Notes on S20230119 and S20230120

Sequences of David J. Sycamore.

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Abstract.

This is a brief study of 2 lexically earliest sequences having to do with inhabited or uninhabited symmetric differences in the sets of prime factors of the immediately preceding 2 terms.

Introduction.

David Sycamore presented a sequence he described as follows:

Let $a(1) = 1$, $a(2) = 2$. Let $i = a(n-2)$ and $j = a(n-1)$.

If $(i, j) = 1$, $a(n)$ = least novel k divisible by $(i \times j)$,

else if $(i, j) > 1$ we follow one of the two following conditions:

1. If one of i, j divides the other, $a(n)$ = least novel multiple of the smallest common prime divisor of i and j.
2. If one does not divide the other, $a(n)$ = least k coprime to both i and j.

We may logically define sequence S20230119 as follows:

$a(1) = 1$, $a(2) = 2$. Let $i = a(n-2)$ and $j = a(n-1)$.

Define function $c(x)$ to be TRUE iff $a(j) = x$ for $j < n$, else FALSE.

For $n > 2$, we define $a(n)$ according to the following axioms:

Axiom 1: $(i, j) = 1$ implies $\boxed{<}k : (i \times j) \mid k \wedge \neg c(k)$.

Axiom 2.1: $(i \mid j \vee j \mid i)$ implies $\boxed{<}k : \mathrm{LPF}((i, j)) \mid k \wedge \neg c(k)$.

Axiom 2.2: $1 < (i, j) < \mathrm{MIN}(i, j)$ implies $\boxed{<}k : (i \times j, k) = 1 \wedge \neg c(k)$.

Algorithmically, we process the conditions in order as presented. Hence we design a function $f(x)$ that applies the above axioms in order, and remap this function on the sequence that starts $\{1, 2\}$, generating Sycamore's sequence.

The first terms of the sequence appear below:

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1, 2, 4, 6, 5, 30, 10, 8, 3, 24, 9, 7, 63, 14, 11, 154,
22, 12, 13, 156, 26, 16, 15, 240, 18, 17, 306, 34, 20,
19, 380, 38, 28, 23, 644, 46, 32, 21, 672, 27, 25, 675,
35, 29, 1015, 58, 31, 1798, 62, 36, 37, 1332, 74, 40,
33, 1320, 39, 41, 1599, 82, 43, 3526, 86, 42, 47, 1974,
94, 44, 45, 1980, 48, 49, 2352, 56, 50, 51, 2550, ...
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This is a prime divisor restricted lexically earliest sequence (PDRLES) or "pearl sequence".

$$a(n) \neq a(i) : i < n, \qquad \text{[AL]}$$

$$a(n) \text{ is the smallest possible solution.} \qquad \text{[AE]}$$

Fundamental Theorem of Lexically Eariest Sequences.

Let $\mathcal{R}$ be the range of $f(n)$ with least element η and let indices $i \in \mathcal{R}$ and $n \in \mathcal{R}$ such that $\eta \le i < n$.

Let $\mathcal{D}$ be the domain of $f(n)$ and let μ be its least element. Axioms [AL] and [AL] imply the partitioning of $\mathcal{D}$ by 2 numbers. The first is u, the "smallest missing number" such that $u \notin a(1 \dots n-1)$. The second is $r = \mathrm{MAX}(a(1 \dots n-1))$.

The saturated range $[\mu \dots u-1] \ni k$ such that $a(i) = k$, which implies $k < u \le a(n)$. The open range $[r+1 \dots \infty] \ni k$ such that $a(i) \neq k$, which implies $a(n) = k$ iff k also satisfies the other axioms. Finally, the semi-saturated range $[u \dots r]$ wherein we must determine the availability of k, that is, whether k hasn't yet appeared in the sequence in addition to whether k satisfies the other axioms.

Hence we have a smallest missing number u, and as a consequence of [AE], equality (state ⑤) is prohibited.

Let's examine the constitutive properties of input and output of $f(x)$ described above:

$$i \text{ ⓪ } j \rightarrow \text{[A1]} \rightarrow \boxed{<}k : k = m(i \times j) \wedge \neg c(k)$$

$$i \text{ ②④⑥⑧ } j \rightarrow \text{[A2.1]} \rightarrow \boxed{<}k : k = mp \wedge \neg c(k) : p = \mathrm{LPF}((i, j))$$

$$i \text{ ①③⑦⑨ } j \rightarrow \text{[A2.2]} \rightarrow \boxed{<}k : (i \times j) \perp k \wedge \neg c(k)$$

Theorem 1: Axiom 1 implies $k = m \times (i \times j) : m \ge 1$.

