The Symmetric Semidivisor Counting Function

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Abstract.

We explore properties of numbers $k < n$ such that both share a squarefree kernel, yet k does not divide n. The smallest example of such is $k = 12$, $n = 18$. We determine the sort of prime decomposition that k and n must have so as to enjoy this rare relationship. Examination of a counting function and records transform follow. The problem touches upon odd prime p-smooth numbers and the nature of the tensor product of prime divisor power ranges bounded by n, which relates to OEIS A010846.

Introduction.

Define an n-regular number k to be a product limited to primes p such that $p \mid n$. Let $\mathrm{RAD}(n) = \mathrm{A7947}(n) = \varkappa$ be the squarefree kernel of n. We note that n-regularity is determined without regard to multiplicity ε of prime power factors $p^\varepsilon \mid n$. It is easy to see that n-regular k are such that $\mathrm{RAD}(k) \mid \mathrm{RAD}(n)$. Furthermore, n-regular k are elements of set $\boldsymbol{R}_\varkappa = \{\, k : \mathrm{RAD}(k) \mid \mathrm{RAD}(n) \,\}$ where $\varkappa = \mathrm{RAD}(n)$. We may also express this as follows:

$$\boldsymbol{R}_\varkappa = \bigotimes_{p\mid\varkappa} \{\, p^\varepsilon : \varepsilon \geq 0 \,\}. \qquad [1.1]$$

This expression implies $|\, \boldsymbol{R}_\varkappa \,| = \aleph_0$ for $\varkappa > 1$, since $|\, \{p^\varepsilon : \varepsilon \geq 0\} \,| = \aleph_0$. A consequence of restricting k to $p \mid n$ allows the following:

$$\boldsymbol{R}_\varkappa = \{\, k : k \mid n^\varepsilon : \varepsilon \geq 0 \,\}. \qquad [1.2]$$

It is clear from [1.2] that $1 \in \boldsymbol{R}_\varkappa$, since $1 \mid n^\varepsilon : \varepsilon = 0$. There are 2 species of n-regular k. Since multiplicity ε of prime power factors $p^\varepsilon \mid n$, we define these species with regard to n such that $\mathrm{RAD}(n) = \varkappa$, rather than $\varkappa$.

Divisors: $\boldsymbol{D}_n = \{\, d : d \mid n^\varepsilon : 0 \leq \varepsilon \leq 1 \,\}$ [1.3]

Semidivisors: $\boldsymbol{Đ}_n = \{\, k : k \mid n^\varepsilon : \varepsilon > 1 \,\}$ [1.4]

The divisor counting function $\tau(n) = \mathrm{A5}(n)$ is defined as follows:

$$\tau(n) = |\, \boldsymbol{D}_n \,| = \prod_{p^\varepsilon \mid n} (\varepsilon{+}1). \qquad [1.5]$$

Since $\boldsymbol{Đ}_n = \boldsymbol{R}_\varkappa \setminus \boldsymbol{D}_n$, $|\, \boldsymbol{Đ}_n \,| = \aleph_0$.

We can construct a "regular counting function" that employs the bound n as follows:

$$\begin{aligned}\theta(n) &= |\, \{\, k : k \mid n^\varepsilon : \varepsilon \geq 0 \wedge 1 \leq k \leq n \,\} \,| \\ &= |\, \mathrm{A162306}(n) \,| \\ &= \mathrm{A010846}(n)\end{aligned} \qquad [1.6]$$

The bound n, though not as natural for n-regular k as it is for divisors of n, is justified by the fact that n itself is n-regular. The computation of $\theta(n)$ is most efficiently achieved through an algorithm related to [1.1]. (Usage of θ comes from Granville.) Define "the semidivisor counting function" $\eth(n)$ to be the following:

$$\begin{aligned}\eth(n) &= |\, \{\, k : k \mid n^\varepsilon : \varepsilon > 1 \wedge 1 \leq k \leq n \,\} \,| \\ &= \theta(n) - \tau(n) \\ &= \mathrm{A010846}(n) - \mathrm{A5}(n) \\ &= \mathrm{A243822}(n)\end{aligned} \qquad [1.7]$$

Consider a symmetric or completely regular relation $k \parallel n$, that is, k is n-regular, and n is k-regular. We may say that k and n are "coregular". It is clear that this relation has the following property:

$$\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa \qquad [1.8]$$

We recognize that within coregular relations, we may have three species. These are symmetric divisibility (i.e., equality, $k = n$), mixed regularity wherein 1 term divides the other, and symmetric semidivisibility, a relation with no divisor relation. An example of symmetric divisibility is the relationship between 12 and 18.

The Symmetric Semidivisor Counting Function.

Define $f(n)$ to be the "symmetric semidivisor counting function" as follows:

$$f(n) = |\, \{\, k : k \mathrel{¦¦} n \wedge 1 < k < n \,\} \,|. \qquad [2.1]$$

Therefore the sequence $a(n) = \mathrm{A355432}(n) = \{\, f(n) \mapsto \mathbb{N} \,\}$ begins:

```
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,
0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,2,0,2,0,0,0,4,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0,0,1,0,0,0,0,1,0,
0,0,0,0,0,0,0,0,1,0,0,0,0,0,4,0,2,0,2,0,0,0,0,0,0,0,4,
0,0,0,1,0,0,0,0,0,0,0,1,...
```

Code [C2] generates the sequence.

