The Symmetric Semicoprime Counting Function

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Abstract.

We examine a species of numbers k in the cototient of n such that k has a divisor p that does not divide n, and n has a divisor q that does not divide k, called symmetric semicoprimality. Particularly, we examine a counting function $f_1(n) = \text{A360480}(n)$ and note the resemblance of this function to $\text{A051953} = n - \phi(n)$.

Introduction.

Consider the cototient of n, that is, those $k < n$ such that $(k, n) > 1$. In other words, if the reduced residue set $\text{RRS}(n)$ includes $k < n$ such that $(k, n) = 1$, then the cototient is defined as follows:

$$c(n) = \{1 \dots n\} \setminus \text{RRS}(n). \quad [1.1]$$

$$\begin{aligned} \text{A051953}(n) &= \left| \, c(n) \, \right| \quad [1.2] \\ &= n - \phi(n). \\ &= n - \text{A10}(n). \end{aligned}$$

Clearly, $\text{A051953}(n) = 1$ for $n = p$, prime.

Within $c(n)$, we have divisors $d \mid n$, therefore we define the neutral cototient, $\Xi(n)$, the set of k neither coprime to n nor divisors of n, as follows:

$$\Xi(n) = c(n) \setminus \{ \, d : d \mid n \, \}. \quad [1.3]$$

$$\begin{aligned} \xi(n) &= \left| \, \Xi(n) \, \right| \quad [1.4] \\ &= \left| \, \text{A133995}(n) \, \right| \\ &= n - \phi(n) - \tau(n) + 1. \\ &= n - \text{A10}(n) - \text{A5}(n) + 1. \\ &= \text{A045763}(n). \end{aligned}$$

As consequence of neutrality, k and n are composite, since primes p either divide n or are coprime to n. Furthermore, for $n = p$, $\xi(n) = 0$.

We may distinguish 2 species of n-neutral k based on the squarefree kernel $\text{RAD}(m) = \text{A7947}(m)$. The case $\text{RAD}(k) \mid \text{RAD}(n)$ implies k is n-regular, meaning that $k \mid n^\varepsilon$, $\varepsilon \geq 0$, that is, all prime factors of k also divide n. The n-regular numbers k are a superset of divisors $d \mid n^\varepsilon$, $\varepsilon = 0 \dots 1$; for $k \leq n$, these numbers are listed in row n of A162306.

$$\begin{aligned} \text{A162306}(n) &= \{ \, k \leq n : \text{RAD}(k) \mid \text{RAD}(n) \, \} \quad [1.5] \\ &= \{ \, k \leq n : k \mid n^\varepsilon, \varepsilon \geq 0 \, \} \\ &= \{ \, d : d \mid n \, \} \cup \{ \, k < n : k \mid n^\varepsilon, \varepsilon > 1 \, \} \\ &= \text{A027750}(n) \cup \text{A272618}(n) \end{aligned}$$

$$\begin{aligned} \text{A010846}(n) &= \left| \, \text{A162306}(n) \, \right| \\ &= \left| \, \text{A027750}(n) \, \right| + \left| \, \text{A272618}(n) \, \right| \\ &= \text{A5}(n) + \text{A243822}(n) \\ &= \tau(n) + \xi_D(n). \quad [1.6] \end{aligned}$$

Nondivisor n-regular k are called semidivisors, and are 1 of the 2 species in the neutral cototient [2], [3]. These are listed in $\Xi_D(n)$, that is, row n of A272618. The semidivisor counting function $\xi_D(n) = \text{A243822}(n)$.