Proof: $(i \times j) \mid k$ implies that k is an integer multiple of $(i \times j)$. Since the smallest missing number $u = 3$ for $n = 3$, given the lexically earliest axioms in a sequence with domain in the naturals, we have some least novel $k = m \times (i \times j) : m \ge 1$. ■

Therefore as to algorithm, we may devise a counter $m(x)$ and set it to 1 for all x a priori. Then when we have Axiom 1, for x we check $c(m(x)\,x)$ and if false, set $k = m(x)\,x$ and increment $m(x)$++. If somehow true, we increment $m(x)$++ until we have $\neg c(m(x)\,x)$. This guarantees a greedy approach that satisfies Axiom 1.

Corollary 1.1: Axiom 1 admits the following constitutive states between $(i \times j)$ and k: ④⑤⑥. We have $(i \times j)$ ⑤ k for $k = (i \times j)$, i.e., $m = 1$. Multiplier m such that $\mathrm{RAD}(m) \mid \mathrm{RAD}(i \times j)$ implies state ⑥, otherwise we have state ④, since m in this case introduces prime factors q coprime to $(i \times j)$ to k.

Observation. Given 2^{20} terms of the sequence, we observe neither $(i \times j)$ ⑥ k nor $(i \times j)$ ④ k. Axiom 1 is predominantly $(i \times j)$ ⑤ k, meaning that $(i \times j)$ is always novel.

Corollary 1.2. Axiom 1 implies composite k.

Proof. The lexical axiom requires distinct i, j, and k. In the singular case of $i = 1$ and $j = 2$, $(i, j) = 1$ hence , we have $k = 4$ since we cannot repeat $k = i \times j = 1 \times 2 = 2$ since $j = 2$. In all other cases, both i and j exceed 1 and are thus both products of at least 1 prime each. Hence k is always composite following Axiom 1. ■

Observationally, we see as well that Axiom 1 seems to imply k that is not a prime power, outside of $a(3) = 4$.

We cannot rule out primes resulting from Axiom 2.1. Indeed, it seems that, though Axiom 2.1 does not prohibit prime output, the same does not yield primes for $n \le 2^{20}$. In that domain, primes result from Axiom 2.2 except for given $a(2) = 2$.

Conjecture 1.3: Axiom 1 implies $k = (i \times j)$ for $n \ge 4$.

Theorem 3: Axiom 2.1 implies $k = m \times \mathrm{LPF}((i, j)) : m \ge 1$. As proof, we operate under similar logic shown in Theorem 1. ■

We may write $(i \mid j \vee j \mid i)$ instead as $(i, j) = \mathrm{MIN}(i, j)$, which may prove a more effective test in algorithm.

Selection Axiom Chain and Repetition.

Lemma 4.1. Invocation of Axiom 2.2 implies invocation of Axiom 1 immediately thereafter.

Proof. Axiom 2.2 yields $a(n) : (a(n-2) \times a(n-1), a(n)) = 1$, therefore, $(a(n-1), a(n)) = 1$, and we have Axiom 1. ■

Lemma 4.2. Invocation of Axiom 1 implies invocation of Axiom 2.1 immediately thereafter.

Proof. Axiom 1 yields $a(n) = m \times a(n-2) \times a(n-1)$, i.e., $a(n-1) \mid a(n)$, thus we have Axiom 2.1. ■

Lemma 4.3. Invocation of Axiom 2.1 prohibits Axiom 1 immediately thereafter.

