Constitutive Basics.

A succinct overview of multiplicative relations between 2 natural numbers.

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Abstract.

This brief encapsulates basic qualities of 3 constitutive relationships between two natural numbers. Constitutive analysis has proven a handy tool in the examination of integer sequences that have some sort of divisibility restriction. The aim of this paper is to clarify the topic in brief so that mathematicians may employ its concepts in their own work.

Introduction.

This paper concerns the multiplicative relationship between nonzero positive integers k and n, principally composite numbers in the mutual cototient, this is to say, numbers such that $(k, n) > 1$. In this work, we write $(k, n) = 1$ as $k \perp n$ according to Knuth [2] on account of its convenience. For $(k, n) > 1$ we write $k \sqcup n$.

Relations Pertaining to Primes.

Let's first address prime numbers as their relations are simple. Prime p either divides n or, if p is not a divisor of n, p is coprime to n [3]. Hence, for prime p, we have the following well-known facts:

$$k \mid p \text{ iff } k = 1 \text{ or } k = p, \text{ and}$$
$$k \perp p \text{ for } 1 \le k < p.$$

Regarding $n > p$:

$$p \mid n \text{ for } n \equiv 0 \pmod p \text{ and all other cases, } p \perp n.$$

Therefore we have evidence of 2 possible "constitutive" states, one of which is coprimality, the other is divisibility. We know that the product of coprime numbers is also coprime [4].

Additional Relations Pertaining to Composites.

Let's consider three cases that apply to composite numbers, products of more than one prime factor, not necessarily distinct.

Let prime $p \mid n$ and prime $q \perp n$. For each appearance of p or q, we consider that p represents at least 1 prime $p \mid n$ and that q represents at least 1 prime q coprime to n.

Consider three cases:

$$pp, \qquad pq, \qquad qq.$$

There can be no other cases, since each symbol represents at least 1 instance of a prime divisor or a prime nondivisor of n.

<u>The case qq</u> simply represents a n-coprime composite, since the product Q of q such that $q \perp n$ implies $\mathrm{Q} \perp n$.

<u>The case pq</u> must have $1 < (pq, n)$ with $pq \ne (pq, n) \ne n$. It is clear that $p \mid n$ so pq is not coprime to n. Prime q does not divide n, hence pq is not a divisor of n. Since pq is neither a divisor of nor coprime to n, we may say that pq is **neutral** to n. Furthermore, since pq is neutral to n, pq cannot be prime, since primes must either divide or be coprime to n. It is clear that pq is the product of at least 1 prime divisor $p \mid n$ and at least 1 prime q such that $q \perp n$ (or, rather, an n-nondivisor q). We shall call this relation pq **semicoprime** to n.

<u>Case pp</u> is a product k of primes $p \mid n$. We may subdivide this case according to $k \mid n$.

We can construct pp such that $pp \mid n$. For example, $2 \times 2 = 4$ and $2 \times 3 = 6$; $4 \mid 8$ and $6 \mid 12$. This case is simply that of a composite divisor. We also can construct cases such as $k = 4$ and $n = 10$, or $k = 12$ and $n = 18$ such that, though all the prime divisors of the former in each pair divide the latter, we find that the number is too rich in copies of at least one prime divisor p so as to divide the latter number.

The case of composite divisors, has $1 < (pp, n) = pp$ while the case of nondivisors has $1 < (pp, n) < pp$. Hence the second case is also n-neutral. It is evident that the second case is that of n-nondivisors k that are products of primes p such that $p \mid n$. We shall call n-nondivisor n-regular k, an n-**semidivisor**. Though primes p may divide n, they may not semidivide n, as semidivisibility is neutral, hence the nondivisor subspecies of case pp applies to composite k. Taking case pp together, we say that $k \mid n$ and n-semidivisor k are n-**regular**.

The empty product 1 must be treated differently, since 1 is both coprime to all numbers n yet divides all numbers n and therefore n-regular. Exclusive of the empty product, there are 3 distinct relations between k and n: k coprime to n, k semicoprime to n, and k regular to n.

For simplicity, we call these relations "constitutive", as they regard the multiplicative constitution of numbers k and n.

Basic Set-Theoretic Constitutive States.

Define the set of distinct prime divisors of n as follows:

$$\mathrm{P}(n) = \{\text{ prime } p : p \mid n \}. \qquad [1.1]$$

Then the cardinality of $\mathrm{P}(n)$ is rendered thus:

$$\omega(n) = \left| \mathrm{P}(n) \right| = \mathrm{A}1221(n), \qquad [1.2]$$

and the squarefree kernel of n as follows:

$$\varkappa = \mathrm{RAD}(n) = \prod_{p \mid n} p = \mathrm{A}7947(n). \qquad [1.3]$$

Let's examine the three states in a basic set-theoretical light. A priori, the following is obvious:

$$\mathrm{P}(n) \cap \mathrm{P}(k) = \varnothing \Rightarrow (k, n) = 1, \text{ coprimality.} \qquad [1.4]$$

$$\mathrm{P}(n) \cap \mathrm{P}(k) \ne \varnothing \Rightarrow (k, n) > 1, \text{ noncoprimality.} \qquad [1.5]$$

It is necessary to consider multiplicity so as to determine divisibility or nondivisibility between k and n. We further distinguish relations by taking into account the multiset $\mathcal{P}(n)$ of prime divisors with multiplicity. The cardinality of this multiset is given by the following:

$$\Omega(n) = \left| \mathcal{P}(n) \right| = \mathrm{A}1222(n), \qquad [1.6]$$

Considering the sets of distinct prime divisors of 2 nonzero positive numbers, we have disjoint and identical sets, or we have an intersection resulting in an inhabited set. Coprimality concerns the disjoint sets, regularity the identical sets, while semicoprimality is present, with or without regularity, in the case of difference between sets results in an inhabited set. If symmetric difference between sets derives from both sides, we have symmetric semicoprimality (◊◊), otherwise we have a mixed cototient relationship.

Table A.

Relation	Setwise	Kernelwise
coprimality	$\mathrm{P}(n) \cap \mathrm{P}(k) = \varnothing$	$\mathrm{RAD}(k) \perp \mathrm{RAD}(n)$
mixed cototient	$\mathrm{P}(n) \ominus \mathrm{P}(k) \ne \varnothing$	$(\mathrm{RAD}(k), \mathrm{RAD}(n)) > 1$
coregularity	$\mathrm{P}(k) = \mathrm{P}(n)$	$\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$

Therefore semicoprimality and regularity comprise 2 species of the cototient, that is, of numbers that are not coprime. (See Figure 1 on page 6.)

Constitutive Symbols.

We now introduce symbols to signify constitutive relations, intended to streamline statements, akin to writing "$k \mid n$" to mean that k divides n, i.e., k is an n-divisor. We have already employed the notation $k \perp n$ to signify $(k, n) = 1$. To signify $(k, n) > 1$, we write $k \sqcup n$, meaning k and n have a common prime divisor.

In the light of Knuth's coprimality symbol and seeing that n-divisibility, i.e., $k \mid n$, is a form of n-regularity, we use $k \parallel n$ to signify k regular to n i.e., k is n-regular. Hence $k \nparallel n$ signifies k is nonregular to n, as $k \nmid n$ means k does not divide n. We propose $k \mathrel{¦} n$ to signify k semidivides n i.e., k is an n-semidivisor. Finally, we use $k \Diamond n$ to signify k is semicoprime to n i.e., k is n-semicoprime.

The following table summarizes constitutive symbols.

Table B.

$k \perp n$	k is coprime to n	$(k, n) = 1$	
$k \Diamond n$	k is semicoprime to n	$1 < (k, n) < \mathrm{MIN}$	$n/(k, n) \nparallel n$
$k \parallel n$	k is regular to n	$1 \le (k, n) \le \mathrm{MIN}$	$k \mid n^{\varepsilon} : \varepsilon \ge 0$
$k \mid n$	k divides n	$1 \le (k, n) = k$	$k \mid n^{\varepsilon} : \varepsilon = 0 \ldots 1$
$k \mathrel{¦} n$	k semidivides n	$1 < (k, n) < \mathrm{MIN}$	$k \mid n^{\varepsilon} : \varepsilon > 1$

Hence, writing $k \parallel\Diamond n$ signifies k regular to n, but n semicoprime to k, while $k \parallel\parallel n$ indicates symmetric regularity.

Within the mixed cototient, we have the following relations:

Table C.

Relation	Setwise	Kernelwise
$k \perp n$	$P(n) \cap P(k) = \varnothing$	$\mathrm{RAD}(k) \perp \mathrm{RAD}(n)$
$k \Diamond\Diamond n$	$P(n) \ominus P(k) \ne \varnothing$: $P(k) \setminus P(n) \ne \varnothing \land$ $P(n) \setminus P(k) \ne \varnothing$	$\mathrm{RAD}(k) \nmid \mathrm{RAD}(n) \land$ $\mathrm{RAD}(n) \nmid \mathrm{RAD}(k)$
$k \Diamond\parallel n$	$P(n) \ominus P(k) \ne \varnothing$: $P(k) \setminus P(n) \ne \varnothing \land$ $P(n) \setminus P(k) = \varnothing$	$\mathrm{RAD}(k) \nmid \mathrm{RAD}(n) \land$ $\mathrm{RAD}(n) \mid \mathrm{RAD}(k)$
$k \parallel\Diamond n$	$P(n) \ominus P(k) \ne \varnothing$: $P(k) \setminus P(n) = \varnothing \land$ $P(n) \setminus P(k) \ne \varnothing$	$\mathrm{RAD}(k) \mid \mathrm{RAD}(n) \land$ $\mathrm{RAD}(n) \nmid \mathrm{RAD}(k)$
$k \parallel\parallel n$	$P(k) = P(n)$	$\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$

Cases of Symmetric Regularity.

