q[k]++];
    Set[{a[n], c[m], i, j, k},
      {m, True, j, m, f[j + m]}],
  {n, 3, nn}], n];
 Array[a, nn]
```

Concerns sequences:

A002110: Product of the smallest n primes, $P(n)$.
A007947: Squarefree kernel of n; RAD(n).
A339557: $a(n)$.

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Table A: Extended data for phase transitions in A359557.

δ	n	A287352(P(δ))	Decimal P(δ)
1	5	1	2
2	42	1^2	6
3	114	1^3	30
4	471	1^4	210
5	1994	1^5	2310
6	2353	1^6	30030
7	4591	1^7	510510
8	6904	1^7.4	15825810
9	9612	1^8.3	300690390
10	10165	1^9.2	6915878970
11	15922	1^9.2.3	297382795710
12	20530	1^9.2^2.1	12192694624110
13	23372	1^9.2^2.1^2	573056647333170
14	26742	1^9.2.1^4	21203095951327290
15	34375	1^15	614889782588491410
16	39302	1^16	32589158477190044730
17	53538	1^16.11	...
18	69757	1^16.4.7	
19	89995	1^16.3.1.7	
20	96260	1^16.3.1.7.1	
21	108911	1^16.3.1.4.3.1	
22	129634	1^16.3.1.4.3.1.6	
23	144879	1^16.3.1.2^2.3.1.6	
24	177605	1^16.3.1.2^2.1.2.1.6	
25	199537	1^16.3.1.2.1^3.2.1.6	
26	232930	1^16.2.1^2.2.1^3.2.1.6	
27	246327	1^16.2.1^7.2.1.6	
28	275596	1^25.2.1.6	
29	303470	1^25.2.1.2.4	
30	331535	1^25.2.1.2^3	
31	383137	1^25.2.1.2^3.8	
32	414878	1^28.2^3.8	
33	452277	1^30.2^2.8	
34	509216	1^30.2^2.1.7	
35	572672	1^30.2^2.1.7.9	
36	621108	1^30.2^2.1.7.5.4	
37	672487	1^30.2^2.1.7.4.1.4	
38	703926	1^30.2^2.1.4.3.4.1.4	
39	790370	1^30.2^2.1.4.3.1.3.1.4	
40	889469	1^30.2.1^3.4.3.1.3.1.4	
41	943629	1^30.2.1^3.4.3.1^2.2.1.4	
42	1027898	1^35.4.3.1^2.2.1.4	
43	1142474	1^35.4.3.1^2.2.1.4.26	
44	1243271	1^35.4.3.1^2.2.1.4.11.15	
45	1344569	1^35.2^2.3.1^2.2.1.4.11.15	
46	1402640	1^35.2^2.3.1^5.4.11.15	
47	1489891	1^35.2^2.1.2.1^5.4.11.15	
48	1560938	1^37.2.1.2.1^5.4.11.15	
49	1636210	1^40.2.1^5.4.11.15	
50	1702618	1^40.2.1^6.3.11.15	
51	1791701	1^40.2.1^6.3.11^2.4	
52	1990426	1^48.3.11^2.4	
53	2054075	1^49.2.11^2.4	
54	2240090	1^49.2.7.4.11.4	
55	2347911	1^49.2.7.3.1.11.4	
56	2442747	1^49.2.6.1.3.1.11.4	
57	2760572	1^49.2.6.1.3.1.10.1.4	
58	2911467	1^49.2.3^2.1.3.1.10.1.4	
59	3218241	1^49.2.3^2.1.3.1.6.4.1.4	
60	3396481	1^49.2^2.1.3.1.3.1.6.4.1.4	
61	3739388	1^49.2.1^3.3.1.3.1.6.4.1.4	
62	3852882	1^54.3.1.3.1.6.4.1.4	
63	4005756	1^54.3.1.3.1.4.2.4.1.4	
64	4222102	1^54.2.1^2.3.1.4.2.4.1.4	
65	4555098	1^54.2.1^2.2.1^2.4.2.4.1.4	
66	4696704	1^58.2.1^2.4.2.4.1.4	
67	4866669	1^62.4.2.4.1.4	
68	5020017	1^62.4.2^3.1.4	
69	5210967	1^62.3.1.2^3.1.4	
