Notes on OEIS A036998

A partition counting function restricted to distinct parts coprime to n.

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Abstract.

This work relates the totient ratio and the sequence A036998.

Introduction.

We have partitions $\lambda \vdash n$ and we are concerned with part $t \in \lambda_T : k \perp n$. For prime $n = p$, $k < p$ are such that $(k, p) = 1$, and for all n, $k = \pm 1 \pmod{n}$ are such that $k \perp n$ since 2 is the smallest prime, etc.

Define the partition counting function restricted to distinct reduced residues t (that is, totatives t) of n, $p_\phi(n)$ to be the number of distinct totative-restricted partitions λ_T.

Let the set of reduced residues of n, $C(n) = \{t : t \perp n \wedge t < n\}$. The Euler totient function $\phi(n) = |\,C(n)\,|$.

Therefore, for prime p we expect many $\lambda_T \vdash p$, since there are many residues $t \pmod{p}$ such that $t \perp p$, but for highly divisible, even better still, primorials, the reduced residue system is minimized. Considering combinations λ_T of $t \in C(n)$, we expect $p_\phi(n)$ to get large quickly. Therefore we aim to define both a lower and upper bound.

If we allow λ_T with repetition, we obtain the following for $n = 7$:

```
(6, 1),
(5, 2), (5, 1, 1),
(4, 3), (4, 2, 1), (4, 1, 1, 1),
(3, 3, 1), (3, 2, 2), (3, 2, 1, 1), (3, 1, 1, 1, 1),
(2, 2, 2, 1), (2, 2, 1, 1, 1), (2, 1, 1, 1, 1, 1),
(1, 1, 1, 1, 1, 1, 1)
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With restriction to distinct reduced residues, we have this:

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(6, 1), (5, 2), (4, 3), (4, 2, 1).
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Thus, $p_\phi(7) = 4$.

The sequence p_ϕ = OEIS A036998 by Wouter Meeussen in the late 1990s. The first terms of this sequence begin as follows:

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1, 0, 1, 1, 2, 1, 4, 2, 3, 2, 11, 2, 17, 3, 5, 5, 37,
3, 53, 5, 12, 7, 103, 5, 70, 10, 42, 11, 255, 4, 339,
23, 59, 22, 130, 11, 759, 32, 115, 22, 1259, 10, 1609,
44, 94, 64, 2589, 22, 1674, 40, 385, 84, 5119, 30, 1309,
79, 665, 162, 9791, 18, 12075, 217, 556, 276, 3511, 35,
22249, 272, 1845, 62, 32991, 77, 40025, 496, 854, 468,
14108, 68, 70487, 236 , 5331, 833, 101697, 67, 20502,
1070, 7342, 735, 173681, 60, 48280, 1292, 11276, 1737,
45803, 232, 345855, 880, 9787, 658, ...
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Figure 1 is a log log scatterplot of $p_\phi(n)$ for $n = 1 \ldots 1000$.

There is a generating function for $p_\phi(n)$ based on the reduced residue system $C(n) = \{\, k : k \perp n \wedge k < n \,\}$:

$$G(p_\phi(n), z) = \prod_{k \in C(n)} (1+z^k)$$

Hence for $n = 12$, we have

$$G(p_\phi(12), z) = (1+z) \times (1+z^5) \times (1+z^7) \times (1+z^{11}) = 2.$$

We see that the totative-restricted partitions λ_T of 12 are of complementary totatives, i.e., $(11, 1)$ and $(7, 5)$, which we might also represent as ± 1 and ± 5, respectively.

This method does not generate the partitions λ_T.