Taking into account multiplicity and knowing we have 2 species of regularity, i.e., the divisor and the semidivisor, we have the following four possible cases:

$\mid\mid$	$\mid¦$ or $¦\mid$	$¦¦$
Symmetric Divisibility	Mixed Regularity	Symmetric Semidivisibility
⑤	⑥⑧	⑨

Let $k = p^{\alpha} m q^{\delta}$, primes $p < q$, $m \ge 1$, and let $n = p^{\beta} m q^{\varepsilon}$, with nonzero exponents α, β, δ, and ε. Such a definition implies k and n both composite and not prime powers, since they are at least squarefree semiprimes pq.

Suppose $\alpha > \beta$. Then it is clear that $n \mid k$, though $k \parallel\parallel n$. Likewise we might also consider $\delta > \varepsilon$ either alone or independently and conclude the same. If $n \mid k \land k \nmid n$, yet , then it is clear that we have nondivisor n-regular k, hence an n-semidivisor k, i.e., $k \mathrel{¦} n$. Hence we may write $k \mathrel{¦\mid} n$ or via state notation, k ⑧ n. If we reverse the inequalities, then clearly we have the reverse relation $k \mathrel{\mid¦} n$, also known by k ⑥ n. In other words we have the case of mixed regularity.

Suppose $\alpha = \beta$ and $\delta = \varepsilon$. Then it is obvious we have $k \mid n \land n \mid k$, i.e., $k \mid\mid n$, also known as k ⑤ n. This is symmetric divisibility, a special case of symmetric regularity, which implies $k = n$, i.e., equality.

Finally, suppose $\alpha > \beta$, but $\delta < \varepsilon$. Then neither k nor n divide the other, though $k \parallel\parallel n$. We have a case analogous to semicoprimality in that there is an algebraic symmetric difference among multiplicities regarding at least 1 common prime factor. Therefore k and n are mutual semidivisors, $k \mathrel{¦¦} n$, also known by k ⑨ n, and we have a case of symmetric semidivisibility.

From this point on, we will refrain from using the phrase "symmetric regularity" and instead say "completely regular". We also refrain from using the symbol "$\parallel$" which represents symmetric regularity, and instead write symbols associated with completely regular states.

Cases Involving Semicoprimality.

We have already shown that there exists a symmetric semicoprime state $k \Diamond\Diamond n$, i.e, k ① n. Such is implied by symmetric difference between $P(k)$ and $P(n)$.

The existence of 2 species of regular numbers (the divisor and the semidivisor) implies corresponding mixed cototient states:

$\Diamond\Diamond$	$\Diamond\mid$ or $\mid\Diamond$	$\Diamond¦$ or $¦\Diamond$
Symmetric Semicoprimality	Lean Divisorship	Mixed Neutrality
①	②④	③⑦

These will be described in a section below.

Constitutive Partitioning of Natural Numbers.

We define 3 infinite subsets of $\mathbb{N}$ with respect to $P(n)$, $n > 1$. These numbers pertain to squarefree kernel $\varkappa = \mathrm{RAD}(n) = \mathrm{A7947}(n)$. These are the sets of $\varkappa$-regular k, $\varkappa$-semicoprime k, and $\varkappa$-coprime k.

The 3 subsets are defined as follows:

$$\boldsymbol{R}_{\varkappa} = \{\, k : P(n) \subseteq P(k) \,\} = \bigotimes_{p \mid \varkappa} \{p^{\varepsilon} : \varepsilon \ge 0\}, \tag{2.1}$$

$$\boldsymbol{S}_{\varkappa} = \{\, k : 0 < |\, P(k) \setminus P(n) \,| < |\, P(n) \cup P(k) \,| \,\},$$

$$= \{\boldsymbol{R}_{\varkappa} \setminus \{1\}\} \otimes \{\boldsymbol{T}_{\varkappa} \setminus \{1\}\} \text{ implied by case } pq \text{, and} \tag{2.2}$$

$$\boldsymbol{T}_{\varkappa} = \{\, k : P(n) \cap P(k) = \varnothing \,\} = \{\, k : (k, n) = 1 \}. \tag{2.3}$$

Hence we have $\boldsymbol{T}_{\varkappa}$ containing k such that $(k, n) = 1$, $\boldsymbol{R}_{\varkappa}$ the tensor product of prime power ranges $p^{\varepsilon} : \varepsilon \ge 0$, where $p \mid n$, and $\boldsymbol{S}_{\varkappa}$ containing k such that $(k, n) > 1$ but prime $q \mid k$ while $q \perp n$. Recognizing that n-semicoprime k is a product of at least 1 prime p such that $p \mid n$, and at least 1 prime q such that $q \perp n$, we can construct a countably infinite set of n-semicoprime numbers as follows:

$$\boldsymbol{S}_{\varkappa} = \{\boldsymbol{R}_{\varkappa} \setminus \{1\}\} \otimes \{\boldsymbol{T}_{\varkappa} \setminus \{1\}\}. \tag{2.4}$$

Regarding the empty product, $\boldsymbol{R}_{\varkappa} \cap \boldsymbol{T}_{\varkappa} = \{1\}$, otherwise the subsets are distinct.

For $n = 1$, $\boldsymbol{R}_1 = \{1\}$, $\boldsymbol{S}_1 = \varnothing$, and $\boldsymbol{T}_1 = \mathbb{N}$.

For $\varkappa > 1$, it is easy to see the following:

$|\, \boldsymbol{T}_{\varkappa} \,| = \aleph_0$ via congruence with reduced residues and induction.

$\boldsymbol{R}_{\varkappa} = \bigotimes_{p \mid \varkappa} \{p^{\varepsilon} : \varepsilon \ge 0\}$ implies $|\, \boldsymbol{R}_{\varkappa} \,| = \aleph_0$.

$\boldsymbol{S}_{\varkappa} = \{\boldsymbol{R}_{\varkappa} \setminus \{1\}\} \otimes \{\boldsymbol{T}_{\varkappa} \setminus \{1\}\}$ implies $|\, \boldsymbol{S}_{\varkappa} \,| = \aleph_0$.

Regular and Coregular Numbers.

Consider squarefree $\varkappa \in \mathrm{A5117}$ and let $\boldsymbol{R}_{\varkappa}$ be the $\varkappa$-regular numbers. It is clear that $r \in \boldsymbol{R}_{\varkappa}$ is such that $\mathrm{RAD}(r) \mid \varkappa$. Then we can generate an infinite list of $\varkappa$-regular numbers. This list is shared by any number n such that $\mathrm{RAD}(n) = \varkappa$, though a finite set of divisors uniquely pertains to n, with nondivisors comprising an infinite subset of n-semidivisors.

Examples:

$$\boldsymbol{R}_6 = \bigotimes_{p|6} \{p^\varepsilon : \varepsilon \ge 0\}$$
$$= \{2^\delta : \delta \ge 0\} \otimes \{3^\varepsilon : \varepsilon \ge 0\}$$
$$= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, \dots\}$$
$$= \text{A3586}.$$
$$\boldsymbol{R}_{10} = \bigotimes_{p|10} \{p^\varepsilon : \varepsilon \ge 0\}$$
$$= \{2^\delta : \delta \ge 0\} \otimes \{5^\varepsilon : \varepsilon \ge 0\}$$
$$= \{1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, \dots\}$$
$$= \text{A3592}.$$

Then $\varkappa$-coregular numbers $R \in \varkappa\boldsymbol{R}_\varkappa$ are such that $\text{RAD}(R) = \varkappa$.

$$6\boldsymbol{R}_6 = \bigotimes_{p|6} \{p^\varepsilon : \varepsilon \ge 0\} \times 6$$
$$= \{2^\delta : \delta \ge 0\} \otimes \{3^\varepsilon : \varepsilon \ge 0\} \times 6$$
$$= \{6, 12, 18, 24, 36, 48, 54, 72, 96, 108, \dots\}$$
$$= 6 \times \{\text{A3586}\}.$$

It is evident that multiplication of $\boldsymbol{R}_\varkappa$ by $\varkappa$ guarantees $\varkappa \mid R$ for all R.

Basic Constitutive Classes.

Let us divide natural numbers $n \in \mathbb{N}$ into 5 categories based upon prime decomposition of n. The number n is said to be squarefree iff $\omega(n) = \Omega(n)$. The number n is said to be prime iff $\omega(n) = \Omega(n) = 1$, and a prime power iff $\omega(n) = 1$. The empty product $n = 1$ occupies a category all to itself, therefore, we may hold that there are actually 4 nontrivial categories. We further distinguish numbers instead with $M(n)$ = the largest multiplicity in n, meaning the largest exponent ε such that any prime power $p^\varepsilon \mid n$.

