# Notes on A357910

## Extending OEIS A019565 to create a permutation of natural numbers.

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Abstract.

We introduce a permutation of natural numbers based on A019565, which itself is a permutation of squarefree numbers.


Introduction.

Marc LeBrun wrote OEIS A019565 in 1996, a permutation of the squarefree numbers ordered lexically according to prime decomposition in decreasing order of magnitude.

We may express a number $n$ in binary as a sum of powers of 2:

$$\mathrm{B}(n) = \sum_{\varepsilon \in \mathcal{B}} 2^{\varepsilon} \qquad [1.1]$$

where $\mathcal{B}$ is the set of exponents $\varepsilon$ corresponding to the place-values of bits in the binary expansion.

Another way to look at this sequence is as the mapping of the following function $f(n)$ across the natural numbers:

$$a(n) = f(n)$$
$$f(n) = \mathrm{B}(n) \Rightarrow \sum_{\varepsilon \in \mathcal{B}} 2^{\varepsilon} \Rightarrow \prod_{\varepsilon \in \mathcal{B}} p_{(\varepsilon+1)} \qquad [1.2]$$

Hence, we have the following construction of the first terms:

$a(0) = 1$ since 1 is the empty product.
$a(1) = 2$ since $\mathrm{B}(1) = 1_2 = 2^0 \Rightarrow p_1 = 2$.
$a(2) = 3$ since $\mathrm{B}(2) = 10_2 = 2^1 \Rightarrow p_2 = 3$.
$a(3) = 6$ since $\mathrm{B}(3) = 11_2 = 2^1+2^0 \Rightarrow p_1\, p_2 = 2 \times 3 = 6$.
$a(4) = 5$ since $\mathrm{B}(4) = 100_2 = 2^2 \Rightarrow p_3 = 5$.
$a(5) = 10$ since $\mathrm{B}(5) = 101_2 = 2^2+2^0 \Rightarrow p_1\, p_3 = 2 \times 5 = 10$.
$a(6) = 15$ since $\mathrm{B}(6) = 110_2 = 2^2+2^1 \Rightarrow p_2\, p_3 = 3 \times 5 = 15$.
$a(7) = 30$ since $\mathrm{B}(7) = 111_2 = 2^2+2^1+2^0 \Rightarrow p_1\, p_2\, p_3 = 2 \times 3 \times 5 = 30$.
$a(8) = 7$ since $\mathrm{B}(8) = 1000_2 = 2^3 \Rightarrow p_4 = 7$, etc.

The sequence begins as follows:

```
1
2
3   6
5  10  15  30
7  14  21  42  35  70  105  210
...
```

It is clear that all numbers in A019565 are squarefree, since the exponents $\varepsilon$ in $\mathrm{B}(n) = \mathcal{B}$ are distinct, thus the primes in the product in [1.2] are likewise distinct.

Also evident are the following formulas:

$$a(2^n) = p_{(n+1)}. \qquad [1.3]$$

$$a(2^n-1) = P(n) = \prod_{j=1}^{n} p_j = \mathrm{A2110}(n). \qquad [1.4]$$

$$\text{For } \varkappa \in a(2^{(n-1)} \dots 2^n-1),\ \mathrm{GPF}(\varkappa) = p_n = \mathrm{PRIME}(n) \qquad [1.5]$$

Therefore, using powers $2^{(n-1)}$, it is clear that we may partition the sequence into rows. We consider sequence A019565 as an irregular triangle with rows ordered according to $\mathrm{B}(n)$ as follows:

$S(0) = \{1\}$, and for $n > 0$,
$S(n) \ni \varkappa : \varkappa = \mathrm{RAD}(\varkappa) = \mathrm{A7947}(\varkappa)\ \wedge$
$\mathrm{PRIME}(n) \le \varkappa < \mathrm{A2110}(n)\ \wedge$
$\mathrm{GPF}(\varkappa) = \mathrm{PRIME}(n)$.