The other species is n-semicoprime k, $k < n$, hence we have called this species a "semitotative" of n. These are listed in $\Xi_T(n)$, that is, row n of A272619. The semidivisor counting function $\xi_T(n) = \text{A243823}(n)$.

$$\Xi_T(n) = \Xi(n) \setminus \Xi_D(n) \quad [1.7]$$

$$\text{A272619}(n) = \text{A133995}(n) \setminus \text{A272618}(n)$$

We can define the sequence $\Xi_T(n)$ from first principles:

$$\Xi_T(n) = \{ \, k : k < n \wedge (k, n) > 1 \wedge \text{RAD}(k) \nmid \text{RAD}(n) \, \} \quad [1.8]$$

$$\begin{aligned} \xi_T(n) &= \left| \, \Xi_T(n) \, \right| \quad [1.9] \\ &= \left| \, \text{A133995}(n) \, \right| - \left| \, \text{A272618}(n) \, \right| \\ &= \text{A243823}(n). \end{aligned}$$

Symmetric Semicoprimality.

Where coprimality between k and n represents disjunct sets of prime divisors of k and n and regularity represents one set a subset of the other, semicoprimality represents an inhabited symmetric difference. Hence we can have n-semicoprime k, yet k-regular n and vice versa, while coprimality is always symmetric.

Definition 1: When we have at least 1 prime p such that $p \mid k$ that does not divide n, and at least 1 prime q such that $q \mid n$ that does not divide k, we have "symmetric semicoprimality".

In [2] we present the following symbols:

Table A.

$k \perp n$	k is coprime to n	$(k, n) = 1$	
$k \lozenge n$	k is semicoprime to n	$1 < (k, n) < \text{MIN}$	$n/(k, n) \nparallel n$
$k \parallel n$	k is regular to n	$1 \leq (k, n) \leq \text{MIN}$	$k \mid n^\varepsilon : \varepsilon \geq 0$
$k \mid n$	k divides n	$1 \leq (k, n) = k$	$k \mid n^\varepsilon : \varepsilon = 0 \dots 1$
$k \,¦\, n$	k semidivides n	$1 < (k, n) < \text{MIN}$	$k \mid n^\varepsilon : \varepsilon > 1$

Symmetric semicoprimality we express through $k \lozenge\lozenge n$, i.e, k ① n per [2]. These are k and n in the cototient absent divisorship between their squarefree kernels. Such is implied by the definition of symmetric semicoprimality. The existence of 2 species of regular numbers (the divisor and the semidivisor) implies corresponding mixed cototient states:

◊◊	◊| or |◊	◊¦ or ¦◊
Symmetric Semicoprimality	Lean Divisorship	Mixed Neutrality
①	②④	③⑦

For $k < n$, it is clear that we cannot have state ②, that is $k \lozenge| \, n$, since that would require $k > n$, a contradiction. The mixed neutral state ⑦, i.e., k ¦◊ n, is not at issue, since it is a kind of semidivisor. The lean divisor state ④, $k \, |\lozenge \, n$, is also immaterial, since it is a kind of divisor. Therefore the category of lean divisorship can be ignored, but we can't use $n \nmid k$ as a means to determine symmetric semicoprimality.

For our purposes, we are only interested in disambiguating states ① and ③.

We have a final step to distinguish symmetric from asymmetric semitotatives. The set of semitotatives, $\Xi_T(n)$, includes both n-semicoprime k for which n is k-regular, (i.e., $\text{RAD}(n) \mid \text{RAD}(k)$) and where n is k-semicoprime. Therefore the following is necessary to create a set S of symmetric semitotatives:

$$\begin{aligned} S &= \{ \, k \in \text{A272619}(n) : \text{RAD}(n) \nmid \text{RAD}(k) \, \} \quad [2.1] \\ &= \{ \, k : k < n \wedge (k, n) > 1 \wedge \\ &\quad \text{RAD}(k) \nmid \text{RAD}(n) \wedge \text{RAD}(n) \nmid \text{RAD}(k) \, \} \\ &= \text{A361098}(n). \end{aligned}$$

The symmetric semicoprime counting function thus is defined as follows:

$$f_1(n) = \text{A360480}(n) = \left| \, \text{A361098}(n) \, \right| \quad [2.2]$$