We know from theorems in [1] that we might rewrite the constitutive expression $k \mathrel{¦¦} n$, that is, k ⑨ n, as follows:

$$k \mathrel{¦¦} n \rightarrow \mathrm{RAD}(k) = \mathrm{RAD}(n) \wedge k \nmid n \wedge n \nmid k. \qquad [2.2]$$

Symmetric semidivisorship is a special case of coregularity. Consequently, $\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$. Therefore it is clear that both $k, n \in \varkappa\boldsymbol{R}_\varkappa$. We also have shown that symmetric semidivisorship pertains to k and n that are both not prime powers, that is, both $k, n \in \mathrm{A024619}$. Therefore, for $n \in \mathrm{A961}$, $a(n) = 0$ and $k : k \in \mathrm{A961}$ do not satisfy $k \mathrel{¦¦} n$.

We might most succinctly generate $a(n)$ thus:

1. Generate $\boldsymbol{R}_\varkappa$ bounded by M for $\varkappa \in \mathrm{A024619} \wedge \varkappa = M$, where M is the largest n we want to compute.
2. For $n : n \in \mathrm{A961}$, $a(n) = 0$.
3. For $n : n \in \mathrm{A024619}$,
$$a(n) = |\, \{\, k : k \in \boldsymbol{R}_\varkappa \wedge 1 < k < n \wedge (k \mid n \mathbin{\overline{\nabla}} n \mid k) \,\} \,|. \qquad [2.3]$$
4. Let $p = \mathrm{LPF}(\varkappa)$. For $k : k \in \boldsymbol{R}_\varkappa \wedge k > n/p$, $(k \mid n \mathbin{\overline{\nabla}} n \mid k)$.

This should serve as cogent pseudocode.

It is evident that $a(n) < \theta(n)$ for the following reason:

$$\begin{aligned}&\{\, k : k \mathrel{¦¦} n \wedge 1 < k < n \,\} \subset \{\, k : k \parallel n \wedge 1 < k < n \,\}. \\ &\{\, k : k \mathrel{¦¦} n \wedge 1 < k < n \,\} \subset \mathrm{A162306}(n).\end{aligned} \qquad [2.4]$$

This, since $\mathrm{A010846}(n) = |\, \mathrm{A162306}(n) \,|$.

We know that 1 is not an element of the former, but is in the latter, so that the former is a proper subset of the latter.

Furthermore, $a(n) \leq \mathrm{A243822}(n)$. An interesting proposition would regard the question of any n such that $a(n) = \mathrm{A243822}(n)$. We might try $n = 18$, and see the following:

$$\begin{aligned}\{\, k : \mathrm{RAD}(k) \mid \mathrm{RAD}(n) \wedge k \nmid n \wedge n \nmid k \wedge 1 < k < n \,\} &\\ = \{\, k : k \in \boldsymbol{R}_\varkappa \wedge k \nmid n \wedge n \nmid k \wedge 1 < k < n \,\}&\\ = \mathrm{A162306}(n) \setminus \mathrm{A027750}(n)&\\ = \mathrm{A162306}(n) \cap \mathrm{A173540}(n)&\\ = \mathrm{A272618}(n)&\\ = \{4, 8, 12, 16\}.&\end{aligned}$$

This, since $\mathrm{A272618}(n) = \mathrm{A162306}(n) \setminus \mathrm{A027750}(n) = \mathrm{A162306}(n) \cap \mathrm{A173540}(n)$. Of these, only 12 does not divide 18, thus the following:

$$a(n) = \left| \text{A272618}(n) \setminus \{\, k : \text{RAD}(k) \ne \text{RAD}(n) \,\} \right|$$
$$a(n) = \left| \{12\} \right| = 1.$$

In this case the sole symmetric semidivisor is 12, and we see the rest of the 18-regular numbers are composite prime powers. Therefore we would need to show the following is true for numbers n that are not prime powers:

$$\{\, k : k \in \text{A246547}(n) \wedge k \in \text{A162306}(n) \,\} = \varnothing. \qquad [2.5]$$

Let's return to the notion of $\boldsymbol{R}_\varkappa$, the infinite set of $\varkappa$-regular numbers where $\varkappa = \text{RAD}(n)$. Example: for squarefree $\varkappa = 6$, we have $\boldsymbol{R}_6$ = A3586, the "3-smooth numbers":

$$\begin{aligned} \boldsymbol{R}_6 &= \bigotimes_{p|6} \{p^\varepsilon : \varepsilon \ge 0\} \\ &= \{2^\delta : \delta \ge 0\} \otimes \{3^\varepsilon : \varepsilon \ge 0\} \\ &= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, \dots\} \\ &= \text{A3586}. \end{aligned}$$

It is clear that for n such that $\text{RAD}(n) = \varkappa$, we may select any $k \in \boldsymbol{R}_6$, and have the relation $\text{RAD}(k) \mid \text{RAD}(n)$. We may also have the inverse $\text{RAD}(n) \mid \text{RAD}(k)$, which is certainly true for $k = n$ and for not all but infinitely many $k > n$.

Coregular Numbers.

It is clear that $\text{RAD}(k) = \text{RAD}(n) = \varkappa$ is a special case of the above. This case implies both k and n are in the set $\{\varkappa\boldsymbol{R}_\varkappa\}$, i.e., both k and n are $\varkappa$-coregular.