The ordering of the elements of $S(n)$ is such that it occurs according to the so-called Heinz number. Seen this way, $\varkappa \in S(n)$ has $n$ bits, with the most significant a 1. Then $S(n)$ contains all the permutations of bits in such a number. Therefore, row $S(3)$ contains the following:

| $k$ | $\varkappa$ | factors | Heinz | decimal |
|---|---|---|---|---|
| 0 | 5 | 5 | 100 | 4 |
| 1 | 10 | 5 × 2 | 101 | 5 |
| 2 | 15 | 5 × 3 | 110 | 6 |
| 3 | 30 | 5 × 3 × 2 | 111 | 7 |

The ordering is according to the decimal equivalent of the Heinz number that encodes the factors of $\varkappa$. This happens to be the very ordering of $\varkappa$ as to magnitude for $S(3)$. For $S(4)$, we have a different ordering:

7, 14, 21, 42, 35, 70, 105, 210

Here of course, according to magnitude, 42 and 35 are transposed. But according to the Heinz number, these are in order from 8 to 15.

We can write a formula using the row $n$ and column $k$:

$$S(n, k) = f(2^n+k) \qquad [1.6]$$

Therefore A019565 represents the infinite catenation of these sets $S$, and thus constitutes a permutation of squarefree numbers.

A Related Permutation of Natural Numbers.

We now propose a similar sequence A357910 that is a permutation of natural numbers. We create sets $T(n)$ that are based on $S(n)$, hence A357910 is an irregular triangle that comprises these sets $T$.

Define function $c(k) = \mathrm{TRUE}$ iff $k \in T(j) : j < n$, else FALSE. It turns out we can elide this function, but we might use it as a failsafe when first programming a solution.

Set $m(\varkappa) = 1$, a modifier for squarefree $\varkappa$.

Define function $g(\varkappa)$ as follows:

$$k \in \mathrm{A961} \Rightarrow k^{m(\varkappa)}\ ;\ m(\varkappa){+}{+}, \qquad [\mathrm{A1}]$$

$$k \in \mathrm{A024619} \Rightarrow k \times m(\varkappa) :$$
$$\mathrm{RAD}(m(\varkappa)) \mid \mathrm{RAD}(k);\ m(\varkappa){+}{+} \qquad [\mathrm{A2}]$$

We note that after applying $m(\varkappa)$, we increment such that upon the next occasion of $\varkappa$, the modifier is larger than the one just applied.

Then we can describe the sequence as follows:

$T(0) = \{1\}$, and for $n > 0$,
$T(n) = g \mapsto S(1 \dots n)$.

This is to say, row $T(n)$ involves the mapping of $f$ across $S(0 \dots n)$. Thus, when we have (squarefree) $\varkappa$, upon $\neg c(\varkappa)$, $\varkappa$ appears in $T$, else we have $k^{m(\varkappa)}$ if $\varkappa$ prime, else $\varkappa \times m(\varkappa) : \mathrm{RAD}(m(\varkappa)) \mid \varkappa$. This has the effect of having $2^k \le k < \mathrm{A2110}(n) \wedge \mathrm{GPF}(k) = \mathrm{PRIME}(n)$.

We have rows as follows:

```
 1
 2
 4   3   6
 8   9  12   5  10  15  30
16  27  18  25  20  45  60   7  14  21  42  35  70  105  210
...
```

It is clear that we are generating nonsquarefree terms through transforming $S(1 \dots n-1)$ and prepending them to $S(n)$.

It is clear that column $T(\_, k) = \varkappa\, \boldsymbol{R}_{\varkappa}$, the strongly $\varkappa$-regular numbers [2]. For instance, for $k = 1$, we have A79; for $k = 3$ we have $\{6 \times \mathrm{A3586}\}$, and for $k = 5$, we have $\{10 \times \mathrm{A3592}\}$. Generally, for $k = 2^{\varepsilon}$ and $p = \mathrm{PRIME}(\varepsilon-1)$, we have $p\, \boldsymbol{R}_p = \{\, p^{\delta} : \delta > 0 \,\}$.

