The Ever-Late Sequence.

A sequence of David Sycamore · Windhoek, Namibia.

Michael Thomas De Vlieger · St. Louis, Missouri · 12 February 2023.

Abstract.

We examine a lexically earliest sequence (LES) yielding the least novel multiple of the smallest nondivisor prime of the immediately previous 2 terms. This cursory analysis relies upon concepts laid out in [2].

Introduction.

Consider the sequence $a(1) = 1$, $a(2) = 2$, and for $n > 2$, $a(n)$ is least novel multiple of the smallest prime p which does not divide the product $i \times j$. Logically, we define S20230212 = a as follows:

$a(n) = n$ for $n \le 2$;
Let $i = a(n-2)$, and $j = a(n-1)$.
Define function $c(x)$ to be TRUE iff $a(j) = x$ for $j < n$, else FALSE.
For $n > 2$, $a(n) =$ ⍃$k : mp \land \neg c(k)$, ⍃$p : p \perp ij =$ A053669(ij).

Code generates 2^{24} terms of the sequence in several minutes. The sequence begins as follows:

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1, 2, 3, 5, 4, 6, 10, 7, 9, 8, 15, 14, 11, 12, 20, 21,
22, 25, 18, 28, 30, 33, 35, 16, 24, 40, 42, 44, 45, 49,
26, 27, 50, 56, 36, 55, 63, 32, 60, 70, 66, 13, 65, 34,
39, 75, 38, 77, 48, 80, 84, 88, 85, 51, 46, 90, 91, 99,
52, 95, 54, 98, 100, 57, 105, 58, 110, 69, 112, 115, ...
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As consequence of definition, we expect PRIME(s) to show generally late, since the selection axiom requires RAD$(ij) =$ A2110$(s-1)$.

Scatterplot Features.

The scatterplot features striations associated with LPF$(a(n))$. Primes appear in order. We divide the scatterplot into zones. The zone with least apparent slope is zone β of the primes, which enter late. The second zone γ contains "catch-up" striations γ_p associated with $p^\varepsilon : p > 13$ and $\varepsilon > 1$. Finally, zone α contains 3 prominent apparent quasi-rays. The latest quasiray α_2 pertains to even nontrine $k \equiv \pm 2 \pmod 6$. The second-latest quasiray α_3 contains numbers divisible by 3 and sometimes by 6. The earliest quasiray A is a superposition of α_5 containing k : LPF$(k) = 5$, α_7 containing k : LPF$(k) = 7$, α_{11} containing k : LPF$(k) = 11$. As n increases, γ_p moves closer to merging with the earliest quasiray.

In this way, we might unify the striations into those ς_p that pertain to least prime factor p, with some even multiples of p included in the striation ς_p. Though $\varsigma_p : p > 11$ seems to exclude tantus numbers (i.e., those in A126706), they do emerge as n increases. For instance, for $n < 2^{20}$, $a(3266) = 2873$, and there are 1376 tantus numbers in ς_{13}. $a(54270) = 48841$, and there are 18 plenus numbers (in A286708) in ς_{13}.

Table A lists positions of primes in a dataset of 2^{26} terms.

Table A

s	n	a(n)
1	2	2
2	3	3
3	4	5
4	8	7
5	13	11
6	42	13
7	347	17
8	3466	19
9	49012	23
10	528231	29
11	717126	31
12	63056215	37

There appears to be no guarantee by sequence definition that primes occur in order, but they do in the first 2^{26} terms. The appearance of PRIME(s) roughly gauges according to the difference PRIME(s) – PRIME$(s-1)$. We anticipate larger primes to appear roughly in order at some point as n increases.

Lemma 1.1. $p \mid ij$ implies either $p \mid i$ or $p \mid j$, or p is a divisor common to both i and j.
Proof. Define function $P(x) = \{ p : p \mid x \}$ to be the set of (distinct) prime divisors of x. It is clear that for RAD(ij), $P(ij) = P(i) \cup P(j)$. Hence, no prime q may appear in $P(ij)$ unless such appears in at least one of $P(i)$ or $P(j)$. In other words, unless $p \mid i$, $p \mid j$, or p divides both i and j, p does not divide the product ij. ■