Theorem 1. The partition $(T, 1)$ is one of those counted by the partition counting function restricted to distinct totatives, $p_\phi(n)$.
Proof. For all n, $k = \pm 1 \pmod{n}$ are coprime to n, since 2 is the smallest prime. Since $T \bmod n = -1$ by definition, we have $T+1 = n$. ∎

In this way $(1, T)$ forms a pair of "complementary totatives".

Theorem 2. The "complementary totatives" are counted by the partition counting function restricted to distinct totatives, $p_\phi(n)$.
Proof: We can construct the reduced residue system $C(n)$ of n by a sieve process on the ring $n/\mathbb{Z}_n$ (i.e., $k \bmod n : 0 \le k \le n-1$) by which we delete all $k = mp$ for some $p \mid n$, leaving those $k : k \perp n$. This way it is clear that there is a symmetric arrangement of reduced residues $\pm t$. Then it is clear that we have $(t, n-t)$ and the sum of those parts is obviously n. This theorem is a generalization of Theorem 1. ∎

Theorem 3. $p_\phi(n) \ge \lfloor \phi(n)/2 \rfloor$.
Proof. This relies on the immediately preceding theorem. This theorem implies that each totative (reduced residue) t has a complementary residue $t' = (n-t)$ such that $t + t' = n$. We know that there cannot be $t = n/2$ else $2 \mid n$ thus $2 \times t = n$, hence $t \mid n$ and is not coprime to n, contradicting definition of t. This said, we have $C(2) = \{1\}$, and such does not sum to 2. $n = 1$ is a special case since $C(1) = \{1\}$ and 1 does sum to 1. ∎

Corollary 3.1: Set $s = n$. For $t \in C(n)$, i.e., t such that $t \perp n \wedge t < n$, $(s-t) \perp n$ implies $(s-t) \in C(n)$.

Corollary 3.2: Beginning candidate λ_c with totative t_1, we add a totative $t_2 < (n - t_1)$. $\sum_{j=1}^{k} (t_j) < n$ implies addition of $t_{(k+1)} < t_k$ to candidate partition λ_c such that the sum $\sum_{j=1}^{k+1} (t_j) \ge n$. $\sum_{j=1}^{k+1} (t_j) = n$ implies λ_T is complete, while the sum $\sum_{j=1}^{k+1} (t_j) > n$ indicates dropping $t_k = t_i$, where t_i is the i-th totative in $C(n)$. In the latter case, we replace t_i with $t_{(i+1)}$ and start the process anew until we have exhausted all $t \in C(n)$. By this process we find all λ_T that sum to n. This is an inductive extension of Theorem 3 and Corollary 3.1.

Corollary 3.3. Set $s = n$, and for each successive strictly decreasing part t_k in λ_T, such that $t_k \in C(n)$, we perform $s \mathrel{-}= t_k$, that is we set $s = (s - t_k)$. The emergence of $s \perp n$ such that $s > n/2$ implies λ_T is complete.

Algorithms.

There is a generating function for $p_\phi(n)$ based on the reduced residue system $C(n) = \{\, k : k \perp n \wedge k < n \,\}$:

$$G(p_\phi(n), z) = \prod_{k \in C(n)} (1+z^k)$$

Hence for $n = 12$, we have

$$G(p_\phi(12), z) = (1+z) \times (1+z^5) \times (1+z^7) \times (1+z^{11}) = 2.$$

We see that the totative-restricted partitions λ_T of 12 are of complementary totatives, i.e., $(11, 1)$ and $(7, 5)$, which we might also represent as ± 1 and ± 5, respectively.

This method does not generate the partitions λ_T.