Looking at incomplete rows (just the first 7 terms) we have the following irregular triangle:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 2 | | | | | | | |
| 4 | 3 | 6 | | | | | |
| 8 | 9 | 12 | 5 | 10 | 15 | 30 | |
| 16 | 27 | 18 | 25 | 20 | 45 | 60 | ... |
| 32 | 81 | 24 | 125 | 40 | 75 | 90 | ... |
| 64 | 243 | 36 | 625 | 50 | 135 | 120 | ... |
| 128 | 729 | 48 | 3125 | 80 | 225 | 150 | ... |
| 256 | 2187 | 54 | 15625 | 100 | 375 | 180 | ... |
| 512 | 6561 | 72 | 78125 | 160 | 405 | 240 | ... |
| 1024 | 19683 | 96 | 390625 | 200 | 675 | 270 | ... |
| 2048 | 59049 | 108 | 1953125 | 250 | 1125 | 300 | ... |

## (Weakly) and Strongly ϰ-Regular Numbers.

We recognize $\varkappa \in S(n)$ is squarefree, i.e., $\varkappa \in$ A5117. Let's define a couple sets related to squarefree $\varkappa$. The first is the finite set of distinct prime divisors of $\varkappa$:

$$\mathrm{P}(\varkappa) = \{\, p : p \mid \varkappa \,\} \qquad [2.1]$$

The second is an infinite set $\boldsymbol{R}_\varkappa$ of numbers $k$ such that $p \mid k$ and $k$ is indivisible by primes $q$ that are coprime to $\varkappa$:

$$\boldsymbol{R}_\varkappa = \{\, k : \mathrm{P}(n) \subseteq \mathrm{P}(k) \,\} = \bigotimes_{p \mid \varkappa} \{p^\varepsilon : \varepsilon \geq 0\}, \qquad [2.2]$$

It is clear by the above definitions that $r \in \boldsymbol{R}_\varkappa$ implies $\mathrm{RAD}(r) \mid \varkappa$.

Examples:

$$\begin{aligned} \boldsymbol{R}_6 &= \bigotimes_{p \mid 6} \{p^\varepsilon : \varepsilon \geq 0\} \\ &= \{2^\delta : \delta \geq 0\} \otimes \{3^\varepsilon : \varepsilon \geq 0\} \\ &= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, \ldots\} \\ &= \text{A3586}. \\ \boldsymbol{R}_{10} &= \bigotimes_{p \mid 10} \{p^\varepsilon : \varepsilon \geq 0\} \\ &= \{2^\delta : \delta \geq 0\} \otimes \{5^\varepsilon : \varepsilon \geq 0\} \\ &= \{1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, \ldots\} \\ &= \text{A3592}. \end{aligned}$$

Then strongly $\varkappa$-regular numbers $\mathrm{R} \in \varkappa\boldsymbol{R}_\varkappa$ are such that $\mathrm{RAD}(\mathrm{R}) = \varkappa$.

$$\begin{aligned} 6\boldsymbol{R}_6 &= \bigotimes_{p \mid 6} \{p^\varepsilon : \varepsilon \geq 0\} \times 6 \\ &= \{2^\delta : \delta \geq 0\} \otimes \{3^\varepsilon : \varepsilon \geq 0\} \times 6 \\ &= \{6, 12, 18, 24, 36, 48, 54, 72, 96, 108, \ldots\} \\ &= 6 \times \{\text{A3586}\}. \end{aligned}$$

It is evident that multiplication of $\boldsymbol{R}_\varkappa$ by $\varkappa$ guarantees $\varkappa \mid \mathrm{R}$ for all $\mathrm{R}$.

Lemma 1.1. $T(n, 0) = 2^n$.

Proof. A consequence of applying $g \mapsto S(0 \ldots n)$, we have $g(S(1,0)) = g(2)$, and since 2 is prime, through [A1] we have $2^{m(2)}$, with $m(2)$ incrementing as $n$ increments. Therefore we produce $T(n, 0) = 2^n$. ■

Lemma 1.2. $T(n, 2^{(n-1)}) = \mathrm{PRIME}(n)$.