Lemma 2.1: For $\varkappa = 1$, $\boldsymbol{R}_1 = \{1\}$.
Proof: For $k > 1$, at least 1 prime $p \mid k$, and all primes are coprime to 1, therefore, k such that $k > 1$ is nonregular to 1. ■

Lemma 2.2: For $\varkappa = p$ prime, prime powers comprise $p\boldsymbol{R}_p$.
proof: For $\varkappa = p$ prime, $\boldsymbol{R}_p = \{\, p^\varepsilon : \varepsilon \ge 0 \,\}$, hence $p\boldsymbol{R}_p = \boldsymbol{R}_p \setminus \{1\}$, and all terms are prime powers. ■

Lemma 2.3: For composite $\varkappa$, the first term of $\varkappa\boldsymbol{R}_\varkappa$ is $\varkappa$, while the remaining terms are neither prime powers nor squarefree (i.e., a "tantus" number, $k \in$ A126706).
Proof: The empty product 1 is n-regular for all n because $1 \mid n$. Therefore, the first term in $\varkappa\boldsymbol{R}_\varkappa$ is $\varkappa \times \boldsymbol{R}_\varkappa(1) = \varkappa \times 1$. With $\varkappa \in$ A120944 since $\varkappa$ is by definition squarefree, the sequence $\varkappa\boldsymbol{R}_\varkappa$ begins with squarefree $\varkappa$ followed by numbers k of the form $m\varkappa$, where $m \in \boldsymbol{R}_\varkappa$ and $m > 1$, as consequence of [1.1]. Therefore, aside from the smallest term, k is neither squarefree nor prime power. ■

Corollary 2.4: The second-smallest number k in $\varkappa\boldsymbol{R}_\varkappa$ is clearly the product $p\varkappa$, where $p = \text{LPF}(\varkappa) = \text{A020736}(\varkappa)$, since p is the successor of 1 in $\boldsymbol{R}_\varkappa$.

Theorem 2: The infinite sequence $\varkappa\boldsymbol{R}_\varkappa$, squarefree $\varkappa > 1$, consists of prime powers for prime $\varkappa$, otherwise, the first term is squarefree composite $\varkappa$ followed by tantus numbers (i.e., $k \in$ A126706). Proof supplied by Lemmas 2.2 and 2.3.

We are concerned with the finite set defined below:

$$S_n = \{\, k \in \{\varkappa\boldsymbol{R}_\varkappa\} : k \le n \wedge k \nmid n \}. \qquad [2.6]$$

This set S_n represents $\{\varkappa\boldsymbol{R}_\varkappa\}$ truncated after the appearance of n. Therefore we arrive at the following equation:

$$a(n) = \left| S_n \right|. \qquad [2.7]$$

Lemma 3.1: $a(n) = 0$ for prime powers $n = p^\varepsilon$.
Proof: $\text{RAD}(k) = \text{RAD}(n) = p$ prime implies $k = p^\delta$ and $n = p^\varepsilon$ such that $\delta \le \varepsilon$. Then $p^\delta \mid p^\varepsilon$, leaving $S_n = \varnothing$, hence $a(n) = \left| S_n \right| = 0$. ■

Lemma 3.2: $a(n) = 0$ for squarefree n.
Proof: $\text{RAD}(k) = \text{RAD}(n) = n$ implies $k = n$. All numbers divide themselves, leaving $S_n = \varnothing$, hence $a(n) = \left| S_n \right| = 0$. ■

Theorem 3.3: $a(n) \ge 0$ for tantus n (i.e., $n \in$ A126706).
Proof: Consequence of Lemmas 3.1 and 3.2.

Lemma 3.4: $a(n) = 0$ for $n = \text{RAD}(n) \times \text{LPF}(n)$ and $\omega(n) > 1$.
Proof: In the sequence $\varkappa\boldsymbol{R}_\varkappa$, only $\varkappa < n$ by Corollary 2.4, and $\varkappa \mid n$ by definition of squarefree kernel. Thus $S_n = \varnothing$, hence $a(n) = \left| S_n \right| = 0$. ■

Theorem 3.5: Let prime $p = \text{LPF}(n)$ and q be the second smallest prime divisor of n. Let p^ε be the largest power of p such that $p^\varepsilon \mid n$. Let $\text{RAD}(n) = \varkappa$, and let $n/\varkappa = m$. For all $n \in$ A126706 such that the ratio $n/\varkappa < q$, $a(n) = 0$.

Proof: Consider $n = p^\delta qQ$ where p and q are as defined and Q is a product of primes greater than q. Clearly, $n = p^{(\delta-1)}\varkappa$. Recalling Lemma 2.3, we may divide $\varkappa\boldsymbol{R}_\varkappa/\varkappa$ and cancel $\varkappa$ to obtain $\boldsymbol{R}_\varkappa$. The first term of $\boldsymbol{R}_\varkappa$, i.e, $\boldsymbol{R}_\varkappa(1)$, is the empty product 1, followed by $\text{LPF}(\varkappa) = p$ and all powers p^i such that $i \le \varepsilon$. After p^ε, we have q. Hence we have the following power range of p bounded by q:

$$\begin{aligned} P &= \{\, p^i : 0 \le i \le \varepsilon \,\}, \\ &= \{\, p^i : 0 \le i \le \lfloor \log_p q \rfloor \,\} \end{aligned} \qquad [2.8]$$

It is sure that we do not have any interposing products pq, since $pq > q$, yet $p^\varepsilon < q$. It is immaterial whether we have multiplicity for q that exceeds 1, since this only makes for larger products in $\boldsymbol{R}_\varkappa$. By same token, any larger prime and any multiplicity of these primes that exceeds 1 also only makes larger products that do not interpose amid terms of P. Within P, all terms divide p^ε. Therefore, all terms in $\varkappa P$ divide n, leaving $S_n = \varnothing$, thus $a(n) = \left| S_n \right| = 0$. ■

Consequently, we may partition A126706 into 2 subsequences:

A "weak tantus" sequence t of numbers k that are neither prime powers nor squarefree semiprimes, where $p^\varepsilon \le p^{\lfloor \log_p q \rfloor}$ such that $p = \text{LPF}(n)$. For $n \in t$, $a(n) = 0$. Code [c4] generates the sequence t. This sequence A360767 begins as follows:

```
12, 20, 28, 40, 44, 45, 52, 56, 60, 63, 68, 76, 84, 88,
92, 99, 104, 116, 117, 124, 132, 136, 140, 148, 152,
153, 156, 164, 171, 172, 175, 176, 184, 188, 204, 207,
208, 212, 220, 228, 232, 236, 244, 248, 260, 261, 268,
272, 275, 276, 279, 280, 284, 292, 296, 297, 304, 308,
315, 316, 325, 328, 332, 333, 340, 344, 348, 351, ...
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A "strong tantus" sequence T of numbers k that are neither prime powers nor squarefree semiprimes, where $p^\varepsilon > p^{\lfloor \log_p q \rfloor}$. (See Figure 2 in the Appendix for a curious pattern that arises in A126706 amid strong and weak tantus numbers.) For $n \in t$, $a(n) > 0$. Code [c5] generates T. This sequence A360768 begins as follows:

```
18, 24, 36, 48, 50, 54, 72, 75, 80, 90, 96, 98, 100,
108, 112, 120, 126, 135, 144, 147, 150, 160, 162, 168,
180, 189, 192, 196, 198, 200, 216, 224, 225, 234, 240,
242, 245, 250, 252, 264, 270, 288, 294, 300, 306, 312,
320, 324, 336, 338, 342, 350, 352, 360, 363, 375, ...
```

The sequence below comprises the first terms of $\{\, f(n) \mapsto T \,\}$:

```
1, 1, 1, 2, 2, 4, 2, 1, 1, 1, 4, 2, 2, 4, 1, 1, 1, 1,
3, 1, 3, 2, 8, 1, 2, 1, 7, 2, 1, 2, 5, 2, 1, 1, 3, 3,
1, 6, 1, 1, 5, 5, 4, 5, 1, 1, 4, 8, 3, 3, 1, 2, 1, 4,
2, 3, 5, 10, 2, 1, 3, 3, 1, 1, 1, 6, 1, 3, 7, 1, 1, 7,
3, 14, 3, 6, 3, 2, 1, 1, 2, 7, 2, 1, 1, 2, 2, 8, 4, 6,
4, 8, 1, 1, 2, 1, 6, 9, 2, 1, 6, 2, 3, 1, 7, 1, 3, ...
```

Generate the above sequence using Code [c7].

Appendix Figure 1 is an example of the pattern of symmetric semidivisors k such that $k < n$ for k, n in $6\boldsymbol{R}_6$.

Highly Symmetrically Semidivisible Numbers.

Let's examine the records transform r of A355432. The list r of record setters, A360589, begins with the following terms:

```
1, 18, 48, 54, 162, 384, 486, 1350, 1458, 2250, 2430,
3750, 6000, 6750, 7290, 11250, 12150, 14580, 15000,
15360, 18750, 21870, 30720, 33750, 36450, 37500, 43740,
56250, 61440, 65610, 93750, 122880, 168750, ...
```

Code [C10] generates the sequence.

Observation 4.1: $r(i)$ = A055932(j), given $n \le 2^{22}$ terms of A355432, essentially for $i \le 80$. Appendix Table A lists $r(1\ldots80)$ along with several parameters described under Table 1. The numbers in A055932 are products of the smallest primes p such that no nondivisor prime $q < p$.

The above observation, if true, implies the following:

$$r \subset \{\, T \cap \text{A055932} \,\}. \qquad [4.1]$$

Confinement of r to sequence A055932 implies the following:

$$k, n \in \boldsymbol{R}_{\text{A2110}(j)} \text{ such that } j > 1. \qquad [4.2]$$

This statement is tantamount to saying that records in A355432 appear in sequences of the p-smooth numbers, where p is odd. In other words, sequences of numbers k where odd p is the largest prime factor. In OEIS, many such sequences of p-smooth numbers appear for small primes p:

A3586: $\boldsymbol{R}_6 = \boldsymbol{R}_{\text{A2110}(2)}$ = 3-smooth numbers.
A051037: $\boldsymbol{R}_{30} = \boldsymbol{R}_{\text{A2110}(3)}$ = 5-smooth numbers.
A2473: $\boldsymbol{R}_{210} = \boldsymbol{R}_{\text{A2110}(4)}$ = 7-smooth numbers.
A051038: $\boldsymbol{R}_{2310} = \boldsymbol{R}_{\text{A2110}(5)}$ = 11-smooth numbers.
A080197: $\boldsymbol{R}_{30030} = \boldsymbol{R}_{\text{A2110}(6)}$ = 13-smooth numbers.
A080681: $\boldsymbol{R}_{510510} = \boldsymbol{R}_{\text{A2110}(7)}$ = 17-smooth numbers.
A080682: $\boldsymbol{R}_{9699690}$, = $\boldsymbol{R}_{\text{A2110}(8)}$ = 19-smooth numbers.
A080683: $\boldsymbol{R}_{223092870} = \boldsymbol{R}_{\text{A2110}(9)}$ = 23-smooth numbers.

Observation 4.2: For $n \le 2^{22}$, $n \in r$ have $\omega(n) \le 4$. Table 1 is a list of the smallest terms that have ω distinct prime factors:

Table 1.

i	n	A067255(n)	j	a(n)
1	1	0	1	0
2	18	1.2	8	1
8	1350	1.3.2	65	16
40	360150	1.1.2.4	554	168
182	507310650	1.2.2.1.5	5468	1524
601	289898148540	2.1.1.1.1.7	31947	8191

In the table we list the index i, followed by $r(i) = n$, where A355432(n) = $a(n)$ is the number of symmetric semidivisors $k < n$. Furthermore, $a(n)$ = A055932(j). We employ "multiplicity notation" A067255(n), which merely notes multiplicities of prime power factors of n where the first multiplicity pertains to 2, the second to 3, the third to 5, and generally the k-th pertains to prime(k).

Thus $n = 18 = 2^1 \times 3^2 = r(2)$ = A055932(8); A355432(18) = 1.

The parenthetic fifth line is a projection given Observation 4.1. We expect a term n with $\omega(n) = 7$, and don't have any reason to suspect that there are any limits against higher numbers of distinct prime factors for "highly symmetrically semidivisible numbers" $n \in r$.

Theorem 4: A360589 ⊂ A055932.
Proof: Theorems in the last section show that A355432$(n) = 0$ for "strong tantus" numbers, which are in A126706, and that A055932 contains prime powers and squarefree numbers that do not appear in A126706 by definition.

The numbers in A055932 are products of the smallest primes p such that no nondivisor prime $q < p$. The proposition thus implies that $n \in$ A360589 are such that $n = m \times$ A2110(j) where $m \in \boldsymbol{R}_{\text{A2110}(j)}$. This follows from [1.1] bounded by n and $\log n \,/\, \log p$ for all $p \mid n$.

The following sequence is a different ordering of A162306(n) related to the definition of $\boldsymbol{R}_\varkappa$ bounded by n, by vectorizing the tensor in order of prime divisor p:

$$\text{A275280}(n) = \{\, k = \{ \bigotimes_{p|\varkappa} \{\, p^\varepsilon : \varepsilon \ge 0 \,\}\} \wedge k \le n \,\}.$$
$$\begin{aligned}\text{A010846}(n) &= \theta(n)\\ &= \left|\ \text{A275280}(n)\ \right|\\ &= \left|\ \text{A162306}(n)\ \right| \qquad [4.3]\end{aligned}$$

In the $\omega(n)$-orthosimplex defined by A275280(n), the origin is the empty product, the axes contain prime powers, etc., but for strongly tantus n, tantus numbers k reside in the interior where multiplicity of at least 1 prime divisor exceeds 1. There is an encrustation of nontantus numbers that requires sufficiently large n to reach a significant number of nondivisor strong tantus k. Hence, as n increases, there are proportionately more tantus numbers in A275280(n). For numbers n that conserve $\varkappa$, larger n tend to proportionally enrich the polytope A275280(n).

Consider 2 large strong tantus numbers that are "comparable" in that they are the largest strongly $\varkappa$-regular numbers less than n:

$$M = p^\varepsilon q^\delta,\ p < q, \text{ and } N = p^d r^e,\ q < r,$$

where the multiplicities are conserved between the 2 numbers. Since $\log n \,/\, \log q > \log n \,/\, \log r$, we have more tantus numbers in M than in N. This implies that the distinct primes that produce n are a set of the smallest consecutive primes, that is, $\varkappa$ is in A2110.

The only numbers that might supersede products of $m \times$ A2110(j) at sufficient scale are numbers of the form $m \times$ A2110$(j{+}1)$.

We would like to write a more rigorous proof of Theorem 4.

Code [C11] generates record setters A360589 and records A360912 much more efficiently based on Theorem 4.

Some Open Questions:

1. Is there a simpler or more rigorous proof of Theorem 4?
2. What is the smallest instance of 17-smooth n that sets a record in A355432?
3. What is the reason for the pattern of weak and strong tantus numbers (i.e., A126706) seen in Figure 1.
4. We have not proved that A355432(n) < A243822(n), though it seems to follow from the nature of n such that A355432$(n) > 0$.

Conclusion.

We have identified numbers n for which we have at least 1 number k such that RAD(k) = RAD(n), yet k does not divide n. These are the "strong tantus" numbers $n \in$ A360768. A symmetric semidivisor counting function was defined in A355432. We explored the records transform, attempting to show the sort of numbers in A360589 that set records in A355432. These numbers are in A055932, which implies that we need only search the odd prime p-smooth numbers for candidates.

Appendix.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved February 2023.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

Code:

[C1] Calculate $R_\varkappa$ bounded by an arbitrary limit m (i.e., calculate A275280(n); flatten and take union to provide A162306)

```
regularsExtended[n_, m_ : 0] :=
  Block[{w , lim = If[m <= 0, n, m]},
   Sort@ ToExpression@
     Function[w,
       StringJoin[
         "Block[{n = ", ToString@ lim,
         "}, Flatten@ Table[",
         StringJoin@
           Riffle[Map[ToString@ #1 <> "^" <>
             ToString@ #2 & @@ # &, w], " * "],
         ", ", Most@ Flatten@ Map[{#, ", "} &, #],
         "]]" ] &@
       MapIndexed[
         Function[p,
           StringJoin["{", ToString@ Last@ p,
             ", 0, Log[",
             ToString@ First@ p, ", n/(",
             ToString@
               InputForm[
                 Times @@ Map[Power @@ # &,
                   Take[w, First@ #2 - 1]]],
             ")]}" ] ]@ w[[First@ #2]] &, w]]@
       Map[{#, ToExpression["p" <>
         ToString@ PrimePi@ #]} &, #[[All, 1]] ] &@
       FactorInteger@ n];
```

[C2] Generate A355432 (needs [C1]):

```
A355432 = Block[{a, c, f, k, s, t, nn},
  nn = 2^20; c[_] = 0;
  f[n_] := f[n] = n regularsExtended[n, Floor[nn/n]];
  s = Select[Range[nn],
    And[CompositeQ[#], SquareFreeQ[#]] &];
  Monitor[
    Do[Set[t[ s[[i]] ], f@ s[[i]]], {i, Length[s]}],
  i];
  Monitor[
    Do[k = t[ s[[j]] ];
      Map[Function[m,
        Set[c[m],
          Count[TakeWhile[k, # <= m &],
          _?(Mod[m, #] != 0 &)]]], k], {j, Length[s]}],
  j];
  Array[c, nn] ];
```

[C3] Generate tantus numbers (A126706):

