On the Completely Regular Scaling Issue

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Abstract.

This brief aggregates thoughts about the rarity of symmetric regularity in prime-divisor restrictive lexically earliest sequences (PDRLES or pearl sequences).

Introduction.

Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$ and consider primes $p \mid n$ and n-nondivisor primes q. It is clear that, if $q \nmid n$, then $(q, n) = 1$, i.e., $q \perp n$, as primes either divide or be coprime to other numbers. Define an n-regular k to be a product restricted to primes p, indivisible by q. Consequently, n-regularity ascribes to the squarefree kernel A7947(n) = RAD(n) = $\varkappa$, since multiplicity of any prime divisor does not affect whether or not k is regular to n.

Define the set of n-regular numbers $\boldsymbol{R}_\varkappa$ to be the tensor product of prime divisor power ranges $\{ p^\varepsilon : \varepsilon \geq 0 \wedge p \mid \varkappa \}$.

$$\boldsymbol{R}_\varkappa = \bigotimes_{p \mid \varkappa} \{ p^\varepsilon : \varepsilon \geq 0 \wedge p \mid n \} \qquad [1.0]$$

For example, for $n = 12$, thus squarefree kernel $\varkappa = 6$, we have the following:

$$\begin{aligned} \boldsymbol{R}_6 &= \bigotimes_{p \mid 6} \{p^\varepsilon : \varepsilon \geq 0\} \\ &= \{2^\delta : \delta \geq 0\} \otimes \{3^\varepsilon : \varepsilon \geq 0\} \\ &= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, \ldots\} \\ &= \text{A3586}. \end{aligned}$$

Theorem 1. There are 2 species of n-regular k: divisors and nondivisor regular k.

Proof. The definition of n-regular k implies $k \mid n^\varepsilon : \varepsilon \geq 0$. Therefore, we may partition the range of ε into $k \mid n^\varepsilon : \varepsilon = 0 \ldots 1$ which may also be written simply as $k \mid n$, and $k \mid n^\varepsilon : \varepsilon > 1$. The former we call a divisor of n, and the latter we call a semidivisor of n. ■

We may express these two species of n-regular k as follows:

Table A.

$k \mid n$	k divides n	$1 \leq (k, n) = k$	$k \mid n^\varepsilon : \varepsilon = 0 \ldots 1$
$k ¦ n$	k semidivides n	$1 < (k, n) <$ MIN(k, n)	$k \mid n^\varepsilon : \varepsilon > 1$

Corollary 1.1. The number $k = 1$ is regular to $n \in \mathbb{N}$, since 1 divides $n \in \mathbb{N}$.

Corollary 1.2. $k ¦ n$ implies $k \in$ A2808, i.e., composite k, since primes p imply either $p \mid n$ or $p \perp n$. Since semidivisibility is neither coprimality nor divisibility, we say it is n-neutral.

Define $P(n) = \{ p : p \mid n \}$ to be the set of (distinct) prime factors of n.

Lemma 1.3. RAD$(k) \mid$ RAD(n) (alternatively, $P(k) \subseteq P(n)$) implies n-regular k.

Proof. A number k such that $P(k) \subseteq P(n)$ is restricted to the prime factors $p \mid n$. Since $q \notin P(k)$, we see that the number k conforms to the definition of an n-regular number. ■

Corollary 1.4. $P(k) \subset P(n)$ and $k > 1$ imply n is nonregular and not coprime to k. (That is, n is k-semicoprime [2].)

Lemma 1.5. RAD(k) = RAD$(n) = \varkappa$ implies symmetric regularity.

Proof. Both $\varkappa \mid k$ and $\varkappa \mid n$ implies that any prime $p \mid k$ also divides n, and vice versa. There is no way that a prime q that divides one does not divide the other. Therefore, k and n are symmetrically regular. ■

Cases of Symmetric Regularity.