Table D.

	$M(n) = 1$	$M(n) > 1$
$\omega(n) > 1$	**multus** 8, 27, 125 A246547	**tantus** 12, 75, 216 A126706
$\omega(n) = 1$	**prime** 2, 17, 101 A40	**varius** 6, 35, 210 A120944

These names derive from Latin. We apply *multus*, "many" to composite prime powers A246547, since we have many copies of the same primes $p \mid n$. The name *varius* means "variegated" and applies to squarefree composites A120944, since we have a diverse set of distinct primes $p \mid n$, but only one copy of any p. We apply the name *tantus*, "so (many)", to numbers neither squarefree nor a prime power (A126706) since we have a diverse set of distinct primes $p \mid n$, and at least one prime p appears more than once, that is, $M(n) > 1$.

We define a subset of tantus numbers for which all prime power factors $p^\varepsilon \mid n$ such that $\varepsilon > 1$. This is tantamount to the powerful numbers A1694 without prime powers A961, i.e., A1694 \ A961. We call these **plenus** ("full") numbers (A286708). Another way to think of plenus numbers is as a product of multus numbers.

It is clear we may partition the natural numbers $n \in \mathbb{N}$, $n > 1$ into the following mutually exclusive classes:

A40 = $\{n : \Omega(n) = \omega(n) = 1\}$, the primes.

A246547 = $\{n : \Omega(n) > \omega(n) = 1\}$, composite prime powers; multus.

A120944 = $\{n : \Omega(n) = \omega(n) > 1\}$, squarefree composites; varius.

A126706 = $\{n : \Omega(n) > \omega(n) > 1\}$, numbers neither squarefree nor prime powers; tantus.

Code [C2] generates sequences discussed in this section.

Constitutive states.

Outside of coprimality, we may have asymmetric constitutive relations between k and n. Therefore we construct a table as follows and assigning enclosed numerals to define constitutive states (which I have called Svitek states). Since coprimality is symmetric, we write $k \perp n$ as k ⓪ n. We could define states that include the empty product 1, but normally fold these into state ⓪. [SEE NOTE 1]

As a convention, we consider semicoprimality (◊), divisibility (|), and semidivisibility (¦) and construct the following table so as to assign enclosed numeral symbols (state numbers) thus:

Table E.

	$k \lozenge n$	$k \mid n$	k ¦ n
$n \lozenge k$	①	④	⑦
$n \mid k$	②	⑤	⑧
n ¦ k	③	⑥	⑨

The cototient states may be grouped with their inverse states into the following 7 categories. The following lays out basic qualities of the 7 cototient states. These descriptions relate to theorems and proofs whose numbers relate to the state numbers. For example, Theorem 68.1 relates to mixed regularity ⑥⑧.

Let $s = d = \text{MIN}(k, n)$ and $t = m = \text{MAX}(k, n)$. We use $\{d, m\}$ iff $d \mid m$, else we use $\{s, t\}$ where relevant. Let $P = P(k) = \{$ prime $p : p \mid k \}$ and $Q = P(n) = \{$ prime $p : p \mid n \}$.

State ⓪ ($\perp$) — (symmetric) coprimality:

Except in the case $1 \perp 1$, state ⓪ is ambidirectional because of its symmetry and the fact k and n must be distinct. $7 \perp 12$, since $12 = 2^2 \times 3$ and 7 doesn't appear among these factors. The prime 7 does not divide 12, hence it is coprime to 12. $1 \perp 5$, since 1, the empty product, is coprime to all n. Primes p, q, $p \ne q$ are coprime, any number and 1 are coprime, and 1 is coprime to itself.

State ① (◊◊) — symmetric semicoprimality:

State ① is ambidirectional and completely neutral, implying both k and n composite and distinct. Furthermore, both k and n must have more than 1 distinct prime divisor, since there must be symmetric difference between the sets of prime divisors of k and n. Examples: 6 ◊ 10 and 10 ◊ 6, since $6 = 2 \times 3$ and $10 = 2 \times 5$, $(6, 10) = 2$, and $3 \perp 5$; both of these exceed 1. 60 ◊ 21 and 21 ◊ 60, since $60 = 2^2 \times 3 \times 5$ and $21 = 3 \times 7$, $(30, 21) = 3$, and $10 \perp 7$; both of these exceed 1.

State ② (◊|) and its reversal, state ④ (|◊) — lean divisorship:

State ② implies $\omega(k) > \omega(n)$, while state ④ implies $\omega(k) < \omega(n)$. Primes are confined to RHS in state ②, and LHS in state ④. Because of divisibility and inherent inequality, the state is directional; d |◊ m implies $d < m$. The nondivisor m in this relationship has $\omega(m) > 1$. Examples: 6 | 30 and 30 ◊ 6, 288 ◊ 96 and 96 | 288.

State ③ (◊¦) and its reversal, state ⑦ (¦◊) — mixed neutrality:

These states are ambidirectional and completely neutral, implying both k and n composite and distinct. Generally, prime powers $p^\varepsilon : \varepsilon > 1$ relate to composite $m : \omega(m) > 1 \wedge p \mid m$ in the mode p^ε ¦◊ $m \vee m$ ◊¦ p^ε. State ③ implies $\omega(k) > \omega(n)$, while state ⑦ implies $\omega(k) < \omega(n)$. Examples: 12 ¦ 30 and 30 ◊ 12, $12 = 2^2 \times 3$ and $30 = 2 \times 3 \times 5$. 126 ◊ 147 and 147 ¦ 126 since $126 = 2 \times 3^2 \times 7$ and $147 = 3 \times 7^2$.

State ⑤ (||) — (symmetric divisibility) equality:

This state implies $k = n$. It is symmetric and completely regular, but not neutral and admits all positive numbers.

State ⑥ ($|¦$) and its reversal, state ⑧ ($¦|$) — MIXED REGULARITY:

Let $d = \text{MIN}(k, n)$ and $m = \text{MAX}(k, n)$. These states are completely regular, occurring entirely within $\boldsymbol{R}_\varkappa$, where $\varkappa = \text{RAD}(m)$, $d \neq m \neq 1$. State ⑥ confines primes to LHS while state ⑧ confines primes to RHS, and m may not be squarefree. Because of divisibility and inherent inequality, the state is directional. Let $p^a < p^b$ be distinct composite powers of the same prime p; therefore we have the relation $p^a \,|¦\, p^b$. Hence, $d = p^\varepsilon : \varepsilon \geq 1$ imply $d \,|¦\, m$ and $d < m$. Examples: $6 \mid 12$ and $12 ¦ 6$, $20 ¦ 10$ and $10 \mid 20$.

State ⑨ ($¦¦$) — SYMMETRIC SEMIDIVISIBILITY:

This state is symmetrical, completely neutral, and completely regular, occuring within $\varkappa\, \boldsymbol{R}_\varkappa$ where $6 \leq \varkappa = \text{RAD}(k) = \text{RAD}(n)$ absent divisibility, $k \neq n \neq 1$, and both k and n composite. The state is ambidirectional in magnitude and as to multiplicity, but flat in terms of $\omega(\varkappa)$. State ⑨ implies symmetric difference concerning multiplicities of at least one prime divisor $p \mid \varkappa$, hence both k and n are restricted to tantus numbers (in A126706) such that $|\, k - n \,| \geq \varkappa$ for $\varkappa \geq 6$. Examples: $12 \,¦¦\, 18$, $182 \,¦¦\, 361$. This state is also known as nondivisor coregularity.

We summarize the constitutive (or Svitek) states in the following table. The symbolic abbreviation is covered in Table A. The column "sym." is checked if the state is symmetric. The column "neut." is checked if the relation is completely neutral, hence reserved for composites. The "reg." column is checked if the relation is completely regular, meaning that for numbers s and t (independent of consideration of RHS or LHS), $\text{RAD}(s) \mid \text{RAD}(t)$. The "rev." column lists that state that is the reversal or inversion of the state originally considered. We add a note that may show a restriction or show a couple examples of the state.

TABLE F.

Svitek	binary relation	abb.	sym.	neut.	reg.	rev.	note
⓪	$k \perp n \land n \perp k$	$\perp$	✓			⓪	$\forall\, p, q, p \neq q, p \perp q$
①	$k \Diamond n \land n \Diamond k$	$\Diamond\Diamond$	✓	✓		①	$6 \Diamond\Diamond 10$
②	$k \Diamond n \land n \mid k$	$\Diamond \mid$				④	$30 \Diamond\mid 6$
③	$k \Diamond n \land n ¦ k$	$\Diamond ¦$		✓		⑦	$30 \Diamond¦ 12$
④	$k \mid n \land n \Diamond k$	$\mid \Diamond$				②	$10 \mid\Diamond 30$
⑤	$k \mid n \land n \mid k$	$\|\|$	✓		✓	⑤	$k \,\|\|\, n \therefore k = n$
⑥	$k \mid n \land n ¦ k$	$\|¦$			✓	⑧	$6 \,\|¦\, 12$
⑦	$k ¦ n \land n \Diamond k$	$¦\Diamond$		✓		③	$20 ¦\Diamond 30$
⑧	$k ¦ n \land n \mid k$	$¦\|$			✓	⑥	$20 \,¦\|\, 10$
⑨	$k ¦ n \land n ¦ k$	$¦¦$	✓	✓	✓	⑨	$12 \,¦¦\, 18$

THEOREMS PERTAINING TO CONSTITUTIVE STATES.