Using Theorem 3 and corollaries 3.1-3.3, we may write an algorithm that furnishes all partitions λ_T that sum to n.

For computation of $p_\phi(n)$, given the magnitude of the function even for $p_\phi(346) > 10^{12}$, therefore it seems that $G(p_\phi(n), z)$ is the best method for computation of the function $p_\phi(n)$, and we will not have an easy solution to compute, for example, $p_\phi(P(6))$, aside from any approximations we might make taking into consideration quasi-curves evident in log-log scatterplot.

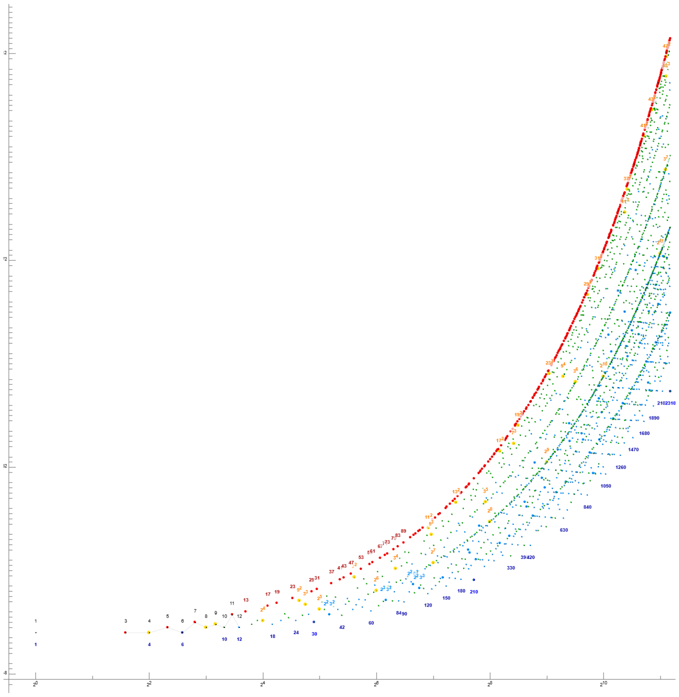

Figure 1: Log-log scatterplot of $a(n)$, $n = 1 \ldots 2310$ showing primes in red, multus numbers (composite prime powers, A246547) in gold, varius numbers (squarefree composites, A120944) in green, tantus (neither squarefree nor prime power, A126706) in blue, highlighting plenus numbers (products of multus numbers, A286708) in large blue, and primorials (A2110) in dark blue. Powerful numbers A1694 = A246547 ∪ A286708. We have labeled numbers in A244052 in blue.
We are interested in "axial quasicurves" which can be seen in green down the apparent center of the graph between the red primes and the blue local near-minima.

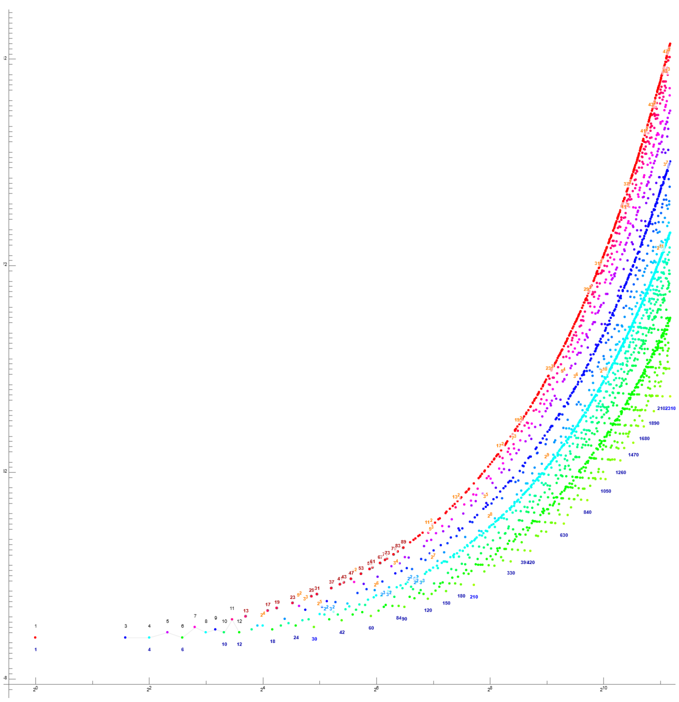

Figure 2: Log-log scatterplot of $a(n)$, $n = 1 \ldots 2310$ with a color function applied that indicates $\phi(n)/n$, where red = 0 and magenta = 1; the smallest values in the plot are around ⅛. Cyan represents $\phi(n)/n$ = ½.

Conjecture 4: Numbers r in A244052 should predominate local minima. This is because these numbers set records for the "regular counting function", RCF(n).

Define an n-regular k as one that is a product limited to the primes $p \mid n$; $k = 1$ is regular to all n, since $1 \mid n$. Divisors are a special case of n-regular k, since divisors are products limited to primes $p \mid n$. In other words, $k \mid n^{\varepsilon} : \varepsilon \geq 0$. We may express n-regular k as $k \parallel n$.

Define regular counting function as follows:

$$\text{RCF}(n) = \text{A010846}(n) = \left|\, \{k : k \parallel n \wedge k < n\} \,\right|.$$

Remark: n-regular $k > 1$ implies k noncoprime to n, since by definition, $(k, n) > 1$. However, n-regular k are only one species of numbers k such that $(k, n) > 1$. We find also n-semicoprime k such that $(k, n) > 1$, with prime $q \mid k$ such that $q \perp n$. We may express n-semicoprime k as $k \between n$. We can also define a function as follows:

$$\text{SCF}(n) = \text{A243823}(n) = \left|\, \{k : k \between n \wedge k < n\} \,\right|$$

Therefore, we have the following equation:

$$n = \phi(n) + \text{RCF}(n) + \text{SCF}(n) - 1$$

Hence richness in n-regular $k < n$ occurs at the expense of $\phi(n)$ + SCF(n). The sequence A244052 begins as follows:

`1, 2, 4, 6, 10, 12, 18, 24, 30, 42, 60, 84, 90, ...`

We observe A244052 ⊂ A2110.

A similar and stronger case might be made for primorials $P(n)$ = A2110(n), since the primorials minimize the totient ratio $\phi(n)/n$.

Totient Ratio and A036998.