70	5498631	1^62.3.1.2^3.1.4.35	
71	5875202	1^62.3.1.2^3.1.4.10.25	
72	6094356	1^62.3.1.2^3.1.4.10.13.12	
73	6299956	1^62.3.1.2^2.1^3.4.10.13.12	
74	6543080	1^62.3.1^3.2.1^3.4.10.13.12	
75	6952357	1^62.2.1^4.2.1^3.4.10.13.12	
76	7178231	1^62.2.1^4.2.1^3.4.10.3.10.12	
77	7420465	1^62.2.1^4.2.1^3.4.10.3.10.2.10	
78	7778834	1^62.2.1^9.4.10.3.10.2.10	
79	8150882	1^62.2.1^9.4.10.3.10.2.3.7	
80	8427085	1^62.2.1^9.4.10.3.2.8.2.3.7	

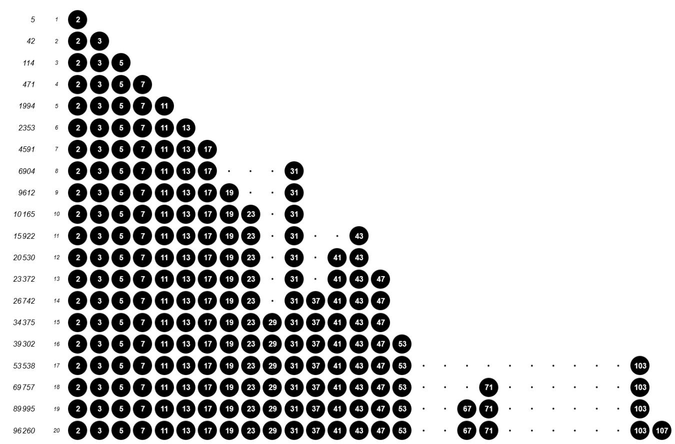

Figure 5: Illustration of the prime divisors of the δ-th sticky factor, δ = 1 … 20.

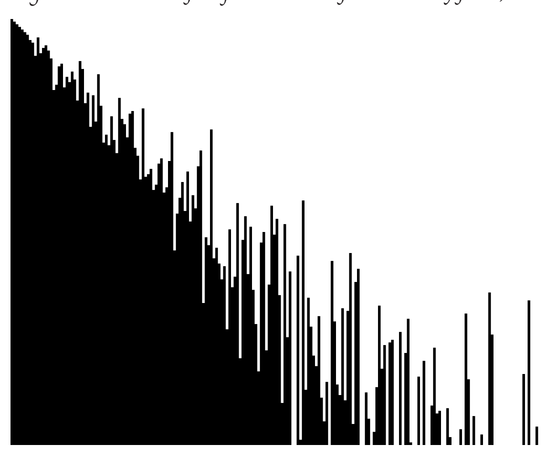

Figure 5: Bitmap of the prime divisors of the δ-th sticky factor, δ = 1 … 162.

81	8976340	1^62.2.1^9.4.7.3^2.2.8.2.3.7
82	9211549	1^62.2.1^9.4.7.3^2.2.5.3.2.3.7
83	9769844	1^62.2.1^9.4.7.3^2.2.5.3.1^2.3.7
84	10033286	1^62.2.1^9.2^2.7.3^2.2.5.3.1^2.3.7
85	10316990	1^62.2.1^9.2^2.7.3.2.1.2.5.3.1^2.3.7
86	10516883	1^62.2.1^9.2^2.7.3.2.1.2.4.1.3.1^2.3.7
87	10687296	1^62.2.1^9.2.1^2.7.3.2.1.2.4.1.3.1^2.3.7
88	11042812	1^62.2.1^9.2.1^2.2.5.3.2.1.2.4.1.3.1^2.3.7
89	11387530	1^73.2.1^2.2.5.3.2.1.2.4.1.3.1^2.3.7
90	12117940	1^73.2.1^2.2.5.3.2.1.2.4.1.3.1^2.3.7.18
91	12547330	1^73.2.1^2.2.5.3.2.1.2.4.1.3.1^2.3.5.2.18
92	12742038	1^73.2.1^4.5.3.2.1.2.4.1.3.1^2.3.5.2.18
93	13058335	1^73.2.1^4.5.3.2.1.2.4.1.3.1^2.3.5.2.11.7
94	13365371	1^73.2.1^5.4.3.2.1.2.4.1.3.1^2.3.5.2.11.7
95	13636415	1^73.2.1^5.2^2.3.2.1.2.4.1.3.1^2.3.5.2.11.7
96	14101127	1^73.2.1^5.2^2.3.2.1.2.4.1.3.1^2.3.5.2.11.7.3
97	14724105	1^73.2.1^5.2^2.3.2.1.2.4.1.3.1^2.3.2.3.2.11.7.3
98	14891178	1^73.2.1^5.2^2.3.2.1^3.4.1.3.1^2.3.2.3.2.11.7.3
99	15184509	1^73.2.1^5.2^3.1.2.1^3.4.1.3.1^2.3.2.3.2.11.7.3
100	15514061	1^73.2.1^7.2^2.1.2.1^3.4.1.3.1^2.3.2.3.2.11.7.3
101	16282338	1^73.2.1^7.2^2.1.2.1^3.4.1.3.1^2.3.2.3.2.11.7.2.1
102	16881956	1^73.2.1^7.2^2.1.2.1^3.4.1.2.1^3.3.2.3.2.11.7.2.1
103	17228214	1^73.2.1^7.2.1^3.2.1^3.4.1.2.1^3.3.2.3.2.11.7.2.1
104	17592980	1^73.2.1^7.2.1^3.2.1^4.3.1.2.1^3.3.2.3.2.11.7.2.1
105	18186030	1^73.2.1^7.2.1^3.2.1^4.3.1.2.1^3.3.2.3.2.11.7.2.1.50
106	18845268	1^73.2.1^7.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2.11.7.2.1.50
107	19417180	1^73.2.1^7.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.9.7.2.1.50
108	20295197	1^73.2.1^7.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.9.7.2.1.50.15
109	20617011	1^82.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.9.7.2.1.50.15