The following theorems, lemmas, and corollaries support the assertions posited in the previous section. Most of these simply apply elementary number theory and basic logic.

THE COTOTIENT

THEOREM G1: Let $(k, n) = g$. Numbers k and n in the cototient (i.e., $g > 1$) have difference $|\, k - n \,| \geq \text{LPF}(g)$.

$$k \sqcup n \Rightarrow |\, k - n \,| \geq \text{A020639}(g). \qquad [\text{G1}]$$

PROOF: We know that $k \perp n : n = k \pm 1$ since 2 is the smallest prime. Suppose that both k and n are odd, and define odd prime q is the smallest prime that divides both. Suppose that $n - k = p = 2$. Therefore, $k \equiv n \equiv 0 \pmod q$. Since q is an odd prime, $p < q$. It is apparent $k = n - 2$ would mean that $k \pmod q > 0$, a contradiction. Furthermore, for all $p < q$, such is true, otherwise $p \mid k \land p \mid n$, contradicting the definition of q as least common prime factor of k and n. ■

COROLLARY G2: Numbers k and n in the cototient (i.e., $g > 1$) have difference $|\, k - n \,| > 1$.

$$k \sqcup n \Rightarrow |\, k - n \,| > 1. \qquad [\text{G2}]$$

SEMICOPRIMALITY

THEOREM S1: Let $P = \{$ prime $p : p \mid k \}$ and $Q = \{$ prime $q : q \mid n \}$. Semicoprimality $k \Diamond n$ implies $|\, P \cap Q \,| > 0$.

$$k \Diamond n \Rightarrow |\, P \cap Q \,| > 0. \qquad [\text{S1}]$$

PROOF. The definition of semicoprimality shows $1 < (k, n)$, with $k \neq (k, n) \neq n$, hence semicoprimality is neither coprimality nor divisorship and pertains to composites. It is clear that we can find at least 1 common prime divisor p such that $p \mid k$ and $p \mid n$. The definition of semicoprimality further shows that there is at least one prime q such that $q \mid k$ but does not divide n, proving n-semicoprime k is n-nonregular. Therefore $P \cap Q \neq \varnothing$; it contains at least 1 prime, but P contains other primes that are not in Q. (Q is not restricted only to those primes in P; there may be primes that divide n but do not divide k.) ■

Therefore symmetric semicoprimality is both ambidirectional in magnitude and completely ambiguous in terms of number of distinct prime divisors ω.

THEOREM S2: Asymmetric semicoprimality $k \Diamond\sqcup n$ implies $P \subset Q$ and $\omega(k) > \omega(n)$.

$$k \Diamond\sqcup n \Rightarrow \text{A1221}(k) > \text{A1221}(n).) \qquad [\text{S2}]$$

PROOF. We know $(k, n) > 1$ since k and n share at least 1 prime divisor p, yet at least 1 prime factor $q \mid k$ does not divide n via definition of semicoprime. Such implies k and n both exceed 1. Given n not semicoprime to k, then we are left with $n \mid k^\varepsilon : \varepsilon > 0$ (with respect to the context of coprime, semicoprime, and regular relations being mutually exclusive outside the empty product with domain $\mathbb{N}$). If $n \mid k$, then $n < k$ and $P \subset Q$, hence $\omega(k) > \omega(n)$. If n does not divide k, yet does divide some larger power of k, then, though we cannot speak to the relative magnitude of k and n, we are left with $P \subset Q$, hence $\omega(k) > \omega(n)$, proving the proposition. ■

Hence asymmetric semicoprime states are omega-directional.

REGULARITY

THEOREM D1: Let $d \mid m$ and $d \neq m$. Asymmetric divisor states imply $d < m$.

$$d \mid m \land d \neq m \Rightarrow d < m. \qquad [\text{D1}]$$

PROOF. Divisors $d \mid m$ must be such that $d \leq m$, however, $d = m$ implies that $m \mid d$ as well. Therefore we are left with $d \mid m$ such that $1 < (d, m) < t$. Since $d \mid m$ implies $(d, m) = d$, we see that indeed, $d < m$. ■

Hence, asymmetric divisibility is directional.

THEOREM R1: Semidivisor states $k ¦ n$ imply $\text{RAD}(k) \mid \text{RAD}(n)$.

$$k ¦ n \Rightarrow \text{A7947}(k) \mid \text{A7947}(n). \qquad [\text{R1}]$$

PROOF. The definition of $k ¦ n$ is $k \mid n^\varepsilon : \varepsilon > 1$. If k divides some power of n but not n itself, it follows that no n-nondivisor prime $q \mid k$, else k would not divide any power $n^\varepsilon : \varepsilon > 1$ at all. Hence k is either an empty product (1) or it is a product of prime divisors $p \mid n$ and, additionally, $P \subseteq Q$. ■

Corollary R2: Semidivisor states $k ¦ n$ imply $\omega(k) \geq \omega(n)$.

$$k ¦ n \Rightarrow \text{A1221}(k) \mid \text{A1221}(n). \qquad [\text{R2}]$$

Theorem R3: Completely regular states ⑤⑥⑧⑨ imply $\text{RAD}(k) = \text{RAD}(n) = \varkappa$.

$$k \,||\, n \lor k \,¦|\, n \lor k \,|¦\, n \lor k \,¦¦\, n \Rightarrow \text{A7947}(k) = \text{A7947}(n). \; [\text{R3}]$$

Proof. We approach this problem in parts. There are 2 species of regular numbers; divisors $k \mid n$ and nondivisors, which we denote as semidivisors $k ¦ n$. Permuting these in binary relations we have the four mentioned in the proposition. Therefore we have 4 cases corresponding to states ⑤⑥⑧⑨, respectively, where state ⑧ is state ⑥ reversed.

State ⑤: Lemma 5.1 shows that $k \,||\, n$ implies $k = n$, hence the squarefree kernels of k and n are identical.

States ⑥⑧: Theorem D1 shows that asymmetric divisibility implies $\text{MIN}(k, n) \mid \text{MAX}(k, n)$, $k \neq n$. Thus we can set $d = \text{MIN}(k, n)$ and $m = \text{MAX}(k, n)$ such that $d < m$. This simplifies cases $k \,¦|\, n$ and $k \,|¦\, n$ to $d \,|¦\, m$ and Lemma 68.2, where the proposition is locally proved.

State ⑨: Finally, we have the case $k \,¦¦\, n$, where the proposition is locally proved via Theorem 9.1.

Taken together, Lemma 5.1, Theorem D1, and Theorem 9.1 prove the assertions of the proposition. ■

Corollary R4: Completely regular states ⑤⑥⑧⑨ imply $\omega(k) = \omega(n)$.

$$k \,||\, n \lor k \,¦|\, n \lor k \,|¦\, n \lor k \,¦¦\, n \Rightarrow \text{A1221}(k) = \text{A1221}(n). \; [\text{R4}]$$

Hence completely regular states are flat in terms of number of distinct prime divisors (ω).

Coregular Numbers.

A special case of k regular to n concerns the condition $\text{RAD}(k) = \text{RAD}(n) = \varkappa$ (coregularity, symmetric regularity). This case implies both $k, n \in \{\varkappa \boldsymbol{R}_\varkappa\}$, i.e., both k and n are $\varkappa$-coregular. Coregularity pertains to completely regular states ⑤⑥⑧⑨, but is most useful regarding symmetric semidivisorship ⑨.

Lemma R5.1: For $\varkappa = 1$, $\boldsymbol{R}_1 = \{1\}$.

Proof: For $k > 1$, at least 1 prime $p \mid k$, and all primes are coprime to 1, therefore, k such that $k > 1$ is nonregular to 1. ■

Lemma R5.2: For $\varkappa = p$ prime, prime powers comprise $p\boldsymbol{R}_p$.

proof: For $\varkappa = p$ prime, $\boldsymbol{R}_p = \{\, p^\varepsilon : \varepsilon \geq 0 \,\}$, hence $p\boldsymbol{R}_p = \boldsymbol{R}_p \setminus \{1\}$, and all terms are prime powers. ■

Lemma R5.3: For composite $\varkappa$, the first term of $\varkappa\boldsymbol{R}_\varkappa$ is $\varkappa$, while the remaining terms are neither prime powers nor squarefree (i.e., a "tantus" number, $k \in \text{A126706}$).

Proof: The empty product 1 is n-regular for all n because $1 \mid n$. Therefore, the first term in $\varkappa\boldsymbol{R}_\varkappa$ is $\varkappa \times \boldsymbol{R}_\varkappa(1) = \varkappa \times 1$. With $\varkappa \in \text{A120944}$ since $\varkappa$ is by definition squarefree, the sequence $\varkappa\boldsymbol{R}_\varkappa$ begins with squarefree $\varkappa$ followed by numbers k of the form $m\varkappa$, where $m \in \boldsymbol{R}_\varkappa$ and $m > 1$, as consequence of [1.1]. Therefore, aside from the smallest term, k is neither squarefree nor prime power. ■

Corollary R5.4: The second-smallest number k in $\varkappa\boldsymbol{R}_\varkappa$ is clearly the product $p\varkappa$, where $p = \text{LPF}(\varkappa) = \text{A020736}(\varkappa)$, since p is the successor of 1 in $\boldsymbol{R}_\varkappa$.