Define "totient ratio" $f(n) = \phi(n)/n$ = A076512(n)/A109395(n). This rational sequence begins as follows:

1, ½, ⅔, ½, ⅘, ⅓, 6⁄7, ½, ⅔, ⅖, 10⁄11, ⅓, 12⁄13, 3⁄7, 8⁄15, ½, 16⁄17, ⅓, ...

We examine the relationship of this function and A036998.

There is a sort of "axial" structure evident in the scatterplot that should associate with the "central" trajectory in the plot (Figure 2).

The distribution away from the axis of the plot seems to have to do with the totient ratio since we observe $n = 2^{\varepsilon} : \varepsilon > 1$ occur roughly in the center of the spread. Below is a table pertaining to $n = 2^{\varepsilon}$.

Table 1.

n	a(n)
1	1
2	0
4	1
8	2
16	5
32	23
64	276
128	11564
256	2824974
512	8304924928
1024	824068326214949
2048	1187072379172925177724l
4096	1953210313462092569188908846997 55

Terms $a(p^{\varepsilon})$ for powers $p^{\varepsilon} : \varepsilon > 1$ and odd p appear "north" of this center. We also notice that $a(n) : 6 \mid n$ appear south of the center. The lower bound seems to contain the primorials, the subject of the following Table 2:

Table 2.

n	P(n)	phi(P(n))	a(P(n))
1	2	1	0
2	6	2	1
3	30	8	4
4	210	48	1155
5	2310	480	112880439714286
6	30030	5760	4596398527000079085441168119028423066698839674041038 6843

The totient ratio (graphed in Figure 4) seems relevant to the scatterplot since the partition function restricted to totatives $(p_\phi(n))$ involves the reduced residue system $C(n)$.

The sequence A307540 (see Figure 3) lists squarefree numbers $\varkappa_i$ with $\text{GPF}(\varkappa_i) = p_i$ from least to greatest $\phi(\varkappa_i)/\varkappa_i$. We consider only squarefree numbers since the totient ratio only regards prime factors and not their multiplicities. Hence, a nonsquarefree m such that $\text{RAD}(m) = \text{A7947}(m) = \varkappa$ has $\phi(m)/m = \phi(\varkappa)/\varkappa$.

We can produce a plot $(i, \phi(\varkappa_i)/\varkappa_i)$ and notice this has a particular arrangement of squarefree kernels $\varkappa$ that seem to map to the plot of A036998. In this plot, concerning row i of A307540, we see the function $\phi(\varkappa_i)/\varkappa_i$ which is maximized in prime $\varkappa_i = p_i$ and minimized in $\varkappa_i = P(i) = \text{A2110}(i)$.

Consider squarefree $\varkappa \in \text{A5117}$ and let $\boldsymbol{R}_\varkappa = \bigotimes_{p|\varkappa} \{p^\varepsilon : \varepsilon \ge 0\}$ be the $\varkappa$-regular numbers. It is clear that $r \in \boldsymbol{R}_\varkappa$ is such that $\text{RAD}(r) \mid \varkappa$.

Examples:

$$\begin{aligned}\boldsymbol{R}_6 &= \bigotimes_{p|6} \{p^\varepsilon : \varepsilon \ge 0\}\\ &= \{2^\delta : \delta \ge 0\} \otimes \{3^\varepsilon : \varepsilon \ge 0\}\\ &= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, \ldots\}\\ &= \text{A3586}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{R}_{10} &= \bigotimes_{p|10} \{p^\varepsilon : \varepsilon \ge 0\}\\ &= \{2^\delta : \delta \ge 0\} \otimes \{5^\varepsilon : \varepsilon \ge 0\}\\ &= \{1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, \ldots\}\\ &= \text{A3592}.\end{aligned}$$