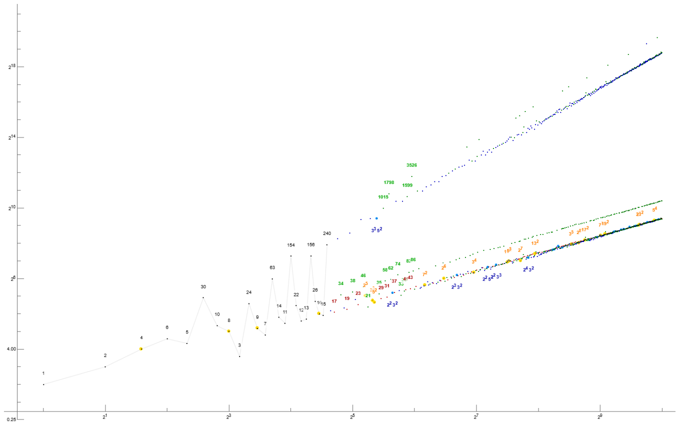

Figure 1: Log-log scatterplot of a(n), n = 1 … 1024 showing primes in red, multus numbers (composite prime powers, A246547) in gold, varius numbers (squarefree composites, A120944) in green, tantus (neither squarefree nor prime power, A126706) in dark blue, highlighting plenus numbers (products of multus numbers, A286708) in large light blue. Powerful numbers A1694 = A246547 ∪ A286708.

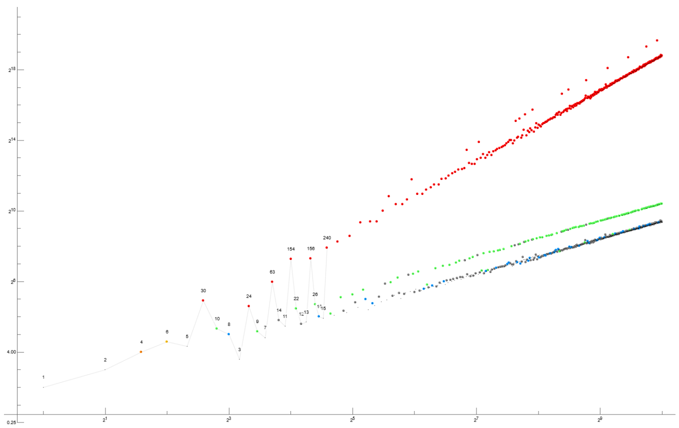

Figure 2: Log-log scatterplot of a(n), n = 1 … 1024 where red indicates terms that derive from Axiom 1, green from Axiom 2.1, and blue from Axiom 2.2.

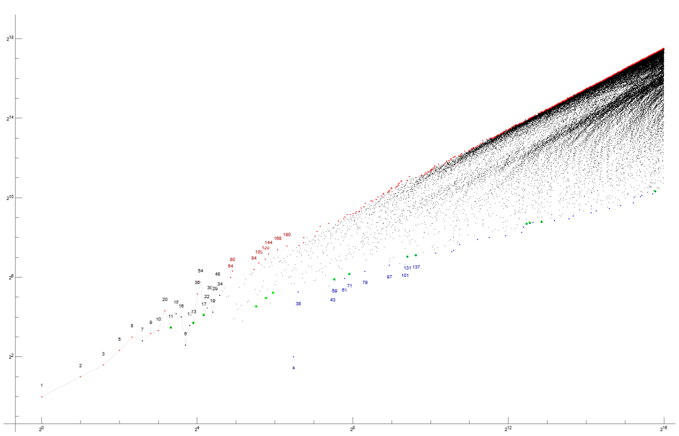

Figure 3: Log-log scatterplot of a(n), n = 1 … 65536 showing records in red, local minima in blue, and terms resulting from Axiom 0 in green. Note the position of a(n) = 4, and the infrequency of Axiom 0.

PROOF. Axiom 2.1 yields $a(n) = m \times p$ such that p divides both $a(n-2)$ and $a(n-1)$, therefore $a(n-1)$ is not coprime to $a(n)$, thus we have either Axiom 2.1 or Axiom 2.2. ■

COROLLARY 4.4. Successive implementations of Axiom 2.1 occur when $a(n-1) \mid a(n)$, $a(n-1) < a(n)$ (states ④ or ⑥; equality is ruled out by the lexical axiom).

COROLLARY 4.5. $a(n-1)$ neutral to $a(n)$ (i.e., states ①③⑦⑨) implies Axiom 2.1 followed by Axiom 2.2.

COROLLARY 4.6. Axiom 2.1 is the only axiom that may repeat; it follows either itself or Axiom 1.