Theorem R5: The infinite sequence $\varkappa\boldsymbol{R}_\varkappa$, squarefree $\varkappa > 1$, consists of prime powers for prime $\varkappa$, otherwise, the first term is squarefree composite $\varkappa$ followed by tantus numbers (i.e., $k \in \text{A126706}$). Proof supplied by Lemmas R5.2 and R5.3.

SYMMETRIC SEMICOPRIMALITY

Lemma 1.1: Symmetric semicoprimality implies both k and n are composite.

$$k \,\Diamond\Diamond\, n \Rightarrow k \in \text{A2808} \land n \in \text{A2808}. \qquad [1.1]$$

Proof: Let $(k, n) = g$. Since $1 < g < k$ and $g < n$, k belongs to the cototient of n yet neither $k \mid n$ nor $n \mid k$. Since primes p must divide or be coprime to other numbers, $k \,\Diamond\Diamond\, n$ is restricted to composite numbers. ■

Lemma 1.2: Symmetric semicoprimality implies both $\omega(k)$ and $\omega(n)$ exceed 1. This is to say that both k and n are not prime powers.

$$k \,\Diamond\Diamond\, n \Rightarrow k \in \text{A024619} \land n \in \text{A024619}. \qquad [1.2]$$

Proof: A number k semicoprime to n is defined as $(k, n) > 1$ yet there exists at least 1 prime q such that $q \mid k$ but $q \nmid n$. Symmetric semicoprimality implies $|\, P \ominus Q \,| > 0$. Since k and n are at least divisible by some common prime p, and since each has at least 1 prime factor q not shared with the other, at least 2 prime factors are implied for both k and n. Hence both have at least 2 distinct prime divisors. ■

Corollary 1.3: Primes and multus numbers (composite prime powers) cannot be symmetrically semicoprime.

Corollary 1.4: Let $(k, n) = g$. The expressions $k/g = u$ and $n/g = v$ imply $(u, v) = 1$ and $u, v > 1$.

LEAN DIVISORSHIP

Lemma 24.1: Let $s = \text{MIN}(k, n)$ and let $t = \text{MAX}(k, n)$. Lean divisorship, an asymmetric semicoprimality, implies both $\omega(s) < \omega(t)$ and $t = ms$ such that integer $m > 1$.

$$s \,|\Diamond\, t \Rightarrow \omega(s) < \omega(t) \land t = ms : m > 1. \qquad [24.1]$$

Proof: $t \,\Diamond\, s$ implies that t is divisible by $\text{Q} > 1 : (\text{Q}, s) = 1$. $s \mid t$ implies $s < t$ (since $s \neq t$ else $s \mid t \land t \mid s$) and $t = ms$. ■

Corollary 24.2: Lean divisorship, an asymmetric semicoprimality, implies t cannot be a composite prime power.

$$s \,|\Diamond\, t \Rightarrow t \notin \text{A246547}. \qquad [24.2]$$

MIXED NEUTRALITY

Lemma 37.1: Mixed neutrality implies both k and n are composite.

$$k \,\Diamond¦\, n \Rightarrow k \in \text{A2808} \land n \in \text{A2808}. \qquad [37.1]$$

Proof. We have shown $k \,\Diamond\, n$ implies composite k. $n ¦ k$ implies composite n since $1 < (k, n) < n$ by definition of semidivisor $n ¦ k$ as nondivisor regular $n \mid k^\varepsilon : \varepsilon > 1$. Hence k and n are neutral in both directions, while a prime must either divide or be coprime to another number. Therefore both k and n are composite. ■

Lemma 37.2: Mixed neutrality is omega-directional, that is, for $k \,\Diamond\, n$ and $n ¦ k$, i.e., k ③ n, $\omega(k) > \omega(n)$, and for $k ¦ n$ and $n \,\Diamond\, k$, i.e., k ⑦ n, $\omega(k) < \omega(n)$.

$$k \,\Diamond¦\, n \Rightarrow \omega(k) > \omega(n) \land k \,¦\Diamond\, n \Rightarrow \omega(k) < \omega(n) \qquad [37.2]$$

Proof: $k \,\Diamond\, n$ implies that k is divisible by $\text{Q} > 1 : (\text{Q}, n) = 1$, yet n is regular to k, meaning that n is a product of primes $p \mid k$ and no prime $q \nmid k$. Further, n does not divide k, yet $\text{RAD}(n) \mid \text{RAD}(k)$, and it is clear that $\omega(k) > \omega(n)$. The same can be said, reversing the relations, so that for $k \text{¦}\Diamond\, n$, i.e., k ⑦ n, $\omega(k) < \omega(n)$. ■

Corollary 37.3: For k, n such that $k \Diamond\text{¦}\, n$, k cannot be multus. Mixed neutrality and $n = p^{\varepsilon}$ implies $n : p^{(\varepsilon - j)} \mid k \wedge j > 0$.

SYMMETRIC DIVISIBILITY (EQUALITY)

Lemma 5.1. Symmetric divisibility implies $k = n$, positive integers.

Proof. Divisors $k \mid n$ are such that $k \le n$. Yet we have both $k \mid n$ and $n \mid k$, and the only solution is that $k = n$, since all positive numbers divide themselves. ■

MIXED REGULARITY

Lemma 68.1. Let $d = \textsc{min}(k, n)$ and let $m = \textsc{max}(k, n)$. Mixed regularity ⑥⑧ implies $d < m$.

$$d \mid\text{¦}\, m \Rightarrow d < m. \qquad [68.1]$$

Proof. Since $d \mid m$ and since $m \nmid d$, $d < m$. Instead $m \mid d^{\varepsilon} : \varepsilon > 1$. ■

Lemma 68.2. Let $S = \textsc{rad}(s)$ and $T = \textsc{rad}(t)$. Mixed regularity ⑥⑧ implies $S = T$ and $\omega(s) = \omega(t)$.

$$\begin{aligned} s \mid\text{¦}\, t \Rightarrow \text{A7947}(s) &= \text{A7947}(t) \wedge \\ \text{A1221}(s) &= \text{A1221}(t). \end{aligned} \qquad [68.2]$$

Proof. We see that $s \mid t$ and $t \mid s^{\varepsilon} : \varepsilon > 1$. Another way to state the latter is that t is a product of $p \mid s$ and not any $q \nmid s$. Therefore t shares some distinct prime factors p with s. In fact, s must have the same number of distinct prime factors p as does t, since $s \mid t$. Suppose prime $P \mid s$ but not t. Then s would not divide t, instead we have $s \Diamond t$. Now suppose prime $P \mid t$ but not s. Then $t \mid s^{\varepsilon} : \varepsilon > 1$ is impossible, since there is no way to introduce the prime factor P in any power of s. Hence, the squarefree kernels of s and t are identical and thereby the number of distinct prime factors of both numbers is the same. ■

Lemma 68.3. Let $s = \textsc{min}(k, n)$ and let $t = \textsc{max}(k, n)$. Mixed regularity ⑥⑧ implies a richness gradient between s and t such that, though $\omega(s) = \omega(t)$, $\Omega(s) < \Omega(t)$.

$$s \mid\text{¦}\, t \Rightarrow \text{A1222}(s) < \text{A1222}(t). \qquad [68.3]$$

Proof. Consider prime power factors $p^{\delta} \mid s$ and $p^{\varepsilon} \mid t$. We know all primes that divide s also divide t and vice versa. In order for t not to divide s, but $s \mid t$, we must have at least 1 prime p for which $\delta < \varepsilon$. ■

Corollary 68.4. $s \mid\text{¦}\, t$ implies t is not squarefree.

$$s \mid\text{¦}\, t \Rightarrow t \in \text{A013929}. \qquad [68.4]$$

SYMMETRIC SEMIDIVISORSHIP

Another way to approach symmetric semidivisors is to consider 2 coregular numbers k and n (that have the same squarefree kernel $\varkappa$) absent divisibility, i.e., both $k \nmid n$ and $n \nmid k$.

Theorem 9.1: Symmetric semidivisibility ⑨ implies $\textsc{rad}(k) = \textsc{rad}(n) = \varkappa$ and $\omega(k) = \omega(n)$.

$$\begin{aligned} k \text{¦¦}\, n \Rightarrow \text{A1221}(k) &= \text{A1221}(n) \wedge \\ \text{A7947}(k) &= \text{A7947}(n) = \varkappa \end{aligned} \qquad [9.1]$$

Proof: $k \text{¦}\, n$ is defined as $k \mid n^{\varepsilon} : \varepsilon > 1$, thus, k must not be a product of primes that do not divide n, otherwise k could not divide any power $n^{\varepsilon} : \varepsilon > 1$ at all. Since we also have $n \text{¦}\, k$, it is clear that primes $p \mid k$ also divide n, hence the set of distinct prime divisors of k are those of n, and the product of the set is the same. Therefore $\textsc{rad}(k) = \textsc{rad}(n) = \varkappa$. Furthermore this set has the same cardinality, hence $\omega(k) = \omega(n)$. ■

Corollary 9.2: Symmetric semidivisibility ⑨ implies $p \mid k$ if and only if $p \mid n$ and $p \mid \varkappa$. Conversely, if $(q, k) = 1$, then $(q, n) = (q, \varkappa) = 1$.