In the case of $k \perp n$ it is clear that coprimality is symmetric. Given the cototient, if $(k, n) > 1$ then through commutative property it is clear that noncoprimality is also symmetric. Through [2] we show that there are 2 species in the cototient, that is, n-regular k as demonstrated, and n-semicoprime k (i.e., $k \lozenge n$) which is described at length in [2]. Corollary 1.4 implies that regularity is not always symmetric; we can have a mixed cototient, meaning that we may have n-regular k, but n itself is nonregular to k, though $(k, n) > 1$.

Taking into account multiplicity and knowing we have 2 species of regularity, i.e., the divisor and the semidivisor, we have the following three possible cases concerning regularity between k and n:

<u>Symmetric Divisibility.</u>

That is, $k \mid n$ and $n \mid k$, hence $k \parallel n$ or k ⑤ n [2, TABLE E] which implies $k = n$, RAD(k) = RAD(n), and $\omega(k) = \omega(n)$.

<u>Mixed Regularity.</u>

That is, $k \mid n$ and $n ¦ k$, hence k |¦ n or k ⑥ n, or the reverse, $k ¦ n$ and $n \mid k$, hence k ¦| n or k ⑧ n. Let d = MIN(k, n) and m = MAX(k, n). These states are completely regular, occurring entirely within $\boldsymbol{R}_\varkappa$, where $\varkappa$ = RAD(m), $d \neq m \neq 1$. State ⑥ confines primes to LHS while state ⑧ confines primes to RHS, and m may not be squarefree. Because of divisibility and inherent inequality, the state is directional. Let $p^a < p^b$ be distinct composite powers of the same prime p; therefore we have the relation p^a |¦ p^b. Hence, $d = p^\varepsilon : \varepsilon \geq 1$ imply d |¦ m and $d < m$. Examples: $6 \mid 12$ and $12 ¦ 6$, $27 ¦ 9$ and $9 \mid 27$.

<u>Symmetric Semidivisibility.</u>

Both $k ¦ n$ and $n ¦ k$, hence k ¦¦ n (i.e., k ⑨ n). This state is symmetrical, completely neutral, and completely regular, occuring within $\varkappa\, \boldsymbol{R}_\varkappa$ where $6 \leq \varkappa$ = RAD(k) = RAD(n) absent divisibility, $k \neq n \neq 1$, and both k and n composite. The state is ambidirectional in magnitude and as to multiplicity, but flat in terms of $\omega(\varkappa)$. State ⑨ implies symmetric difference concerning multiplicities of at least one prime divisor $p \mid \varkappa$, hence both k and n are restricted to tantus numbers (in A126706) such that $|\, k - n \,| \geq \varkappa$ for $\varkappa \geq 6$. Examples: 12 ¦¦ 18, 182 ¦¦ 361.

(See [2] for more information and theorems.)

Therefore we summarize as follows:

‖	|¦ or ¦|	¦¦
Symmetric Divisibility	Mixed Regularity	Symmetric Semidivisibility
⑤	⑥⑧	⑨

Let $k = p^\alpha m q^\delta$, primes $p < q$, $m \geq 1$, and let $n = p^\beta m q^\varepsilon$, with nonzero exponents α, β, δ, and ε. Such a definition implies k and n both composite and not prime powers, since they are at least squarefree semiprimes pq.

Suppose $\alpha > \beta$. Then it is clear that $n \mid k$, though k and n are symmetrically regular. Likewise we might also consider $\delta > \varepsilon$ either alone or independently and conclude the same. If $n \mid k \wedge k \nmid n$, yet , then it is clear that we have nondivisor n-regular k, hence an n-semidivisor k, i.e., $k ¦ n$. Hence we may write k ¦| n or via state notation, k ⑧ n. If we reverse the inequalities, then clearly we have the reverse relation k |¦ n, also known by k ⑥ n. In other words, we have constructed the case of mixed regularity.