Observation. Axiom 2.1 occurs at most twice in a row for $n \le 2^{20}$. Such occurs often after a prime (e.g.: after $a(5) = 5$) or prime power (first instance: $a(72) = 49$), even a varius number (e.g., after $a(324) = 221$). The usual mode for the second occasion is that the factor m delivers the least common prime factor for the second iteration, usually 2. However, $a(127344) = 84177$ followed by 83763, both divisible by 3^2. There are 54236 occasions of duplex Axiom 2.1 for $n \le 2^{20}$ and all the rest have both terms even. In the same dataset, we see (i, j) in $\{2, 4, 8, 9\}$.

THEOREM 4. Axiom 2.2 implies Axiom 1 which in turn implies Axiom 2.1 in a chain. PROOF: Lemmas 4.1, 4.2, and 4.3.

$$\text{Axiom 2.2} \rightarrow \text{Axiom 1} \rightarrow \text{Axiom 2.1}$$

In detail, we have the following:

i ①③⑦⑨ j → [A2.2] → ⧄$k : (i \times j)$ ⓪ $k \wedge \neg c(k)$,
i ⓪ j → [A1] → ⧄$k : (i \times j)$ ②④⑥⑧ $k \wedge \neg c(k)$
i ②④⑥⑧ j → [A2.1] → ⧄$k : k = mp \wedge \neg c(k) : p = \mathrm{LPF}((i, j))$

Thereafter, if i and j are completely neutral, then [A2.2], else [A2.1].

Examining a dataset of 2^{20} terms, we see duplex Axiom 2.1 with the constitutive pattern ②① or ②③ between $(i \times j)$ and k. Singleton occasions of Axiom 2.1 furnish states ①, ②, or ③, with the first-mentioned the commonest state. Therefore we cannot detect the occasion of duplex or repeated Axiom 2.1 through examination of the constitutive relation between $(i \times j)$ and k.

PROPOSITION 5. Axiom 1 sets records for $n > 4$. This would prove Conjecture 1.3.

We employ an argument thus: Axiom 1 requires a multiple of the composite product $(i \times j)$. The greedy axiom requires the smallest k that satisfies all definitional axioms. Axiom 2.1 requires the least product of a prime, while Axiom 2.2 requires the smallest k that is coprime to $(i \times j)$. Because we have 2 axioms that together often generate composite multiples, yet the multiples of Axiom 2.1 are those smallest multiples of a smallest common prime factor shared by i and j, while Axiom 1 requires a multiple of the product $(i \times j)$, it seems clear that Axiom 1 yields a number larger than Axiom 2.1 given input of similar magnitude. Furthermore, Axiom 2.2 does not require a multiple and is often prime. For this reason we presume the proposition true, but the argument is not rigorous.

Axiom 1 requires i ⓪ j, while Axiom 2.1 requires the divisor states ②④⑥⑧ except ⑤ which is prohibited by the lexical axiom. Axiom 2.2 requires the neutral states ①③⑦⑨ between i and j.

Outside of the first 4 terms, we observe the constitutive states ⓪①②③⑤ between $(i \times j)$ and k. State ⓪ strictly pertains to Axiom 2.2 and provided Proposition 5 is true, ⑤ pertains strictly to Axiom 1.

Exceptions: for $n = 3$ we have $(i \times j)$ ⑥ k through Axiom 1, and for $n = 4$ we have $(i \times j)$ ⑦ k through Axiom 2.1, but these states are not seen for the remainder of the dataset of 2^{20} terms. The former may only result from Axiom 1 when $i = 1$, hence it shall never recur through that axiom.

If Proposition 5 proves false, then we might see states ④ or ⑥ from Axiom 1 in addition to ⑤. The former would arise if m is coprime to $(i \times j)$. The latter would arise if $p \mid (i \times j)$ and $p \mid m$. So long as Axiom 1 yields $k = (i \times j)$ we have the equality state ⑤.

The Completely Regular Scaling Issue.