110	21291006	1^82.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.9.7.2.1.8.42.15
111	22104149	1^82.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.9.4.3.2.1.8.42.15
112	22818193	1^82.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.9.4.2.1.2.1.8.42.15
113	23367755	1^82.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.9.4.2.1.2.1.8.33.9.15
114	24085122	1^82.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.4.5.4.2.1.2.1.8.33.9.15
115	24498483	1^82.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.4.5.4.2.1.2.1.8.11.22.9.15
116	24935798	1^82.2.1^3.2.1^4.3.1.2.1^4.2^2.3.2^2.4.5.1.3.2.1.2.1.8.11.22.9.15
117	25608809	1^82.2.1^3.2.1^5.2.1.2.1^4.2^2.3.2^2.4.5.1.3.2.1.2.1.8.11.22.9.15
118	26107809	1^82.2.1^3.2.1^5.2.1.2.1^4.2^2.3.2^2.1.3.5.1.3.2.1.2.1.8.11.22.9.15
119	26673228	1^87.2.1^5.2.1.2.1^4.2^2.3.2^2.1.3.5.1.3.2.1.2.1.8.11.22.9.15
120	27541487	1^87.2.1^5.2.1.2.1^4.2^2.3.2^2.1.3.5.1.3.2.1.2.1.8^2.3.22.9.15
121	28400122	1^87.2.1^5.2.1.2.1^4.2^2.3.2^2.1.3.5.1.3.2.1.2.1.8^2.3.22.9.1.14
122	28757505	1^87.2.1^5.2.1.2.1^4.2.1^2.3.2^2.1.3.5.1.3.2.1.2.1.8^2.3.22.9.1.14
123	29830387	1^87.2.1^5.2.1.2.1^4.2.1^2.3.2^2.1.3.5.1.3.2.1.2.1.8.5.3^2.22.9.1.14
124	30382362	1^87.2.1^5.2.1.2.1^4.2.1^2.3.2^2.1.3.5.1.3.2.1.2.1.8.4.1.3^2.22.9.1.14
125	30824197	1^87.2.1^5.2.1.2.1^4.2.1^2.3.2^2.1.3.5.1.3.2.1.2.1.8.2^2.1.3^2.22.9.1.14
126	31569544	1^87.2.1^5.2.1.2.1^4.2.1^2.3.2^2.1.3.5.1.3.2.1.2.1.8.2^2.1.3^2.10.12.9.1.14
127	32210273	1^87.2.1^5.2.1^7.2.1^2.3.2^2.1.3.5.1.3.2.1.2.1.8.2^2.1.3^2.10.12.9.1.14
128	32946711	1^87.2.1^5.2.1^7.2.1^2.3.2^2.1.3.5.1.3.2.1.2.1.8.2^2.1.3.2.1.10.12.9.1.14
129	34470176	1^87.2.1^5.2.1^7.2.1^2.3.2^2.1^2.2.5.1.3.2.1.2.1.8.2^2.1.3.2.1.10.12.9.1.14
130	35049555	1^94.2.1^7.2.1^2.3.2^2.1^2.2.5.1.3.2.1.2.1.8.2^2.1.3.2.1.10.12.9.1.14
131	36071317	1^94.2.1^7.2.1^2.3.2^2.1^2.2.5.1.3.2.1.2.1.8.2^2.1.3.2.1.6.4.12.9.1.14
132	37381817	1^94.2.1^7.2.1^2.3.2^2.1^2.2.5.1.3.2.1.2.1.8.2^2.1.3.2.1.6.4.12.9.1.14.6
133	37800533	1^94.2.1^7.2.1^2.3.2^2.1^4.5.1.3.2.1.2.1.8.2^2.1.3.2.1.6.4.12.9.1.14.6
134	38935172	1^94.2.1^7.2.1^2.3.2^2.1^4.5.1.3.2.1.2.1.8.1^2.2.1.3.2.1.6.4.12.9.1.14.6
135	39687117	1^103.2.1^2.3.2^2.1^4.5.1.3.2.1.2.1.8.1^2.2.1.3.2.1.6.4.12.9.1.14.6
136	40845076	1^103.2.1^2.3.2^2.1^4.5.1.3.2.1.2.1.8.1^2.2.1.3.2.1.6.4.12.9.1.12.2.6
137	41594779	1^103.2.1^2.3.2^2.1^4.5.1.3.2.1.2.1.8.1^2.2.1.3.2.1.4.2.4.12.9.1.12.2.6