Then strongly $\varkappa$-regular numbers $\text{R} \in \varkappa\boldsymbol{R}_\varkappa$ are such that $\text{RAD}(\text{R}) = \varkappa$.

$$\begin{aligned}6\boldsymbol{R}_6 &= \bigotimes_{p|6} \{p^\varepsilon : \varepsilon \ge 0\} \times 6\\ &= \{2^\delta : \delta \ge 0\} \otimes \{3^\varepsilon : \varepsilon \ge 0\} \times 6\\ &= \{6, 12, 18, 24, 36, 48, 54, 72, 96, 108, \ldots\}\\ &= 6 \times \{\text{A3586}\}.\end{aligned}$$

It is evident that multiplication of $\boldsymbol{R}_\varkappa$ by $\varkappa$ guarantees $\varkappa \mid \text{R}$ for all R.

Proposition: For n of similar magnitude, $p_\phi(n)$ varies according to $\phi(n)/n$. What this means to say is that $p_\phi(n)$ is a sort of remapping of the graph of $\phi(n)/n$. Therefore we can define a "curve" attributable to the following:

$\phi(n)/n = ½$ (powers of 2),
$\phi(n)/n = ⅓$ (strongly 6-regular numbers $6\,\boldsymbol{R}_6$),
$\phi(n)/n = ⅔$ (powers of 3),
$\phi(n)/n = ⅘$ (powers of 5),
$\phi(n)/n = 4/15$ (strongly 15-regular numbers $15\,\boldsymbol{R}_{15}$), etc.

The proposition remains to be proved. It would show that $\text{R} \in \varkappa\boldsymbol{R}_\varkappa$, presented in their order and transformed by $p_\phi(\text{R})$, trace a quasi-curve in the scatterplot of A036998 that appeared "parallel" so to speak, to the quasi-curve of $a(n) : n = 2^\varepsilon$ for $\varepsilon > 1$. This would then explain the "axial" quasi-curves in the log-log scatterplot of A036998.

This proposition also seems to support the conjecture that the highly regular numbers (A244052) comprise locally small terms, but A2110 comprise the local minima of A036998.

Conclusion.

We have drawn a few connections between A036998 and numbers of certain multiplicative species, including the totient ratio $\phi(n)/n$. It appears that there is a map between the totient ratio and A036998 examining features of the plot $(n, \phi(n)/n)$ and that of A036998. We may be able to write bounds and describe curves pertaining to $\phi(n)/n$ associated with certain squarefree kernels $\varkappa$ and thereby estimate the value of $\text{A036998}(n)$ such that $\text{RAD}(n) = \varkappa$.

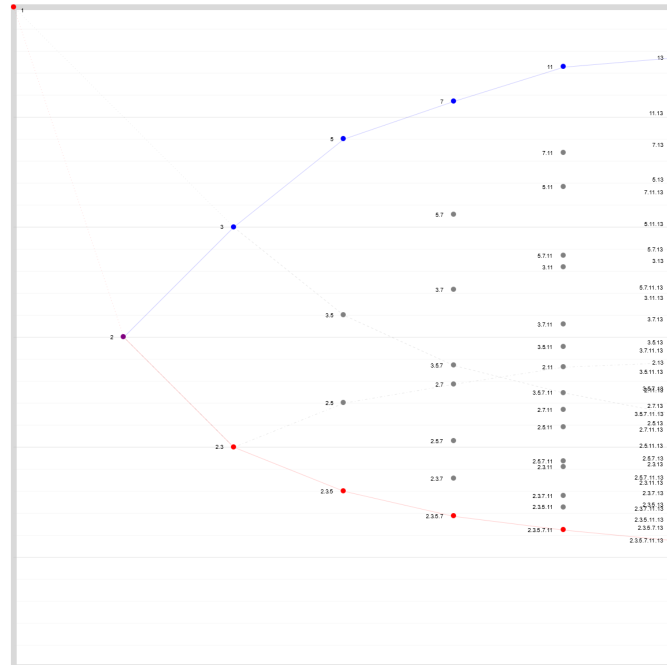

Figure 3: Plot squarefree $\varkappa_i$ with $\text{GPF}(\varkappa_i) = p_i$ at $(i, \phi(\varkappa_i)/\varkappa_i)$, where i increases to the right and $0 < \phi(\varkappa_i)/\varkappa_i < 1$ increases from bottom. Labels indicate prime factors of $\varkappa_i$. Primorials appear at bottom in red; primes appear at top in blue.

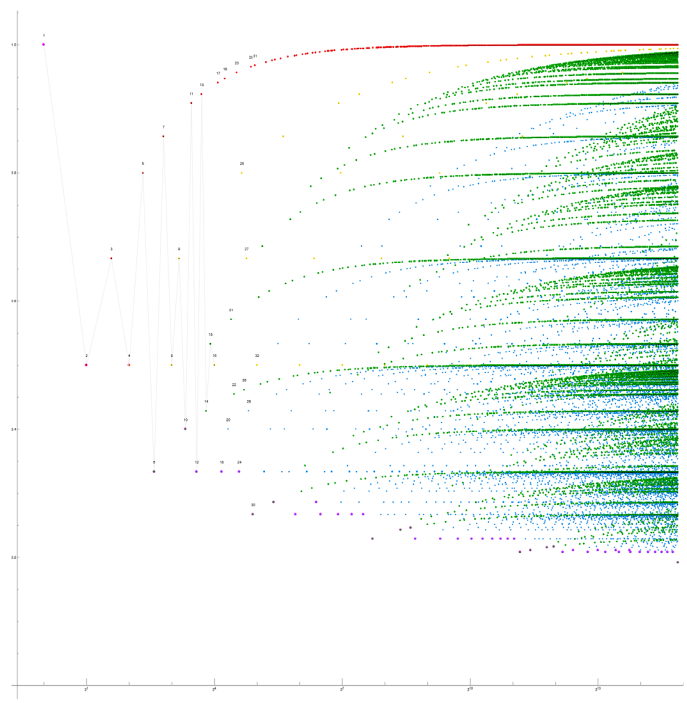

Figure 4: Plot n at $(n, \phi(n)/n)$ for $n = 1 \ldots 2^{15}$, where n increases to the right and $0 < \phi(n)/n < 1$ increases from bottom. Color scheme matches Figure 1, except that magenta highlights numbers in A244092.
This is tantamount to plotting A076512(n)/A109395(n).

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved November 2022.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

Code:

[c1] Syntactically short and naïve algorithm:

```
Table[
  Count[IntegerPartitions[n, All,
    Select[Range[n],
     CoprimeQ[#, n] &]],
       _?(# == Reverse@ Union@ # &)], {n, 50}]
```

[c2] Generating function:

```
Table[Coefficient[
  Series[Times @@ ((1 + z^#) & /@
    Select[Range[q], Coprime[#, q] &]), {z, 0, q}],
     z^q], {q, 2^8}]
```

Concerns oeis sequences:

A002110: Primorials $P(n)$: products of the smallest n primes.
A003586: Numbers of the form $2^i \times 3^j$, $i \geq 0$, $j \geq 0$.
A003592: Numbers of the form $2^i \times 5^j$, $i \geq 0$, $j \geq 0$.
A005117: Squarefree numbers.
A007947: Squarefree kernel of n; $\mathrm{RAD}(n)$.
A010846: Regular counting function $\mathrm{RCF}(n)$.
A036998: Totative-restricted partition counting function $p_\phi(n)$.
A076512: Numerator of $\phi(n)/n$.
A109395: Denominator of $\phi(n)/n$.
A120944: "Varius" numbers; squarefree composites.
A126706: "Tantus" numbers neither prime power nor squarefree.
A243823: Semitotative counting function $\mathrm{SCF}(n)$.
A244052: Highly regular numbers.
A246547: "Multus" numbers; composite prime powers $p^\varepsilon : \varepsilon \geq 1$.
A307540: Row i lists squarefree numbers $\varkappa_i$ with $\mathrm{GPF}(\varkappa_i) = p_i$ from least to greatest $\phi(\varkappa_i)/\varkappa_i$.

Document Revision Record:

2023 0124: Version 1
2023 0127: Extension of A036998 dataset to 2310 terms.