Proof. $T(n, 2^{(n-1)}) = S(n{+}1, 0) = f(2^{n+1}) = \mathrm{PRIME}(n)$. ■

Lemma 1.3. For $j = 1 \ldots j$. $T(n, 2^{(n-j-1)}) = \mathrm{PRIME}(n{-}j)^j$.

Proof. Lemma 1.2 gives $T(n, 2^{(n-1)}) = S(n{+}1, 0) = f(2^{n+1}) = \mathrm{PRIME}(n)$. This is so, because $g \circ f(2^{n+1}) = f(2^{n+1})^m$ for $m = 1$. through [A1]. For $T(n{+}1, 2^{(n-1)})$, we have $g \circ f(2^{n+1}) = f(2^{n+1})^m$ for $m = 2$, generally for $T(n{+}i, 2^{(n-1)})$, we have $g \circ f(2^{n+1}) = f(2^{n+1})^m$ for $m = i{+}1$ and we prove the proposition through induction. ■

Corollary 1.4. $T(n, 2^\varepsilon{-}1) = \{p^\delta : p = \mathrm{PRIME}(\varepsilon) \wedge \delta > 0\}$.

Theorem 1. Let $\varkappa =$ A019565$(k)$. Column $k = \varkappa\, \boldsymbol{R}_\varkappa$, the strongly $\varkappa$-regular numbers $\mathrm{R}$.

Proof. Corollary 1.4 covers the case of [A1] to yield a prime power range, $p\, \boldsymbol{R}_p$ that, for $p = 2$, is complete, but for odd primes is missing the empty product. Generally, we introduce $\varkappa$ according to its position in A019565, a permutation of A5117. Indeed, $\varkappa$ such that $\mathrm{GPF}(\varkappa) = \mathrm{PRIME}(n)$ appears at $S(n, k)$. Therefore, as $n$ increases, we advance $m(\varkappa)$ as described in [A1] or [A2] in the mapping $g \mapsto S(0 \ldots n)$. Prime $\varkappa = p$ implies $T(n, 2^\varepsilon{-}1) = p^{m(p)}$, with $m(p)$ incrementing as $n$ increments. In other cases we have a squarefree composite $\varkappa$ constructed by function $f$ and modified by function $g$ in a manner similar to Lemma 1.3, hence we have $\varkappa \times m(\varkappa)$. In this manner, through induction, we create $\varkappa\, \boldsymbol{R}_\varkappa$ for $\varkappa$ that appears at the head of each column.

From these, it is evident that A357910 is a permutation of natural numbers.

## Conclusion.

This permutation of natural numbers, generated from a permutation of squarefree numbers $\varkappa$, has the interesting property of generating columns that list the strongly $\varkappa$-regular numbers in order.

## References:


[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved November 2022.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.


## Code:

[C1] Generate 20 rows of A357910:

```
nn = 20;
rad[n_] := rad[n] =
  Times @@ FactorInteger[n][[All, 1]];
  q[_] = 1; r = t[0] = {1};
  Do[Set[s, Join[r, Prime[n]*r]];
    Set[t[n],
     Map[
      If[PrimeQ[#],
        Set[m, #^q[#]]; q[#]++; m,
        Set[m, q[#] #]; q[#]++;
        While[! Divisible[#, rad[q[#]]], q[#]++]; m]&,
      Rest[r]]];
    r = s, {n, nn}];
  {{1}}~Join~Rest@ Array[t, nn]] // Flatten
```

## Concerns sequences:

A002110: Primorials $P(n)$: products of the smallest $n$ primes.

A003586: Numbers of the form $2^i \times 3^j$, $i \geq 0$, $j \geq 0$.

A003592: Numbers of the form $2^i \times 5^j$, $i \geq 0$, $j \geq 0$.

A005117: Squarefree numbers.

A007947: Squarefree kernel of $n$; $\mathrm{RAD}(n)$.

A019565: permutation of the squarefree numbers ordered lexically according to prime decomposition in decreasing order of magnitude

A357910: permutation of the natural numbers ordered lexically according to prime decomposition in decreasing order of magnitude, where columns contain numbers strongly regular to squarefree number that begins each column.

## Document Revision Record:

2023 0127: Version 1