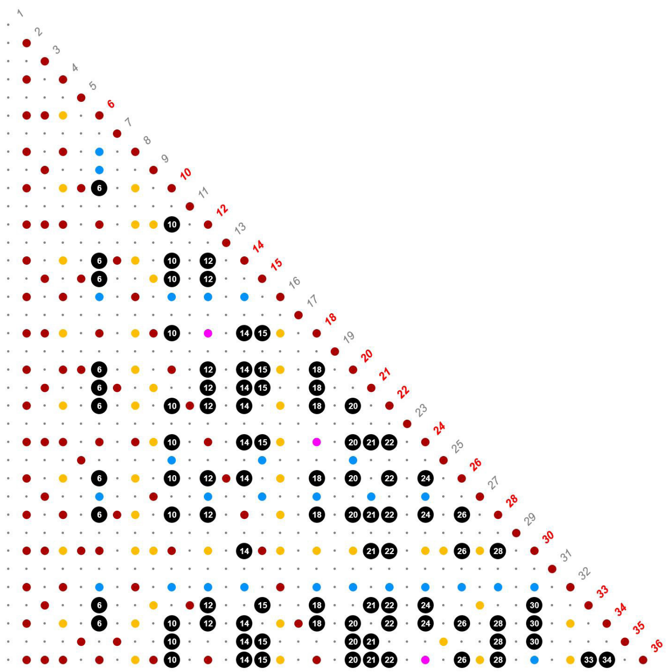

Figure 1: A map of constitutive states in the cototient between k and n for k ≤ 36 and n ≤ 36. Black circles are in state ①, while gray dots represent coprimality (state ⓪). Red dots represent divisor states ④⑤⑥, notably excepting k = 1. Blue represents state ③ while yellow represents state ⑦. Finally, magenta represents symmetric semidivisibility, state ⑨, which requires rad(k) = rad(n).

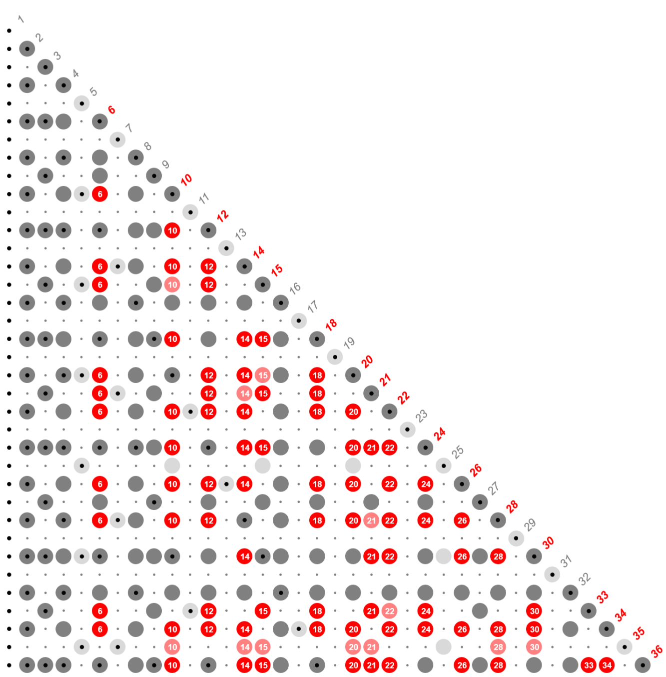

Figure 2: Relationship of symmetric n-semicoprime k to "quincunx" numbers and the cototient in general. Plot k and n for k ≤ 36 and n ≤ 36 at (k, −n). We show "quincunx" numbers Q(n, k) = [OR(2|k, 3|k, 2|n, 3|n)] in dark gray, T(n, k) = [(k, n) > 1] in light gray, k coprime to n with a gray dot, and k | n with a black dot. For k ① n : Q(n, k) = 1, we highlight in red, and for k ① n : T(n, k) = 1, we highlight in pink, in both cases labeling k in each row.