Theorem 1. RAD$(ij) =$ A2110$(s-1)$ implies $k = p = \text{prime}(s)$.
Proof. Given Lemma 1.1 and the fact that novel divisor $p =$ A053669(ij) implies $m = 1$, and since we begin the sequence with $a(1 \dots 2) = \{1, 2\}$, we may only have $p = k$ if and only if $P(ij) = \{$ prime $q : q < p \}$. Suppose this is false. Then we would have $p < r$, the greatest prime factor $r \mid ij$. Could we have $m = 1$ for such nonrecord p? This implies that p has not occurred as a divisor of a previous term. If $p < r$ is novel, then r must not have been the least prime that does not divide some $a(t)$, $t < n$, and that p would have been a smaller solution. Hence, novel prime divisor $p = \text{prime}(s)$ results from RAD$(ij) =$ A2110$(s-1)$, and $k = mp$ with $m = 1$, hence $k = p$. ■

Corollary 1.2. Primes p appear in order in the sequence.

Corollary 1.3. Products mp, $m > 1$ appear after p appears naked in the sequence.

Corollary 1.4. The sequence is a permutation of natural numbers as a consequence of primes appearing in order, and multiples mp following the appearance of p itself in the sequence, with m increasing monotonically upon every instance of $p =$ A053669(ij).

Constitutive Qualities between Adjacent Terms.

We examine the constitutive relationship between adjacent terms j and k. Since we have $a(n) =$ ⍃$k : mq \land \neg c(k)$, ⍃$q : q \perp ij$, and since $q \perp ij$ implies $q \perp i$ and $q \perp j$, i and j are nonregular to k for $n > 1$. The lexical axiom forbids equality, hence there is no state ⑤. Therefore the sequence is confined to completely nonregular states ⓪①④⑦. We surmise, due to the nature of regular relations, that states ⓪ (coprimality) and ① (symmetric semicoprimality) are most common. This is borne out by observation given 2^{24} terms. Table B shows the cardinality of states that first emerge at $j = a(n-1)$ and $k = a(n)$.

Table B

State	card.	n	j	k
(0)	10293110	2	1	2
(1)	6483964	7	6	10
(4)	1	43	13	65
(7)	140	6	4	6

Some of the same scatterplot features seem to arise in related sequence S20230211. Similarity concerns the appearance of the quasiray in S20230211 associated with even numbers, and the "percolation" of products mp of primes p into an earlier appearance as m