Lemma 9.3: Consider prime power factors $p^{\delta} \mid k$ and $p^{\varepsilon} \mid n$. Symmetric semidivisibility ⑨ implies $\delta \ne \varepsilon$ for at least 2 distinct primes that divide both k and n.

$$k \text{¦¦}\, n \Rightarrow |\, P \ominus Q \,| > 0. \qquad [9.2]$$

Proof. Theorem 9.1 shows that k and n share the same squarefree kernel $\varkappa$, hence every prime $p \mid k$ also divides n. If there were no further differences between k and n, then we would have $k \,||\, n$, therefore $k = n$ (state ⑤). Multiplicatively, the only way k and n differ, having the same squarefree kernel $\varkappa$, is if multiplicities are as follows. Let us also consider prime power factors $p^{\alpha} \mid k$ and $p^{\beta} \mid n$, and $q^{\delta} \mid k$ and $q^{\varepsilon} \mid n$, $p \ne q$. We must have $\alpha < \beta$ and $\delta > \varepsilon$ so that k does not divide n (thus state ⑥) and n does not divide k (thus state ⑧). If we had only 1 prime (say, p) such that the multiplicities pertaining to k and n were equal, we would again have asymmetric divisibility (mixed regularity). Hence, at least 2 prime power factors must have different multiplicities in k and n so as to have symmetric semidivisibility. ■

Corollary 9.4: Let $(k, n) = m\varkappa = g$, $m \ge 1$. The expressions $k/g = u$ and $n/g = v$ imply $(u, v) = 1$ and $u, v > 1$. Furthermore, $uv = \varkappa$ for $\omega(\varkappa) = 2$ and for $\omega(\varkappa) > 2$, $\textsc{rad}(uv) \mid \varkappa$.

Corollary 9.5: Let prime $p = \textsc{lpf}(n)$ and q be the second smallest prime divisor of n. For varius $\varkappa \in \text{A120944}$, the smallest case of $k \text{¦¦}\, n$ is given by $\{\, k = \varkappa p,\ n = \varkappa q \,\}$.

Corollary 9.6. Symmetric semidivisibility ⑨ implies k such that $p\varkappa \le k < (n - \varkappa)$, i.e., $\textsc{lpf}(u) \le u < (v - 1)$.

$$k \text{¦¦}\, n \Rightarrow p\varkappa \le k < (n - \varkappa). \qquad [9.6]$$

Theorem 9.7: Let p^{ε} be the largest power of p such that $p^{\varepsilon} \mid n$. Let $\textsc{rad}(n) = \varkappa$, and let $n/\varkappa = m$. For all $n \in \text{A126706}$ such that $n/\varkappa < q$, i.e., $n \in \text{A360767}$, $a(n) = 0$.

Proof: Consider $n = p^{\delta} q Q$ where p and q are as defined and Q is a product of primes greater than q. Clearly, $n = p^{(\delta - 1)}\varkappa$. Recalling Lemma 2.3, we may divide $\varkappa \boldsymbol{R}_{\varkappa}/\varkappa$ and cancel $\varkappa$ to obtain $\boldsymbol{R}_{\varkappa}$. The first term of $\boldsymbol{R}_{\varkappa}$, i.e, $\boldsymbol{R}_{\varkappa}(1)$, is the empty product 1, followed by $\textsc{lpf}(\varkappa) = p$ and all powers p^{i} such that $i \le \varepsilon$. After p^{ε}, we have q. Hence we have the following power range of p bounded by q:

$$\begin{aligned} P &= \{\, p^{i} : 0 \le i \le \varepsilon \,\}, \\ &= \{\, p^{i} : 0 \le i \le \lfloor \log_p q \rfloor \,\} \end{aligned} \qquad [9.7]$$

It is sure that we do not have any interposing products pq, since $pq > q$, yet $p^{\varepsilon} < q$. It is immaterial whether we have multiplicity for q that exceeds 1, since this only makes for larger products in $\boldsymbol{R}_{\varkappa}$. By same token, any larger prime and any multiplicity of these primes that exceeds 1 also only makes larger products that do not interpose amid terms of P. Within P, all terms divide p^{ε}. Therefore, all terms in $\varkappa P$ divide n, leaving $S_n = \varnothing$, thus $a(n) = |\, S_n \,| = 0$. ■

Corollary 9.8: Symmetric semidivisibility ⑨ implies $k \in \text{A360768}$ and $n \in \text{A126706}$, where $\text{A360768} \subset \text{A126706}$.

Lemma 9.9: Symmetric semidivisibility ⑨ implies $|\, k - n \,| \ge 6$.

$$k \text{¦¦}\, n \Rightarrow |\, k - n \,| \ge 6. \qquad [9.9]$$

Proof. Since $\varkappa$ is squarefree and symmetric semidivisibility is completely neutral, $\varkappa$ is furthermore a varius number (i.e., squarefree composite, in A120944). Since the smallest varius number is 6 and given Lemma 9.4, the smallest difference between symmetric semidivisors $|\, k - n \,| = 6$. ■

<u>GENERAL COTOTIENT</u>

Theorem 10: For $k > 1$ and $n > 1$, $k \parallel n$ or $k \between n$ implies $n \sqcup k$.
Proof. For $k > 1$ and $n > 1$, $k \parallel n$ implies $\mathrm{RAD}(k) \mid \mathrm{RAD}(n)$ hence $(k, n) > 1$. Likewise, $k \between n$ implies $P(k) \cup P(n) \neq \varnothing$, hence $(k, n) > 1$. The definition of $n \sqcup k$ is $(k, n) > 1$. ■

Summary of the Partition of the Cototient.

We may partition $\{\, \mathbb{N} \setminus \{1\} \,\} \ni k$ by $(k, n) = 1$ into the following:

$T_n = \{\, k : (k, n) = 1 \,\}$, a state we call coprimality,
$\Theta_n = \{\, k : (k, n) \neq 1 \,\}$, a state we call the cototient.

Consider $\mathrm{RRS}(n) = \{\, k : (k, n) = 1 \wedge k < n \,\}$ as the reduced residue system of n. Then $|\, \mathrm{RRS}(n) \,| = \phi(n)$, the Euler totient function. The formula for the totient function appears below:

$$\phi(n) = \prod_{p \mid n} (1 - 1/p). \qquad [3.1]$$

From this formula, it is plain to see that we need only consider $\mathrm{RAD}(n) = \varkappa$ because multiplicity of any prime power factor of n or k is immaterial. Hence we hereinafter write T_n instead as $T_\varkappa$.

Consider the residues mod n that are multiples of $p \mid n$:

$$\{\, mp : p \mid n \wedge 1 \leq m \leq n/p \,\}. \qquad [3.2]$$

We can construct $\mathrm{RRS}(n)$ thus:

$$\mathrm{RRS}(n) = \{\, k = 0 \ldots n{-}1 \,\} \setminus \{\, mp : p \mid n \wedge 1 \leq m \leq n/p \,\}. \qquad [3.3]$$

It is well-known that we may also express $T_\varkappa$ as

$$T_\varkappa = \{\, k : (k \bmod n) \in \mathrm{RRS}(\varkappa) \,\}. \qquad [3.4]$$

From this we recognize $|\, T_\varkappa \,| = \aleph_0$, since there are an infinite number of k such that $k \bmod n \equiv t$, where $t \in \mathrm{RRS}(\varkappa)$. Some call $\mathrm{RRS}(n)$ the "totient" of n, extending it to $T_\varkappa$. By this, we arrive at a "cototient" that may be extended to the set $\Theta_\varkappa$ of numbers k such that $k \bmod n \not\equiv t$, where $t \in \mathrm{RRS}(\varkappa)$. Therefore, $|\, \Theta_\varkappa \,| = \aleph_0$ as well, and we can express $\Theta_\varkappa$ as follows:

$$\Theta_\varkappa = \{\, k : (k \bmod n) \notin \mathrm{RRS}(\varkappa) \,\} \qquad [3.5]$$

Define function $P(n) = \{\, p : p \mid n \,\}$,
$\omega(n) = |\, P(n) \,|$, and
$\mathrm{RAD}(n) = \mathrm{A7947}(n) = \prod P(n)$.

We determine precisely 2 species in the cototient:

$R_n = \{\, k : P(k) \subseteq P(n) \,\}$, k is n-regular,
$S_n = \{\, k : P(k) \ominus P(n) \neq \varnothing \,\}$, k is n-semicoprime.