```
a126706 = Block[{k}, k = 0;
   Reap[Monitor[Do[
       If[And[#2 > 1, #1 != #2] & @@
         {PrimeOmega[n], PrimeNu[n]},
        Sow[n]; Set[k, n] ],
       {n, 2^21}], n]][[-1, -1]]] (* Tantus *);
```

[C4] Generate "weak tantus" numbers (A360767):

```
Select[a126706[[1 ;; 120]], #1/#2 < #3 & @@
  {#1, Times @@ #2, #2[[2]]} & @@
  {#, FactorInteger[#][[All, 1]]} &]
```

[C5] Generate "strong tantus" numbers (A360768):

```
Select[a126706[[1 ;; 120]], #1/#2 >= #3 & @@
  {#1, Times @@ #2, #2[[2]]} & @@
  {#, FactorInteger[#][[All, 1]]} &]
```

[C6] Select strongly tantus terms of a sequence:

```
stantusSelect[w_List] :=
 Select[
  Select[w,
    Nor[PrimePowerQ[#], SquareFreeQ[#]] &],
  #1/#2 >= #3 & @@
  {#1, Times @@ #2, #2[[2]]} & @@
  {#, FactorInteger[#][[All, 1]]} &];
```

[C7] Generate $\{ f(n) \mapsto T \}$, effectively eliminating 0's from A355432 :

```
A355432[[#]] & /@
 Select[a126706[[1 ;; 2^10]],
   #1/#4 >= #3 & @@
   {#1, #2[[1]], #2[[2]], Times @@ #2} & @@
   {#, FactorInteger[#][[All, 1]]} &]
```

[C8] Generate a table of A360589 and corresponding values in A355432:

```
With[{s = A355432},
  Map[{FirstPosition[s, #][[1]], #} &,
   Union@ FoldList[Max, s]]] // TableForm
```

[C9] Function that generates A055932:

```
A055932[n_, l_ : 0, o_ : 0] :=
  Block[{lim, ww, dec},
  dec[x_] := Apply[Times,
    MapIndexed[Prime[First@ #2]^#1 &, x]];
  Set[{lim, ww},
   If[l < 1,
    {Product[Prime@ i, {i, n}],
      NestList[Append[#, 1] &, {1}, n - 1]},
    {n, NestList[Append[#, 1] &, {1}, # - 1]} &[
       -2 + Length@
       NestWhileList[NextPrime@ # &,
         1, Times @@ {##} <= n &, All] ] ] ];
  {{{Boole[o == 0]}}}~Join~Map[Block[{w = #, k = -1},
      Sort@
         Apply[Join, {{If[o > 0, #, dec@ #] &@
             ConstantArray[1, Length@ w]},
           If[Length@ # == 0, #, #[[1]]] }] &@
         Reap[Do[
           If[# <= lim,
             Sow[If[o > 0, w, #]]; k = -1,
             If[k <= -Length@ w, Break[], k--]] &@
               dec@ Set[w,
                If[k == -1,
                  MapAt[# + 1 &, w, k],
                  PadRight[#, Length@ w, 1] &@
                    Drop[MapAt[# + Boole[i > 1] &,
                      w, k], k + 1] ]],
           {i, Infinity}] ][[-1]] ] &, ww]]
```

[C10] Generate A360589 and A360912 via A355432 (syntactically concise version):

```
Set[{a360589, a360912},
 With[{s = A355432[[1 ;; 2^16]]},
  Transpose@
  Map[{FirstPosition[s, #][[1]], #} &,
   Union@ FoldList[Max, s] ] ] ]
```

[C11] Efficiently generate A360589 and A360912:

```
Set[{a360589, a360912},
  Block[{a, c, f, k, s, t, pp, nn},
   nn = 2^20; pp = 5; c[_] = 0;
   f[n_] := f[n] = n regularsExtended[n, Floor[nn/n]];
   s = Rest@ FoldList[Times, Prime@ Range[pp]];
   Monitor[
    Do[Set[t[s[[i]]], f@ s[[i]]], {i, Length[s]}], i];
   Transpose@
    Sort@ Reap[
       Monitor[Do[k = t[s[[j]]];
         Map[Function[m,
           If[# > 0, Sow[{m, #}]] &@
            Count[TakeWhile[k, # <= m &],
              _?(Mod[m, #] != 0 &)]], k],
         {j, Length[s]}], j]][[-1, -1]] ] ];
```

Table A.

i = index in A360589.
j = index in A055932.
n = index in A355432.

i	r(i) = n	A067255(n)	j	a(n)	i
1	1	0	1	0	1
2	18	1.2	8	1	2
3	48	4.1	13	2	3
4	54	1.3	14	4	4
5	162	1.4	25	8	5
6	384	7.1	37	10	6
7	486	1.5	42	14	7
8	1350	1.3.2	65	16	8
9	1458	1.6	67	21	9
10	2250	1.2.3	81	23	10
11	2430	1.5.1	85	26	11
12	3750	1.1.4	99	33	12
13	6000	4.1.3	122	34	13
14	6750	1.3.3	127	39	14
15	7290	1.6.1	131	44	15
16	11250	1.2.4	154	51	16
17	12150	1.5.2	161	52	17
18	14580	2.6.1	172	54	18
19	15000	3.1.4	174	55	19
20	15360	10.1.1	176	58	20
21	18750	1.1.5	190	67	21
22	21870	1.7.1	201	70	22
23	30720	11.1.1	229	76	23
24	33750	1.3.4	237	77	24
25	36450	1.6.2	244	80	25
26	37500	2.1.5	248	83	26
27	43740	2.7.1	261	84	27
28	56250	1.2.5	286	95	28
29	61440	12.1.1	296	98	29
30	65610	1.8.1	304	104	30
31	93750	1.1.6	345	119	31
32	122880	13.1.1	381	124	32
33	168750	1.3.5	426	133	33
34	182250	1.6.3	436	134	34
35	187500	2.1.6	443	142	35
36	196830	1.9.1	450	148	36
37	245760	14.1.1	486	153	37
38	281250	1.2.6	509	160	38
39	328050	1.8.2	536	164	39
40	360150	1.1.2.4	554	168	40
41	375000	3.1.6	564	169	41
42	393660	2.9.1	573	172	42
43	425250	1.5.3.1	588	174	43
44	430080	12.1.1.1	589	178	44
45	459270	1.8.1.1	602	186	45
46	468750	1.1.7	607	191	46
47	504210	1.1.1.5	622	197	47
48	590490	1.10.1	659	201	48
49	648270	1.3.1.4	680	210	49
50	656250	1.1.6.1	682	217	50
51	765450	1.7.2.1	718	223	51
52	833490	1.5.1.3	738	229	52
53	860160	13.1.1.1	746	235	53
54	918540	2.8.1.1	762	236	54
55	918750	1.1.5.2	763	243	55
56	1008420	2.1.1.5	787	252	56
57	1071630	1.7.1.2	804	253	57
58	1152480	5.1.1.4	824	255	58
59	1181250	1.3.5.1	832	262	59
60	1275750	1.6.3.1	852	266	60
61	1286250	1.1.4.3	853	273	61
62	1312500	2.1.6.1	860	276	62
63	1377810	1.9.1.1	872	284	63
64	1512630	1.2.1.5	902	294	64
65	1720320	14.1.1.1	941	303	65
66	1800750	1.1.3.4	954	309	66
67	1944810	1.4.1.4	979	314	67
68	1968750	1.2.6.1	984	319	68
69	2016840	3.1.1.5	991	320	69
70	2296350	1.8.2.1	1032	333	70
71	2500470	1.6.1.3	1062	340	71
72	2521050	1.1.2.5	1066	350	72
73	2755620	2.9.1.1	1096	353	73
74	3010560	12.1.1.2	1130	358	74
75	3025260	2.2.1.5	1132	364	75
76	3214890	1.8.1.2	1154	373	76
77	3281250	1.1.7.1	1163	386	77
78	3529470	1.1.1.6	1188	397	78
79	4033680	4.1.1.5	1242	402	79
80	4133430	1.10.1.1	1252	415	80

Concerns sequences:

A000040: Prime numbers.
A000961: Prime powers.