Suppose $a = \beta$ and $\delta = \varepsilon$. Then it is obvious we have $k \mid n \wedge n \mid k$, i.e., $k \,||\, n$, also known as k ⑤ n. This is symmetric divisibility, a special case of symmetric regularity, which implies $k = n$, i.e., equality.

Finally, suppose $a > \beta$, but $\delta < \varepsilon$. Then neither k nor n divide the other, though k and n are symmetrically regular. We have a case analogous to semicoprimality in that there is an algebraic symmetric difference among multiplicities regarding at least 1 common prime factor. Therefore k and n are mutual semidivisors, k ¦¦ n, also known by k ⑨ n, and we have a case of symmetric semidivisibility.

From this point on, we will refrain from using the phrase "symmetric regularity" and instead say "completely regular".

The Completely Regular Scaling Issue.

The usual scaling issues seen in lexically earliest sequences (LES) seem to make the completely regular states ⑥⑧⑨ impossible as n increases. Let us define a squarefree kernel as follows:

$$\varkappa = \mathrm{RAD}(n) = \prod_{p|n} p = \mathrm{A7947}(n). \qquad [2.1]$$

Recall the definition of completely regular states outside symmetric divisibility (equality) which is prohibited:

$k \,\|\|\, n$	k |¦ n or k ¦| n	k ¦¦ n
Symmetric Divisibility	Mixed Regularity	Symmetric Semidivisibility
⑤	⑥⑧	⑨

What these states have in common is that $\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$, which implies that, outside of state ⑤ and for k and n that both exceed 1, k and n are distinct elements of the infinite set (or list) $\boldsymbol{R}_\varkappa$ of $\varkappa$-regular numbers. Given the prime decomposition of $\varkappa$, we have the following set-building formula for $\boldsymbol{R}_\varkappa$:

$$\boldsymbol{R}_\varkappa = \bigotimes_{p|\varkappa} \{p^\varepsilon : \varepsilon \geq 0\} \qquad [1.2]$$

Suppose we have the term $a(n) = k = \boldsymbol{R}_\varkappa(v)$ resulting from a given selection axiom A in the middle of an interval $n \pm \eta$ of terms resulting from A. Let $r = \boldsymbol{R}_\varkappa(v-1)$ and $\mathrm{R} = \boldsymbol{R}_\varkappa(v+1)$ such that $r < k < \mathrm{R}$.

The scaling issue has to do with the likelihood of finding r or R not already in the sequence given the circumstance of selection axiom A within the interval $n + \eta$, as n increases. This is dependent on the density of $\boldsymbol{R}_\varkappa$ in the vicinity of k, and the dilation of η as n increases. Usually, it seems that as n increases, $\boldsymbol{R}_\varkappa$ becomes too sparse to furnish solutions for selection axiom A, even for primorials $\varkappa$. Furthermore, symmetric semidivisibility implies k and n both tantus numbers (i.e., numbers neither squarefree nor prime powers, A126706) neither of which divide the other, hence state ⑨ proves rare.

Consequences of the scaling issue for completely regular relations include rarity outside a few early terms if the states appear at all. Normally the terms in completely regular relation have squarefree kernel 6, 10, or 30, for example; small even kernels with small prime factors.

Lemma 2.1: Strongly $\varkappa$-regular $n \in \varkappa\boldsymbol{R}_\varkappa : n > \varkappa$ implies d ⑥ $n : d \leq n/p$ where $p = \mathrm{LPF}(\varkappa)$.

Proof: Consider nontrivial divisors d of a composite number n. Let D be the largest nontrivial divisor of n. Since 2 is the smallest prime, then $\mathrm{D} \leq n/2$. More precisely, if $p = \mathrm{LPF}(n)$, then $\mathrm{D} = n/p$. Furthermore, if $\mathrm{RAD}(\mathrm{D}) \mid \mathrm{RAD}(n) = \varkappa$, then $p = \mathrm{LPF}(\varkappa)$, hence $\mathrm{D} = n/p$.