We present some theorems from [2] having to do with semicoprimality and its relevant varieties.

<u>SEMICOPRIMALITY</u>

Theorem S1: Let $P = \{$ prime $p : p \mid k \}$ and $Q = \{$ prime $q : q \mid n \}$. Semicoprimality $k \lozenge n$ implies $|\, P \cap Q \,| > 0$.

$$k \lozenge n \Rightarrow |\, P \cap Q \,| > 0. \qquad [2.3]$$

Proof. The definition of semicoprimality shows $1 < (k, n)$, with $k \neq (k, n) \neq n$, hence semicoprimality is neither coprimality nor divisorship and pertains to composites. It is clear that we can find at least 1 common prime divisor p such that $p \mid k$ and $p \mid n$. The definition of semicoprimality further shows that there is at least one prime q such that $q \mid k$ but does not divide n, proving n-semicoprime k is n-nonregular. Therefore $P \cap Q \neq \varnothing$; it contains at least 1 prime, but P contains other primes that are not in Q. (Q is not restricted only to those primes in P; there may be primes that divide n but do not divide k.) ■

Therefore symmetric semicoprimality is both ambidirectional in magnitude ($k \lesseqgtr n$) and completely ambiguous in terms of number of distinct prime divisors ($\omega(k) \lesseqgtr \omega(n)$).

Theorem S2: Asymmetric semicoprimality $k \lozenge\!\!\mid n$ implies $P \subset Q$ and $\omega(k) > \omega(n)$.

$$k \lozenge\!\!\mid n \Rightarrow \text{A1221}(k) > \text{A1221}(n).) \qquad [2.4]$$

Proof. We know $(k, n) > 1$ since k and n share at least 1 prime divisor p, yet at least 1 prime factor $q \mid k$ does not divide n via definition of semicoprime. Such implies k and n both exceed 1. Given n not semicoprime to k, then we are left with $n \mid k^{\varepsilon} : \varepsilon > 0$ (with respect to the context of coprime, semicoprime, and regular relations being mutually exclusive outside the empty product with domain $\mathbb{N}$). If $n \mid k$, then $n < k$ and $P \subset Q$, hence $\omega(k) > \omega(n)$. If n does not divide k, yet does divide some larger power of k, then, though we cannot speak to the relative magnitude of k and n, we are left with $P \subset Q$, hence $\omega(k) > \omega(n)$, proving the proposition. ■

Hence asymmetric semicoprime states are omega-directional.

<u>SYMMETRIC SEMICOPRIMALITY</u>

Lemma 1.1: Symmetric semicoprimality implies both k and n are composite.

$$k \lozenge\lozenge n \Rightarrow k \in \text{A2808} \wedge n \in \text{A2808}. \qquad [2.5]$$

Proof: Let $(k, n) = g$. Since $1 < g < k$ and $g < n$, k belongs to the cototient of n yet neither $k \mid n$ nor $n \mid k$. Since primes p must divide or be coprime to other numbers, $k \lozenge\lozenge n$ is restricted to composite numbers. ■

Lemma 1.2: Symmetric semicoprimality implies both $\omega(k)$ and $\omega(n)$ exceed 1. This is to say that both k and n are not prime powers.

$$k \lozenge\lozenge n \Rightarrow k \in \text{A024619} \wedge n \in \text{A024619}. \qquad [2.6]$$

Proof: A number k semicoprime to n is defined as $(k, n) > 1$ yet there exists at least 1 prime q such that $q \mid k$ but $q \nmid n$. Symmetric semicoprimality implies $|\, P \ominus Q \,| > 0$. Since k and n are at least divisible by some common prime p, and since each has at least 1 prime factor q not shared with the other, at least 2 prime factors are implied for both k and n. Hence both have at least 2 distinct prime divisors. ■

Corollary 1.3: Primes and multus numbers (composite prime powers) cannot be symmetrically semicoprime.