A001221: Number of distinct prime divisors of n, $\omega(n)$.
A002473: $R_{210} = R_{A2110(4)}$ = 7-smooth numbers.
A003586: $R_6 = R_{A2110(2)}$ = 3-smooth numbers.
A005117: Squarefree numbers.
A007947: Squarefree kernel of n; RAD(n).
A013929: Numbers that are not squarefree.
A024619: Numbers that are not prime powers.
A051037: $R_{30} = R_{A2110(3)}$ = 5-smooth numbers.
A051038: $R_{2310} = R_{A2110(5)}$ = 11-smooth numbers.
A080197: $R_{30030} = R_{A2110(6)}$ = 13-smooth numbers.
A080681: $R_{510510} = R_{A2110(7)}$ = 17-smooth numbers.
A080682: $R_{9699690}, = R_{A2110(8)}$ = 19-smooth numbers.
A080683: $R_{223092870} = R_{A2110(9)}$ = 23-smooth numbers.
A120944: "Varius" numbers; squarefree composites.
A126706: "Tantus" numbers neither prime power nor squarefree.
A162306: Truncation of $R_\varkappa$: row $n = \{ k \in R_\varkappa : k \le n \}$, RAD$(n) = \varkappa$.
A275280: $\{ k = \{ \bigotimes_{p \mid \varkappa} \{ p^\varepsilon : \varepsilon \ge 0 \} \} \land k \le n \}$.
A355432: $a(n)$ = symmetric semidivisor counting function.
A359929: Row n lists symmetric semidivisors of A360768(n).
A360589: Record setters in A355432.
A360767: Weakly tantus numbers.
A360768: Strongly tantus numbers.
A360912: Records in A355432.

Document Revision Record:

2023 0219: Draft 1. 2023 0222: Draft 2.
2023 1025: Minor edits to align with notation in later papers.

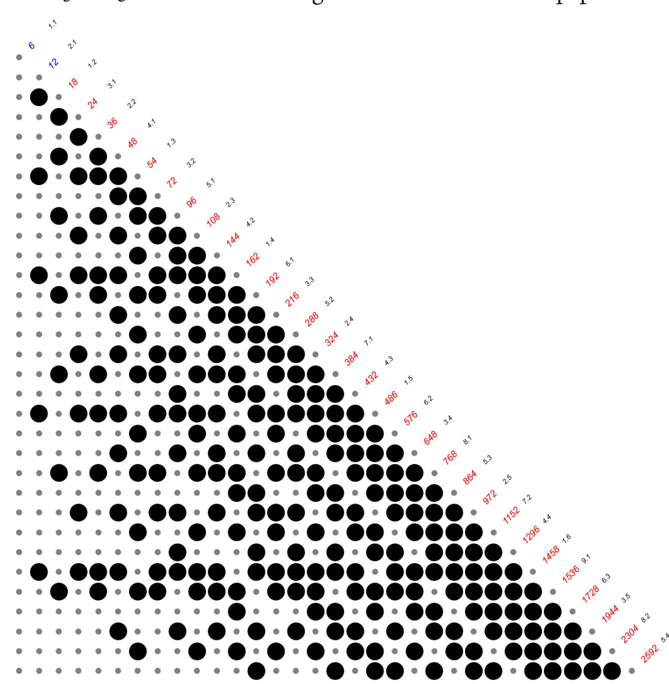

Figure 1: Pattern of symmetric semidivisors shown in large black dots, versus divisors in gray, for k and n both in the sequence $\varkappa R_\varkappa$ such that $\varkappa = 6$, i.e., OEIS A3586. Numbers that are strong tantus are printed in red, in row and column that springs from the gray dot "southwest" of the diagonal index. The exponents of 2 and 3 appear in black to the right of the index. Sorting lexically by A067255(n), we see a pattern shared by numbers of similar prime power decomposition, incrementing only one of the exponents.

This work is dedicated to my son Karl
on the occasion of his 16th birthday.

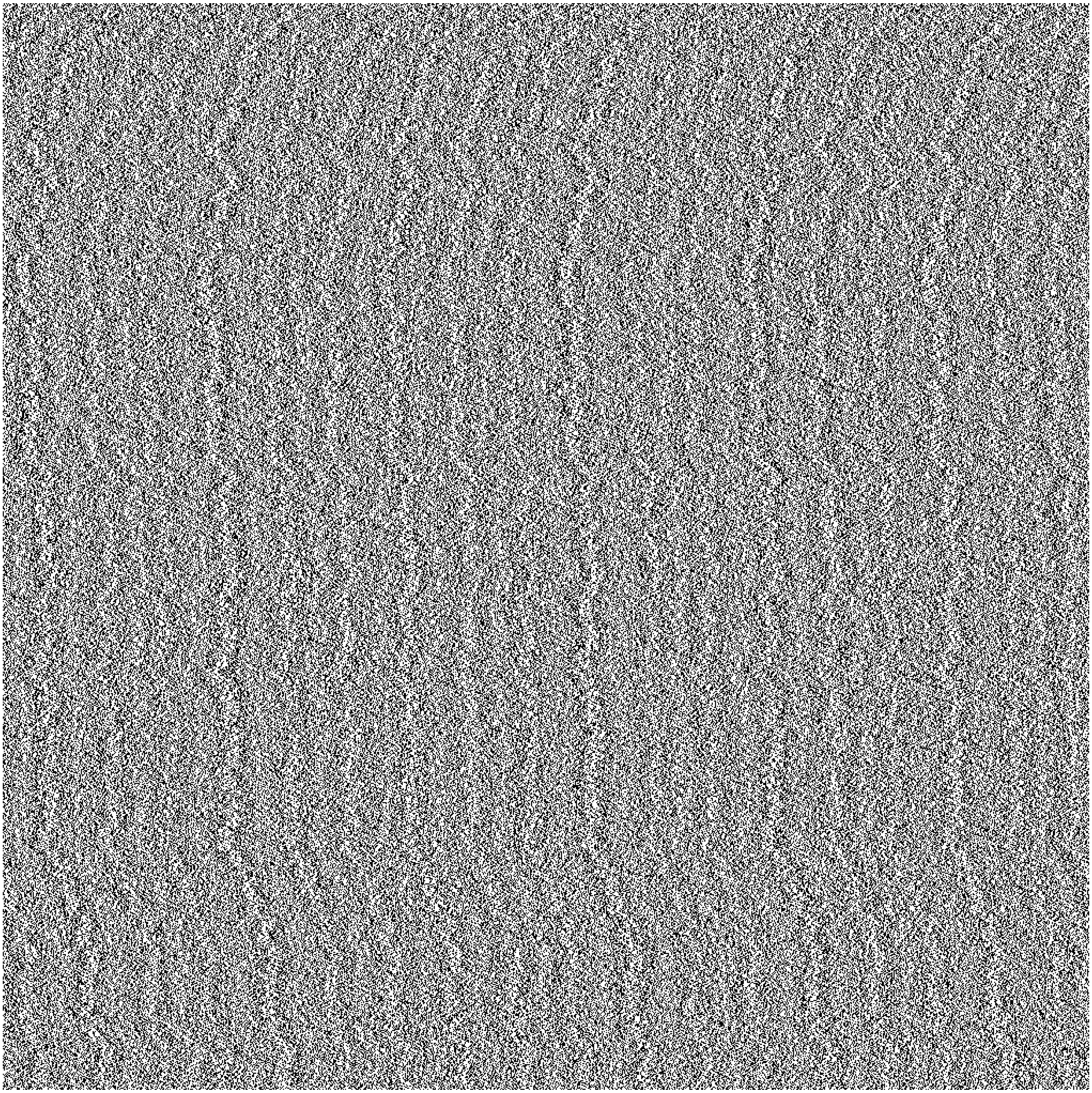

Figure 2: OEIS A126706 *is the sequence of tantus numbers: neither prime power nor squarefree. Consider 2 smallest prime factors p and q, $p < q$, and define a "strong tantus" number n to be such that $p^{\varepsilon} > q$. Define a sequence $b(n)$ that is a characteristic function of n such that* A126706*(n) is a strong tantus number, where white represents weak and black strong tantus numbers. This is an image of $b(1 \ldots 1032256)$, $1032256 = 1016^2$, exhibiting a curious interference pattern (the reason for 1016 terms per row). Perhaps the rarefaction delimiated by compression features to make "sand ripple" shapes pertains to congruence relations with numbers in the cototient of 6 or 12. The ripples are still not explained.*