Therefore in the strongly $\varkappa$-regular numbers $\varkappa\boldsymbol{R}_\varkappa : \mathrm{LPF}(\varkappa) = p$, suppose we select an element $n > \varkappa$. Then for composite n, there is an element $d : d \mid n$, $1 < d < n$, and $d \leq n/p$. We know from theorem that $d : d \mid n$ and $d \in \varkappa\boldsymbol{R}_\varkappa$. All divisors d appear before n in the sequence $\varkappa\boldsymbol{R}_\varkappa$. Therefore, most of the numbers $k \in \varkappa\boldsymbol{R}_\varkappa$ are not such that k ⑥ n, but depending on $n/\varkappa$ as it remains small, we may have saturated k ⑥ n for $k < n$. ■

Lemma 2.2: Strongly $\varkappa$-regular $n \in \varkappa\boldsymbol{R}_\varkappa : n > \varkappa$ implies n ⑧ k such that $k = mn : m \in \varkappa\boldsymbol{R}_\varkappa$ and $m \geq p$ where $p = \mathrm{LPF}(\varkappa)$. ■

Proof: We pursue an argument similar to Lemma 2.1.

Corollary 2.3: For all other $k \in \varkappa\boldsymbol{R}_\varkappa$, we have n ⑨ k.

Theorem 2: Except n itself, $(1/p)n < k < pn$ implies k ⑨ n.

Proof: Consequence of Lemmas 2.1, 2.2, and Corollary 2.3. That is, $n/p < k < n$ implies k ⑨ n and $n < k < pn$ implies k ⑨ n, while n ⑤ n. ■

Here we define $\{k : k \in \varkappa\boldsymbol{R}_\varkappa \wedge n/p < k < pn \wedge k \neq n\}$ to be the set of strongly $\varkappa$-regular numbers k that are "similar" to n (in magnitude). We distinguish $k = n$ as being "the same" or identical, hence, nonsimilar. Therefore we can say that strongly n-regular k similar to n implies symmetric semicoprimality.

Theorem 3: Squarefree $\varkappa$ and $n \in \varkappa\boldsymbol{R}_\varkappa : k > \varkappa$ implies $\varkappa$ ⑥ k.

Proof. Squarefree n implies $n = \varkappa$. Since $\varkappa$ is the minimum of $\varkappa\boldsymbol{R}_\varkappa$, we have $\varkappa$ ⑥ k, because $\varkappa$ divides all elements in $\varkappa\boldsymbol{R}_\varkappa$ by definition. Recall that state ⑨ implies tantus numbers, and that $\varkappa$ is the sole squarefree number in $\varkappa\boldsymbol{R}_\varkappa$. ■

Hence, given tantus i and j "in their vicinity", with same squarefree kernel $\varkappa$, more precisely, within a factor of $\mathrm{LPF}(\varkappa)$, we have state ⑨.

Next Steps.

The next step in this paper is to join the properties of a greedy lexical selection axiom to the nature of $\varkappa\boldsymbol{R}_\varkappa$.

Proof Sketch. What we have to show is that for some sufficiently large k, the cototient axiom selection function can always (or nearly always) find a legal solution $m < n$ where k and n are completely regular. Adjunct to this is the notion that, if k and n are such that they are "similar" in $\varkappa\boldsymbol{R}_\varkappa$, implies state ⑨.

Note: this work derives from thoughts in SA20230119 and SA20230125. These related thoughts appear here so as to attempt to prove what is called the scaling issue, meaning the relation between the cototient axiom selection function and completely regular relations between input k and output n.

Updates will be made available upon further development.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved November 2022.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

Concerns sequences:

A002808: Composite numbers.
A005117: Squarefree numbers.
A007947: Squarefree kernel of n; $\mathrm{RAD}(n)$.
A126706: Numbers neither squarefree nor prime powers.

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