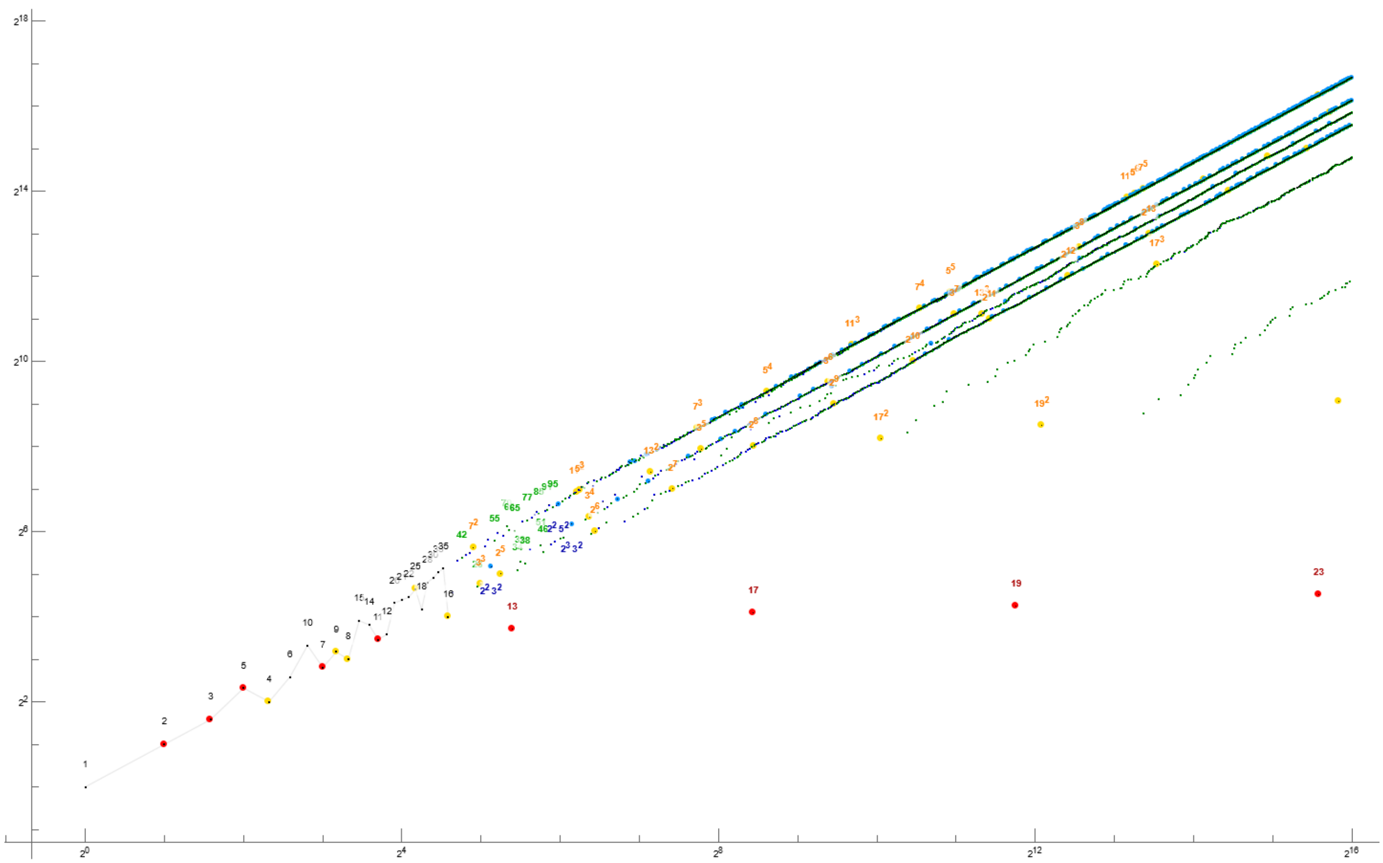

Figure 1: Log log scatterplot of s20230212 *for $n = 1 \ldots 2^{16}$, showing primes in red, multus (composite prime powers in* A246547*) in gold, varius (squarefree composites in* A120944*) in green, tantus (neither squarefree nor prime power in* A126706*) in blue, and plenus (products of multus in* A286708*, a subset of tantus) in light blue. Powerful numbers (*A1694*) are the union of $\{1\}$, the multus, and the plenus numbers. Essentially, zone β appears in red, zone γ beginning with prime powers in gold followed by green, with everything else in zone α.*