The usual scaling issues seen in LES seem to make the completely regular states ⑥⑧⑨ impossible as n increases. Let us define a squarefree kernel as follows:

$$\varkappa = \mathrm{RAD}(n) = \prod_{p \mid n} p = \mathrm{A7947}(n). \qquad [1.1]$$

Recall the definition of completely regular states outside symmetric divisibility (equality) which is prohibited:

$k \mid\mid n$	$k \mid ¦\, n$ or $k\, ¦ \mid n$	$k\, ¦¦\, n$
Symmetric Divisibility	Mixed Regularity	Symmetric Semidivisibility
⑤	⑥⑧	⑨

What these states have in common is that $\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$, which implies that, outside of state ⑤ and for k and n that both exceed 1, k and n are distinct elements of the infinite set (or list) $\boldsymbol{R}_\varkappa$ of $\varkappa$-regular numbers. Given the prime decomposition of $\varkappa$, we have the following set-building formula for $\boldsymbol{R}_\varkappa$:

$$\boldsymbol{R}_\varkappa = \bigotimes_{p \mid \varkappa} \{p^\varepsilon : \varepsilon \geq 0\} \qquad [1.2]$$

Suppose we have the term $a(n) = k = \boldsymbol{R}_\varkappa(\nu)$ resulting from a given selection axiom A in the middle of an interval $n \pm \eta$ of terms resulting from A. Let $r = \boldsymbol{R}_\varkappa(\nu - 1)$ and $\mathrm{R} = \boldsymbol{R}_\varkappa(\nu + 1)$ such that $r < k < \mathrm{R}$.

The scaling issue has to do with the likelihood of finding r or R not already in the sequence given the circumstance of selection axiom A within the interval $n + \eta$, as n increases. This is dependent on the density of $\boldsymbol{R}_\varkappa$ in the vicinity of k, and the dilation of η as n increases. Usually, it seems that as n increases, $\boldsymbol{R}_\varkappa$ becomes too sparse to furnish solutions for selection axiom A, even for primorials $\varkappa$. Furthermore, symmetric semidivisibility implies k and n both tantus numbers (i.e., numbers neither squarefree nor prime powers, A126706) neither of which divide the other, hence state ⑨ proves rare.

Consequences of the scaling issue for completely regular relations include rarity outside a few early terms if the states appear at all. Normally the terms in completely regular relation have squarefree kernel 6, 10, or 30, for example; small even kernels with small prime factors.

Quasirays in Scatterplot.

There are 4 principal features in scatterplot neatly tied to axioms. From lowest apparent slope to the highest, we have the following:

Quasiray α resulting from Axiom 2.2. This contains numbers of all omega-multiplicity classes. Odd primes appear in quasiray α, along with some multus numbers (i.e., composite prime powers).

Quasiray β results from Axiom 2.1 preceded by a prime p ahead of Axiom 1, hence contains $2p$. Varius numbers (i.e., squarefree composites) dominate the quasiray, though some multus numbers appear. The ray appears to parallel quasiray α in log-log scatterplot.

A second quasiray, γ, superposes quasiray α and is comprised of terms that result from duplex Axiom 2.1.

Finally, quasirays δ_1 and δ_2 comprise an echo of quasiray α wrought by Axiom 1. Quasiray δ_2 arises when singleton Axiom 2.1 doubles a prime. This echo is limited to numbers that are not prime powers.

We summarize the provenance of quasirays in scatterplot below:

$$\begin{array}{lll}
i \text{ ①③⑦⑨ } j \to [\mathrm{A2.2}] \to \boxed{\leq} k : (i \times j) \text{ ⓪ } k \land \neg c(k), & & \alpha \\
i \text{ ⓪ } j \to [\mathrm{A1}] \to \boxed{\leq} k : (i \times j) \text{ ②④⑥⑧ } k \land \neg c(k) & & \delta \\
i \text{ ②④⑥⑧ } j \to [\mathrm{A2.1}] \to \boxed{\leq} k : k = mp \land \neg c(k) : p = \mathrm{LPF}((i, j)) & & \beta \\
i \text{ ②④⑥⑧ } j \to [\mathrm{A2.1}] \to \boxed{\leq} k : k = mp \land \neg c(k) : p = \mathrm{LPF}((i, j)) & & \gamma
\end{array}$$

Permutation of Natural Numbers.

We surmise the sequence is a permutation of $\mathbb{N}$ for the following reasons. The axioms cover coprimality and complementary species of the cototient, including neutral and divisor states as input. The neutral input axiom [A2.2] yields coprime pairs that instigate [A1] which in turn results in a divisor state or a neutral state between i and j. Axiom [A1] sets records and the other two axioms fill in the gaps, saturating the sequence. Smallest missing u is often prime, if not multus; the coprimality axiom [A2.2] admits smallest missing u often and is the agent of insurance for a permutation.

Thus, we surmise the sequence is a permutation of $\mathbb{N}$.