138	42586367	1^103.2.1^2.3.2^2.1^4.5.1.3.2.1.2.1.8.1^2.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
139	43492540	1^103.2.1^2.3.2^2.1^4.3.2.1.3.2.1.2.1.8.1^2.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
140	44069546	1^103.2.1^2.3.2^2.1^4.3.2.1^2.2^2.1.2.1.8.1^2.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
141	44561936	1^103.2.1^2.3.2^2.1^4.3.2.1^2.2^2.1.2.1.7.1^3.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
142	45149055	1^103.2.1^2.3.2.1^6.3.2.1^2.2^2.1.2.1.7.1^3.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
143	45955721	1^103.2.1^2.3.2.1^6.3.2.1^2.2^2.1.2.1.3.4.1^3.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
144	46739504	1^103.2.1^2.3.2.1^6.3.2.1^4.2.1.2.1.3.4.1^3.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
145	47403070	1^103.2.1^2.3.2.1^7.2^2.1^4.2.1.2.1.3.4.1^3.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
146	48464579	1^103.2.1^2.3.2.1^7.2^2.1^7.2.1.3.4.1^3.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
147	48790892	1^107.3.2.1^7.2^2.1^7.2.1.3.4.1^3.2.1.3.2.1.4.2.4.12.1.8.1.12.2.6
148	49396660	1^107.3.2.1^7.2^2.1^7.2.1.3.4.1^3.2.1.3.2.1.4.2.3.1.12.1.8.1.12.2.6
149	50354546	1^107.3.2.1^7.2^2.1^7.2.1.3.4.1^3.2.1.3.2.1.4.2.3.1.5.7.1.8.1.12.2.6
150	51034611	1^107.3.2.1^7.2^2.1^7.2.1.3.4.1^3.2.1.3.2.1.4.2.3.1.2.3.7.1.8.1.12.2.6
151	51889451	1^107.3.2.1^7.2^2.1^7.2.1.3.4.1^3.2.1.3.2.1.4.2.3.1^3.3.7.1.8.1.12.2.6
152	52783440	1^107.3.2.1^7.2^2.1^7.2.1.3.4.1^3.2.1.3.2.1.4.2.3.1^3.3.7.1.2.6.1.12.2.6
153	54854079	1^107.3.2.1^7.2^2.1^7.2.1.3.1.3.1^3.2.1.3.2.1.4.2.3.1^3.3.7.1.2.6.1.12.2.6
154	56475218	1^107.3.2.1^9.2.1^7.2.1.3.1.3.1^3.2.1.3.2.1.4.2.3.1^3.3.7.1.2.6.1.12.2.6
155	57109585	1^107.3.2.1^9.2.1^10.3.1.3.1^3.2.1.3.2.1.4.2.3.1^3.3.7.1.2.6.1.12.2.6
156	58525354	1^107.3.2.1^9.2.1^10.3.1.3.1^3.2.1.3.2.1.4.2.3.1^3.3.7.1.2.6.1.12.2.3^2
157	59556588	1^107.3.2.1^9.2.1^10.3.1.3.1^3.2.1.3.2.1.4.2.3.1^3.3.5.2.1.2.6.1.12.2.3^2
158	60504726	1^107.3.2.1^9.2.1^10.3.1.2.1^4.2.1.3.2.1.4.2.3.1^3.3.5.2.1.2.6.1.12.2.3^2
159	61762523	1^107.3.2.1^9.2.1^10.3.1.2.1^4.2.1.3.2.1.4.2.3.1^3.3.5.2.1.2.3^2.1.12.2.3^2
160	63153476	1^107.3.2.1^9.2.1^10.3.1.2.1^4.2.1.3.2.1.4.2.3.1^3.3.1.4.2.1.2.3^2.1.12.2.3^2
161	63777117	1^107.3.1^11.2.1^10.3.1.2.1^4.2.1.3.2.1.4.2.3.1^3.3.1.4.2.1.2.3^2.1.12.2.3^2
162	65306860	1^107.3.1^11.2.1^10.3.1.2.1^4.2.1.3.2.1^2.3.2.3.1^3.3.1.4.2.1.2.3^2.1.12.2.3^2