We may also rewrite $P(k) \subseteq P(n)$ as $\mathrm{RAD}(k) \mid \mathrm{RAD}(n)$, thus:

$$R_n = \{\, k : \mathrm{RAD}(k) \mid \mathrm{RAD}(n) \,\}, \qquad [3.6]$$

and it is clear that we need only consider $\mathrm{RAD}(n) = \varkappa$ because multiplicity of any prime power factor of n or k is immaterial to n-regularity. Hence we hereinafter write R_n instead as $R_\varkappa$. Additionally we remark that n-regularity restricts k to a product exclusive of primes q that do not divide n, hence includes the empty product. We may also express this as follows:

$$R_\varkappa = \bigotimes_{p \mid \varkappa} \{\, p^\varepsilon : \varepsilon \geq 0 \,\}. \qquad [3.7]$$

This expression implies $|\, R_\varkappa \,| = \aleph_0$ for $\varkappa > 1$, since $|\, \{p^\varepsilon : \varepsilon \geq 0\} \,| = \aleph_0$. A consequence of restricting k to $p \mid n$ allows the following:

$$R_\varkappa = \{\, k : k \mid n^\varepsilon : \varepsilon \geq 0 \,\}. \qquad [3.8]$$

As regards S_n and the expression $P(k) \ominus P(n) \neq \varnothing$, the latter can be rewritten as follows:

$$0 < |\, P(k) \cap P(n) \,| < |\, P(k) \cup P(n) \,|. \qquad [3.9]$$

Essentially, k is n-semicoprime if and only if k and n share at least 1 prime factor p, but at least 1 prime factor $q \mid k$ yet $q \nmid n$. Such implies that n-semicoprime k is composite, since at minimum, $k = pq$. Furthermore, it is plain to see that multiplicity of any prime power factor of n or k is immaterial to n-semicoprimality. Hence we hereinafter write S_n instead as $S_\varkappa$. Finally, we may construct $S_\varkappa$ with $\varkappa > 1$ as follows:

$$S_\varkappa = \{R_\varkappa \setminus \{1\}\} \otimes \{T_\varkappa \setminus \{1\}\}. \qquad [3.10]$$

This formula implies $|\, S_\varkappa \,| = \aleph_0$ since we have shown that $R_\varkappa$ and $T_\varkappa$ are infinite.

We partition the n-regular numbers $k \in R_\varkappa$ based on divisibility. Define $D_n = \{\, d : d \mid n \,\}$. The divisor counting function $\tau(n)$ is defined as follows:

$$\tau(n) = \prod_{p^\varepsilon \mid n} (\varepsilon{+}1). \qquad [3.11]$$

It is clear from this equation that D_n is finite, and indeed cannot be substituted with $D_\varkappa$ since multiplicity ε is part of the formula. The equation also implies the following formula for D_n:

$$D_n = \bigotimes_{p \mid n} \{\, p^\varepsilon : 0 \leq \varepsilon \leq \delta \,\}, \qquad [3.12]$$

where p^δ is the largest power of p that divides n. The following is a consequence of this expression of D_n:

$$Đ_n = R_n \setminus D_n \qquad [3.13]$$
$$Đ_n = \bigotimes_{p \mid n} \{\, p^\varepsilon : \varepsilon \geq 0 \,\} \setminus \bigotimes_{p \mid n} \{\, p^\varepsilon : 0 \leq \varepsilon \leq \delta \,\}. \qquad [3.14]$$

where $\mathrm{RAD}(n) = \varkappa$. We call $k \in Đ_n$ a "semidivisor" of n. An n-semidivisor is a composite product restricted to primes p such that $p \mid n$, but k itself does not divide n. There exists at least one prime power divisor $p^\varepsilon \mid k$ such that both $p^\delta \mid n$ and $p^\varepsilon > p^\delta$, i.e., $\varepsilon > \delta$. This implies the following:

$$Đ_n = \{\, k : k \mid n^\varepsilon : \varepsilon > 1 \,\}. \qquad [3.15]$$

Figure 1 shows $n = p^4 q^3$ in bold, with a box drawn around the divisors of n, set within $R_\varkappa$ where $\mathrm{RAD}(n) = \varkappa$. It is clear for instance that Mintz [7] exploits the relationship of D_n to $R_\varkappa$ for $\varkappa = 6$. It is also evident that the principles in [7] extend to any squarefree semiprime $\varkappa$.

<u>Figure 1.</u>

⋮							
5	q^5	pq^5	p^2q^5	p^3q^5	p^4q^5	p^5q^5	
4	q^4	pq^4	p^2q^4	p^3q^4	p^4q^4	p^5q^4	
3	q^3	pq^3	p^2q^3	p^3q^3	$\boldsymbol{p^4q^3}$	p^5q^3	
2	q^2	pq^2	p^2q^2	p^3q^2	p^4q^2	p^5q^2	
1	q	pq	p^2q	p^3q	p^4q	p^5q	
0	1	p	p^2	p^3	p^4	p^5	
	0	1	2	3	4	5	…

Hence the n-semidivisor and the n-divisor are 2 species of n-regular numbers $R_\varkappa = R_n$, $\mathrm{RAD}(n) = \varkappa$.

Let's address the empty product, 1, which is coprime to all n and also divides all n, where $n \in \mathbb{N}$. Hence $1 \in T_\varkappa$ and $1 \in D_n$, and since $D_n \subseteq R_\varkappa$, $1 \in R_\varkappa$. Finally, for $n = 1$, a number that is the product of no primes at all, $T_1 = \mathbb{N}$, $R_1 = \{1\}$ since $1 \mid 1$, and $S_n = \varnothing$.

Figure 2 is a diagram of the partition of the cototient and the partition of regular numbers into divisors and semidivisors. The empty product is ignored in this diagram for simplicity and clarity.

Conclusion.

This paper lays out basic parameters of the relation of two sets of primes, P and Q, and therefore their products, k and n, respectively, using elementary number theory.

We have described three basic relations: coprimality, regularity, and semicoprimality. These govern three infinite subsets of the natural numbers with regard to a number $n > 1$. For $n = 1$, we only have 1 regular to 1, and all natural numbers are coprime to 1.

We have partitioned the cototient into 2 species; regularity and semicoprimality, and we have partitioned regularity into divisors and semidivisors. Coprimality and regularity are distinct except as regards the empty product 1, which is at once coprime to all natural numbers, as well as a divisor of and hence regular to all natural numbers.

Along with prime numbers, we have divided the composites into subsets based on the number of distinct prime factors, and whether or not these numbers are squarefree. These are the multus, varius, tantus, and plenus numbers, which serve as aids toward constitutive analysis on account of limiting the number of distinct prime factors and the multiplicity of any prime power factors.

We described the nature of six basic relationships in the cototient, and proved certain aspects of these relationships.

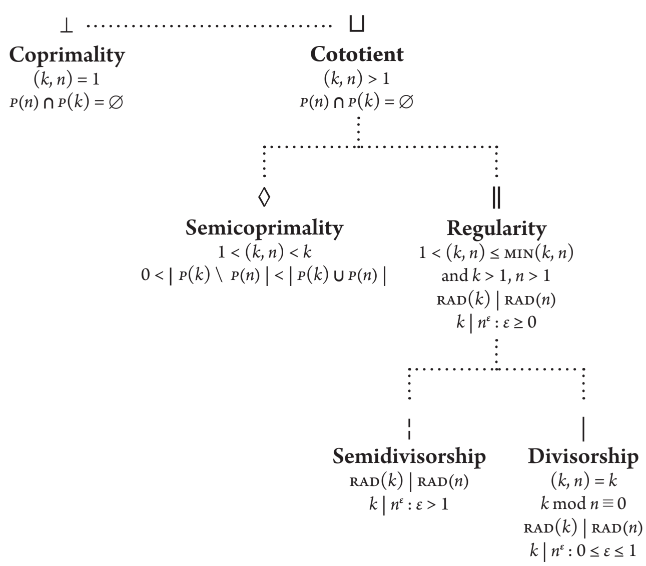

Figure 2: Diagram of the partition of the cototient into n-semicoprimality and n-regularity, and partition of n-regularity into n-semidivisorship and n-divisorship. From the "free ends" of the partitions of the cototient, and considering coprimality, we derive eight principal relational states between k and n. Semicoprimality regarding n, being neither coprimality nor n-divisorship, is n-neutral, along with n-regularity, hence these relations are the province of composites.
We note that the empty product $k = 1$ presents a special case as both n-regular and coprime to n, hence not technically of the cototient of n, though $1 \mid n$ and is hence n-regular.

Appendix Table 1.

The following chart proves handy in creating individual tests for each constitutive state. Note well: we generally deem either $k = 1$ or $n = 1$ to be state ⓪, and $k = n = 1$ to be state ⑤, if not in separate states entirely that do not appear in this chart.