Study 2.

Let's examine a related sequence $b(n)$.

Let $b(n) = n$ for $n \leq 3$ and let $i = b(n-2)$ and $j = b(n-1)$,

If $\mathrm{RAD}(i) \neq \mathrm{RAD}(j)$ then $b(n)$ is the least novel k divisible by the sum of all primes which divide i but not j, and all the primes which divide j but not i. If $\mathrm{RAD}(i) = \mathrm{RAD}(j)$ then there are no such primes and we define $b(n)$ as the least k prime to both i and j.

We can define the sequence S20230120 logically as follows:

> $b(n) = n$ for $n \leq 3$; let $i = b(n-2)$ and $j = b(n-1)$,
> and define sets $S = \{ p : p \mid i \}$ and $T = \{ p : p \mid j \}$.
> Define function $c(x)$ to be TRUE iff $b(j) = x$ for $j < n$, else FALSE.
> For $n > 3$, we define $b(n)$ according to the following axioms:
> Axiom 0: $\mathrm{RAD}(i) = \mathrm{RAD}(j)$ implies $\boxed{\leq} k : (i \times j, k) = 1 \land \neg c(k)$.
> Axiom 1: $\mathrm{RAD}(i) \neq \mathrm{RAD}(j)$ implies $\boxed{\leq} k$:
> for $s = \sum q : q \in \{ S \ominus T \}$, $k = ms \land \neg c(k)$, where
> $S \ominus T = S \triangle T = (S \setminus T) \cup (T \setminus S)$.