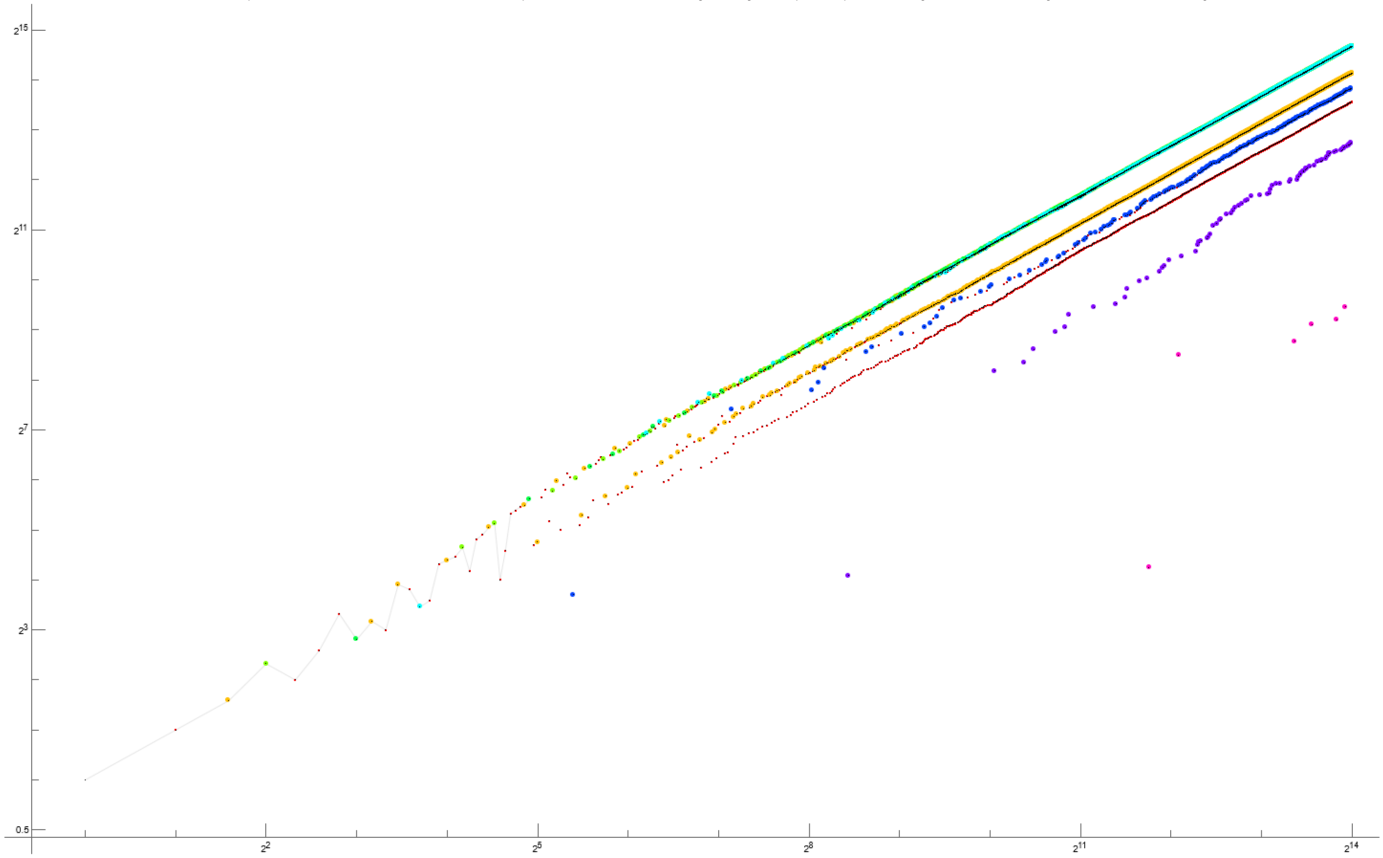

Figure 2: Log log scatterplot of s20230212 *for $n = 1 \ldots 2^{14}$, showing k : LPF$(k) = 2$ in red, LPF$(k) = 3$ in gold, LPF$(k) = 5$ in chartreuse, LPF$(k) = 7$ in green, LPF$(k) = 11$ in cyan, LPF$(k) = 13$ in blue, LPF$(k) = 17$ in purple, and LPF$(k) = 19$ in magenta.*

increases. Therefore this simpler sequence may serve as a guide toward examination of S20230211.

Open questions:

1. What is the significance of $p = 13$, which seems to emerge late enough to require a long process of 13-rough numbers in the striation ς_{13} to merge with zone α.
2. Why do $2 \times 13 \times M$ appear among $13 \times M$ in ς_{13}, and does same happen to $\varsigma_p : p > 13$ when ς_p traverses α_2?
3. Do all $\varsigma_p : p > 11$ merge with A?

Conclusion.

We have shown that primes occur in order in this sequence, that multiples of the primes follow the primes themselves, and consequently the sequence is a permutation of natural numbers. Constitutively, the sequence has k completely nonregular to ij. There remain a number of minor questions regarding this interesting LES.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved November 2022.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

Code:

[C1] Generate S20230212:

```
nn = 2^20; c[_] = False; q[_] = 1;
Array[Set[{a[#], c[#]}, {#, True}] &, 2];
Set[{i, j}, {a[1], a[2]}]; u = 3;
Monitor[Do[
  (k = q[#]; While[c[k #], k++]; k *= #;
     While[c[# q[#]], q[#]++]) &[(p = 2;
    While[Divisible[i j, p], p = NextPrime[p]]; p)];
  Set[{a[n], c[k], i, j}, {k, True, j, k}];
  If[k == u, While[c[u], u++]], {n, 3, nn}], n];
Array[a, nn]
```

Concerns sequences:

A000040: Prime numbers.
A002110: Primorials.
A006530: Least prime factor GPF(n).
A007947: Squarefree kernel of n; RAD(n).
A020639: Least prime factor GPF(n).
A053669: Smallest prime nondivisor of n.
A126706: “Tantus” numbers neither prime power nor squarefree.
A286708: “Plenus” numbers, products of multus numbers.

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