State	Tests	Description
⓪	$(k, n) = 1$.	Coprimality.
①	$k, n \in$ A024619 $\wedge$ $k \neq n \wedge$ $(k, n) > 1 \wedge$ $\mathrm{rad}(k) \nmid \mathrm{rad}(n) \wedge$ $\mathrm{rad}(n) \nmid \mathrm{rad}(k)$.	Symmetric semicoprimality. Non prime powers that are unequal, in cototient, absent divisibility among their squarefree kernels.
②	$k \in$ A024619 $\wedge$ $k > n \wedge$ $n \mid k \wedge$ $\omega(k) > \omega(n)$.	Lean Divisorship. n divides k yet k has a factor $q > 1$ that does not divide n.
③	$k \in$ A024619 $\wedge$ $n \in$ A2808 $\wedge$ $k \neq n \wedge$ $\mathrm{rad}(n) \mid \mathrm{rad}(k) \wedge$ $\omega(k) > \omega(n)$.	Mixed Neutrality. Both k and n are composite and unequal, but k is not a prime power. k has more distinct prime factors than n, and the squarefree kernel of n divides that of k.
④	$n \in$ A024619 $\wedge$ $k < n \wedge$ $k \mid n \wedge$ $\omega(k) < \omega(n)$.	Lean Divisorship. k divides n yet n has a factor $q > 1$ that does not divide k.
⑤	$k = n$.	Equality.
⑥	$n \in$ A013929 $\wedge$ $k < n \wedge$ $k \mid n \wedge$ $\mathrm{rad}(k) = \mathrm{rad}(n) = \varkappa$.	Mixed Regularity. $k < n$, k divides n, n not squarefree, and shares the squarefree kernel of n.
⑦	$k \in$ A2808 $\wedge$ $n \in$ A024619 $\wedge$ $k \neq n \wedge$ $\mathrm{rad}(k) \mid \mathrm{rad}(n) \wedge$ $\omega(k) < \omega(n)$.	Mixed Neutrality. Both k and n are composite and unequal, but n is not a prime power. n has more distinct prime factors than k, and the squarefree kernel of k divides that of n.
⑧	$k \in$ A013929 $\wedge$ $k > n \wedge$ $n \mid k \wedge$ $\mathrm{rad}(k) = \mathrm{rad}(n) = \varkappa$.	Mixed Regularity. $k > n$, n divides k, k not squarefree, and shares the squarefree kernel of k.
⑨	$k, n \in$ A126706 $\wedge$ $k \neq n \wedge$ $k \nmid n \wedge$ $n \nmid k \wedge$ $\mathrm{rad}(k) = \mathrm{rad}(n) = \varkappa$.	Symmetric Semidivisibility. (Coregularity absent divisibility) k and n have the same squarefree kernel, absent divisibility.

Note 1.

Handling the emergence of the empty product 1 may necessitate separate states as follows:

⑩ $k = n = 1$. Usually rendered as k ⑤ n.

⑪ $k = 1 \wedge n > 1$. Usually rendered as k ⓪ n.

⑫ $k > 1 \wedge n = 1$. Usually rendered as k ⓪ n.

These states accommodate the property of 1 both as divisor of n and coprime to n.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved November 2022.

[2] Ronald Graham, Donald Knuth, and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley Professional, 2nd ed., 1994. See p. 115.

[3] Øystein Ore, *Number Theory and Its History*. Mineola, NY: Dover, 1988. Chapter 4, "Prime Numbers", page 50, specifically Lemma 4-1.

[4] William J. LeVeque, *Elementary Theory of Numbers*. Mineola, NY: Dover, 1990. Chapter 2, "The Euclidean Algorithm and Its Consequences", Section 2–2, "The Euclidean algorithm and greatest common divisor", page 24.

[5] Michael Thomas De Vlieger, Omega multiplicity classes, *Simple Sequence Analysis*, 20220518.

[6] Michael Thomas De Vlieger, Constitutive Relations, *Simple Sequence Analysis*, 20211122.

[7] Donald J. Mintz, 2,3 sequence as a binary mixture, *Fibonacci Quarterly*, Vol. 19, №. 4, Oct 1981, 351-360.

[8] Natalia da Silva, Serban Raianu, and Hector Salgado, Differences of Harmonic Numbers and the *abc*-Conjecture, arXiv:1708.00620 [math.NT], 2017.

[9] Raphael Schumacher, The Formulas for the Distribution of the 3-Smooth, 5-Smooth, 7-Smooth and all other Smooth Numbers, arXiv:1608.06928 [math.NT], 2016.

Code:

[c1] Program assigns constitutive states to a pair of positive numbers:

```
conState[j_, k_] :=
 Which[j == k, 5, GCD[j, k] == 1, 0, True,
  1 + FromDigits[
    Map[Which[Mod[##] == 0, 1,
      PowerMod[#1, #2, #2] == 0, 2, True,
         0] & @@ # &, Permutations[{k, j}]], 3]]
```

[c2] Generate powerful, multus, varius, tantus, and plenus numbers:

```
a1694 = With[{nn = 2^40},
   Union@ Flatten@
     Table[a^2*b^3, {b, nn^(1/3)}, {a,
       Sqrt[nn/b^3]}]] (* Powerful *);
a246547 = Select[a1694, PrimePowerQ] (* Multus *);
a286708 = Rest@ Select[a1694, Not@*PrimePowerQ]
  (* Plenus *);
a126706 = Block[{k}, k = 0;
   Reap[Monitor[Do[
       If[And[#2 > 1, #1 != #2] & @@
         {PrimeOmega[n], PrimeNu[n]},
        Sow[n]; Set[k, n] ],
       {n, 2^21}], n]][[-1, -1]]] (* Tantus *);
a120944 = Block[{k}, k = 0;
   Reap[Monitor[Do[
       If[And[CompositeQ[n], SquareFreeQ[n]], Sow[n];
         Set[k, n] ],
       {n, 2^21}], n]][[-1, -1]]] (* Varius *);
```

[c3] Tests designed for states except ⓪ and ⑤:

```
state1[j_, k_] :=
 Which[j == k, False, GCD[j, k] == 1, False,
  AnyTrue[{j, k}, PrimePowerQ], False, True,
  Nor[Divisible[#1, #2], Divisible[#2, #1]] & @@
   Map[Times @@ FactorInteger[#][[All, 1]] &, {j, k}]]

state2[j_, k_] :=
 And[! PrimePowerQ[j], Divisible[j, k], PrimeNu[j] >
PrimeNu[k]]

state3[j_, k_] :=
 Which[j == k, False, GCD[j, k] == 1, False, AnyTrue[{j,
k}, PrimeQ],
  False, PrimePowerQ[j], False, True,
  And[PrimeNu[j] > PrimeNu[k], Divisible[#1, #2]] & @@
   Map[Times @@ FactorInteger[#][[All, 1]] &, {j, k}]]

state4[j_, k_] :=
 And[! PrimePowerQ[k], Divisible[k, j], PrimeNu[j] <
PrimeNu[k]]

state6[j_, k_] :=
 Which[j == k, False, GCD[j, k] == 1, False, SquareFre-
eQ[k], False,
  True, And[Divisible[k, j], Divisible[#1, #2]] & @@
   Map[Times @@ FactorInteger[#][[All, 1]] &, {j, k}]]

state7[j_, k_] :=
 Which[j == k, False, GCD[j, k] == 1, False, AnyTrue[{j,
k}, PrimeQ],
  False, PrimePowerQ[k], False, True,
  And[PrimeNu[j] < PrimeNu[k], Divisible[#2, #1]] & @@
   Map[Times @@ FactorInteger[#][[All, 1]] &, {j, k}]]

state8[j_, k_] :=
 Which[j == k, False, GCD[j, k] == 1, False, SquareFre-
eQ[j], False,
  True, And[Divisible[j, k], Divisible[#2, #1]] & @@
   Map[Times @@ FactorInteger[#][[All, 1]] &, {j, k}]]

state9[j_, k_] :=
 Which[j == k, False, GCD[j, k] == 1, False,
   AnyTrue[{j, k}, Or[PrimePowerQ[#], SquareFreeQ[#]]
&], False, True,
  And[Nor[Divisible[j, k], Divisible[k, j]], #1 == #2]
& @@
   Map[Times @@ FactorInteger[#][[All, 1]] &, {j, k}]]
```

Concerns sequences:

A000040: Prime numbers.
A000961: Prime powers.
A001221: Number of distinct prime divisors of n, $\omega(n)$.
A001694: Powerful numbers.
A002808: Composite numbers.
A003586: Numbers of the form $2^i \times 3^j$, $i \ge 0$, $j \ge 0$.
A003592: Numbers of the form $2^i \times 5^j$, $i \ge 0$, $j \ge 0$.
A005117: Squarefree numbers.
A007947: Squarefree kernel of n; $\mathrm{RAD}(n)$.
A013929: Numbers that are not squarefree.
A024619: Numbers that are not prime powers.
A120944: “Varius” numbers; squarefree composites.
A126706: “Tantus” numbers neither prime power nor squarefree.
A246547: “Multus” numbers; composite prime powers $p^\varepsilon : \varepsilon \ge 1$.
A286708: “Plenus” numbers, products of multus numbers.

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This work is dedicated to Mrs. Betty Svitek, junior high mathematics teacher at Holy Family School on Larkin at Ingalls Avenues in Joliet, IL in the 1970s and 1980s. She had encouraged me through mathematics enrichment. Therefore this work is offered in gratitude for her teaching and encouragement. Mrs. Svitek passed away in the nineties.

The constitutive states described in this paper should be called “Svitek states” in her honor. Therefore, constitutive state ① should be called “Svitek-1”, and a constitutive state function should be called the Svitek function, etc. The term “De Vlieger states”, etc., if ever considered, should never be used. Instead the constitutive states should be named for Mrs. Svitek.

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2023 0125: Draft 1.
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2023 0210: Add Figure 2.
2023 0215: Add summary of the partition of the cototient.
2023 0301: Add Appendix Table 1, Note 1, [c3].
2023 0309: Corrected errata.
2023 0401: Add symmetric semidivisor studies.
2023 1025: Integration of systemic nomenclature.