The first terms of the sequence appear below:

```
1, 2, 3, 5, 8, 7, 9, 10, 20, 11, 18, 16, 6, 12, 13, 36,
54, 17, 22, 30, 19, 29, 48, 34, 40, 44, 32, 33, 64, 80,
15, 25, 21, 45, 24, 14, 50, 60, 27, 28, 72, 70, 75, 84,
42, 23, 35, 105, 39, 100, 46, 56, 90, 120, 31, 41, 144,
92, 26, 108, 96, 37, 126, 49, 55, 69, 168, 128, ...
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We note $\mathrm{RAD}(i) = \mathrm{RAD}(j) = \varkappa$ signifies symmetric regularity. There are three degrees of symmetric regularity shown in the previous section; these are symmetric divisibility (⑤), mixed regularity (⑥⑧), and symmetric semidivisibility (⑨). From earlier work, we see that the latter two cases are restrictive whereas state ⑤ admits any natural number. See [2], Mixed Regularity, pp. 5-6. As stated in the previous section, state is restricted to i and j both tantus and absent divisibility.

Mixed regularity requires the semidivisor to be composite, further, not a prime power. Symmetric semidivisibility is open only to tantus numbers (i.e., both numbers neither squarefree nor a prime power).

Therefore Axiom 0 occurs for numbers in the same strongly regular sequence $\varkappa\boldsymbol{R}_\varkappa$. For instance, suppose we have both $i \geq 6$ and $j \geq 6$ in the sequence of 3-smooth numbers $6\boldsymbol{R}_6 = \{6 \times \mathrm{A3586}\}$. Then we could see $i = j$ (which is prohibited by the lexical axiom) hence state ⑤. This aside, we may have either $i \mid j$ or $j \mid i$, or both $i \mid j$ and $j \mid i$. Asymmetric divisibility in regularity arises when all multiplicities of the divisor are accommodated by the multiple, that is as follows:

As stated in the previous section, scaling issues normally act against the emergence of state ⑨. Also rare but more common for early terms in $\boldsymbol{R}_\varkappa$ are states ⑥⑧.

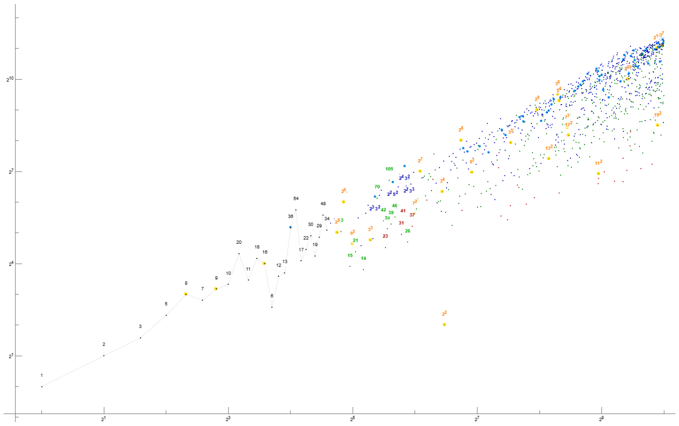

Figure 4: Log-log scatterplot of $a(n)$, $n = 1 \dots 1024$ showing primes in red, multus numbers (composite prime powers, A246547) in gold, varius numbers (squarefree composites, A120944) in green, tantus (neither squarefree nor prime power, A126706) in dark blue, highlighting plenus numbers (products of multus numbers, A286708) in large light blue. Powerful numbers A1694 = A246547 ∪ A286708.

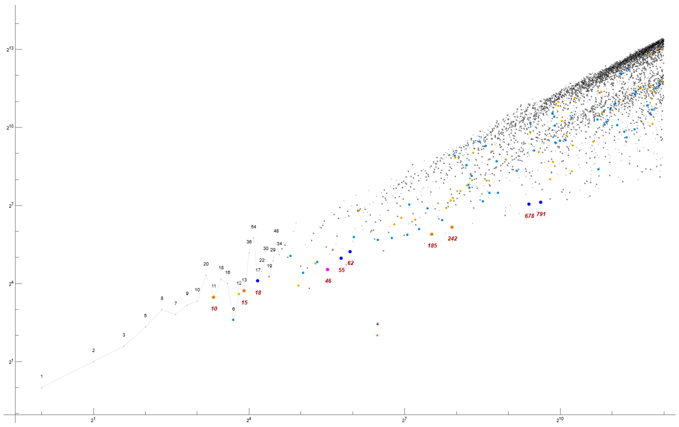

Figure 5: Log-log scatterplot of $a(n)$, $n = 1 \dots 4096$ where the color function indicates the constitutive state $Sv(j, k)$.

Axiom 0 can be rewritten as follows:

$$i \text{ ⑥⑧⑨ } j \text{ implies } ⧄k : (i \times j) \perp k \wedge \neg c(k).$$

Therefore on account of the "scaling issue" we expect Axiom 0 to prove rare. This table relating to k resulting from Axiom 0 shows data that pertain to the smallest 2^{24} terms:

n	state	i	j	k	K
8	(6)	10	20	11	10
13	(6)	6	12	13	6
16	(9)	36	54	17	6
44	(8)	84	42	23	42
53	(9)	90	120	31	30
60	(9)	108	96	37	6
183	(6)	182	364	59	182
240	(6)	170	340	71	170
676	(9)	1456	1274	131	182
789	(9)	2052	1824	137	114
5703	(9)	14850	14520	409	330
6032	(9)	15392	12506	421	962
7437	(9)	20580	20160	439	210
56182	(9)	158400	159720	1277	330

We note that many of these strongly regular relations that trigger Axiom 0 concern state ⑨, having declared the state rare.

Lemma 6.1: Strongly $\varkappa$-regular $n \in \varkappa\boldsymbol{R}_\varkappa : n > \varkappa$ implies d ⑥ $n : d \le n/p$ where $p = \text{LPF}(\varkappa)$.
Proof: Consider nontrivial divisors d of a composite number n. Let D be the largest nontrivial divisor of n. Since 2 is the smallest prime, then $D \le n/2$. More precisely, if $p = \text{LPF}(n)$, then $D = n/p$. Furthermore, if $\text{RAD}(D) \mid \text{RAD}(n) = \varkappa$, then $p = \text{LPF}(\varkappa)$, hence $D = n/p$.

Therefore in the strongly $\varkappa$-regular numbers $\varkappa\boldsymbol{R}_\varkappa : \text{LPF}(\varkappa) = p$, suppose we select an element $n > \varkappa$. Then for composite n, there is an element $d : d \mid n$, $1 < d < n$, and $d \le n/p$. We know from theorem that $d : d \mid n$ and $d \in \varkappa\boldsymbol{R}_\varkappa$. All divisors d appear before n in the sequence $\varkappa\boldsymbol{R}_\varkappa$. Therefore, most of the numbers $k \in \varkappa\boldsymbol{R}_\varkappa$ are not such that k ⑥ n, but depending on $n/\varkappa$ as it remains small, we may have saturated k ⑥ n for $k < n$. ■

Lemma 6.2: Strongly $\varkappa$-regular $n \in \varkappa\boldsymbol{R}_\varkappa : n > \varkappa$ implies n ⑧ k such that $k = mn : m \in \varkappa\boldsymbol{R}_\varkappa$ and $m \ge p$ where $p = \text{LPF}(\varkappa)$. ■
Proof: We pursue an argument similar to Lemma 6.1.

Corollary 6.3: For all other $k \in \varkappa\boldsymbol{R}_\varkappa$, we have n ⑨ k.

Theorem 6: Except n itself, $(1/p)n < k < pn$ implies k ⑨ n.
Proof: Consequence of Lemmas 6.1, 6.2, and Corollary 6.3. That is, $n/p < k < n$ implies k ⑨ n and $n < k < pn$ implies k ⑨ n, while n ⑤ n. ■

Theorem 7: Squarefree $\varkappa$ and $n \in \varkappa\boldsymbol{R}_\varkappa : k > \varkappa$ implies $\varkappa$ ⑥ k.
Proof. Squarefree n implies $n = \varkappa$. Since $\varkappa$ is the minimum of $\varkappa\boldsymbol{R}_\varkappa$, we have $\varkappa$ ⑥ k, because $\varkappa$ divides all elements in $\varkappa\boldsymbol{R}_\varkappa$ by definition. Recall that state ⑨ implies tantus numbers, and that $\varkappa$ is the sole squarefree number in $\varkappa\boldsymbol{R}_\varkappa$. ■

Hence, given tantus i and j "in their vicinity", with same squarefree kernel $\varkappa$, more precisely, within a factor of $\text{LPF}(\varkappa)$, we have state ⑨.

In scatterplot we observe the following:

1. Squarefree numbers enter late while non squarefree predominate early terms.
2. There appears to be a clustering or dense set of terms around the line $b(n) = n$.
3. The number $b(89) = 4$ occurs conspicuously late, following 81 and 162. This attributes to the first emergence of 2 as a factor of the latter that does not divide the former. Since $b(2) = 2$, we have $2m_2$ where $m_2 = 2$. ⁂

Acknowledgement

Thanks to David Sycamore for reviewing this work and recommending minor changes.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved November 2022.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

CODE:

[C1] Generate S20230119:

```
nn = 2^20; c[_] = False; q[_] = 1;
Array[Set[{a[#], c[#]}, {#, True}] &, 2];
i = a[1]; j = a[2]; u = 3;
Monitor[Do[If[CoprimeQ[i, j],
    (k = q[#]; While[c[k #], k++]; k *= #;
      While[c[# q[#]], q[#]++]) &[i j],
    If[Or[Divisible[i, j], Divisible[j, i]],
     (k = q[#]; While[c[k #], k++]; k *= #;
        While[c[# q[#]], q[#]++]) &
        [FactorInteger[GCD[i, j]][[1, 1]]],
      k = u; While[Nand[! c[k], CoprimeQ[#, k]], k++] &[
      i j] ] ];
  Set[{a[n], c[k], i, j}, {k, True, j, k}];
  If[k == u, While[c[u], u++]], {n, 3, nn}], n];
Array[a, nn]
```

[C2] Generating function:

```
nn = 2^20; c[_] = False; q[_] = 1;
f[n_] := FactorInteger[n][[All, 1]];
Array[Set[{a[#], c[#]}, {#, True}] &, 3];
Set[{i, j, S, T},
    {a[2], a[3], f[a[2]], f[a[3]]}];
Set[{ri, rj}, {Times @@ S, Times @@ T}]; u = 4;
Monitor[Do[If[S == T,
    k = u; While[Nand[! c[k], CoprimeQ[#, k]], k++] &[
      i j],
   (k = q[#]; While[c[k #], k++]; k *= #;
      While[c[# q[#]], q[#]++]) &[
     Total@ SymmetricDifference[S, T] ] ];
  Set[{a[n], c[k], i, j}, {k, True, j, k}];
  Set[{S, T}, {T, f[j]}];
  Set[{ri, rj}, {Times @@ S, Times @@ T}];
  If[k == u, While[c[u], u++]], {n, 4, nn}], n];
Array[a, nn]
```

CONCERNS SEQUENCES:

A007947: Squarefree kernel of n; RAD(n).

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