The definition of symmetric semicoprimality implies $\omega(n) \geq 2$ with the following consequences:

A360480(n) > 0 for $n \in$ A024619.
A360480(n) = 0 for $n \in$ A961.
A360480(6) = 0 since $k < 6$ are prime powers.

The sequence A360480(n) begins as follows:

```
0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 3, 3, 0, 0, 3,
0, 5, 5, 6, 0, 6, 0, 8, 0, 9, 0, 5, 0, 0, 8, 11, 7, 10,
0, 13, 10, 13, 0, 12, 0, 16, 13, 17, 0, 16, 0, 18, 14,
20, 0, 19, 11, 21, 16, 23, 0, 19, 0, 25, 19, 0, 13, ...
```

Relation between Symmetric Semicoprimality and the Cototient.

The sequence A361098 includes the following terms (where 0 represents a null row):

```
10:  6  .  .  .  .
11:  .  .  .  .  .  .
12:  .  .  .  .  10 .  .
13:  .  .  .  .  .  .  .  .
14:  6  .  .  .  10 .  12 .  .
15:  6  .  .  .  10 .  12 .  .  .
16:  .  .  .  .  .  .  .  .  .  .  .
17:  .  .  .  .  .  .  .  .  .  .  .  .
18:  .  .  .  .  10 .  .  .  14 15 .  .  .
19:  .  .  .  .  .  .  .  .  .  .  .  .  .  .
20:  6  .  .  .  .  .  12 .  14 15 .  .  18 .  .
21:  6  .  .  .  .  .  12 .  14 15 .  .  18 .  .  .
22:  6  .  .  .  10 .  12 .  14 .  .  .  18 .  20 .  .
23:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
24:  .  .  .  .  10 .  .  .  14 15 .  .  .  .  20 21 22 .  .
25:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
26:  6  .  .  .  10 .  12 .  14 .  .  .  18 .  20 .  22 .  24 .  .
```

Figures 1 and 3 enlarge the above triangle and lend context. It is clear there are increasingly many symmetrically n-semicoprime k as n increases.

We might remark on the "quincunx" pattern of semitotatives of n. The pattern arises given that of the cototient. Let us define the "quincunx" pattern as follows:

$$\text{A349297}(n) = \{\, Q(n,k) : k \leq n \,\},$$
$$Q(n,k) = [2 \mid n \lor 2 \mid k \lor 3 \mid n \lor 3 \mid k]. \qquad [3.1]$$

In other words, we have all even or trine k for even or trine n, where trine signifies $m \bmod 3 \equiv 0$.

We use the name quincunx for the 5-die pattern " ⁙ " that forms part of the plot of A349297(n, k). The sequence A349297 stands at issue because it comprises a significant part of the cototient; symmetrically semicoprime $k < n$ occur in the nondivisor cototient. The cototient has the pattern described in A349317 as follows:

$$\text{A349317}(n) = \{\, T(n,k) : k \leq n \,\},$$
$$T(n,k) = [\,(n,k) > 1\,]. \qquad [3.2]$$

We may write a sequence as follows:

$$\text{A349298}(n) = \{\, T(n,k) - Q(n,k) : k \leq n \,\}. \qquad [3.3]$$

Let Q(n) represent the cardinality of A349297(n):

$$\text{Q}(n) = |\, \{\, Q(n,k) : k \leq n \,\} \,| \qquad [3.4]$$

The first terms of Q(n), arranged mod 6, appear as follows:

```
0,  1,  1,  2, 0,  4,
0,  4,  3,  5, 0,  8,
0,  7,  5,  8, 0, 12,
0, 10,  7, 11, 0, 16,
0, 13,  9, 14, 0, 20,
0, 16, 11, 17, 0, 24, ...
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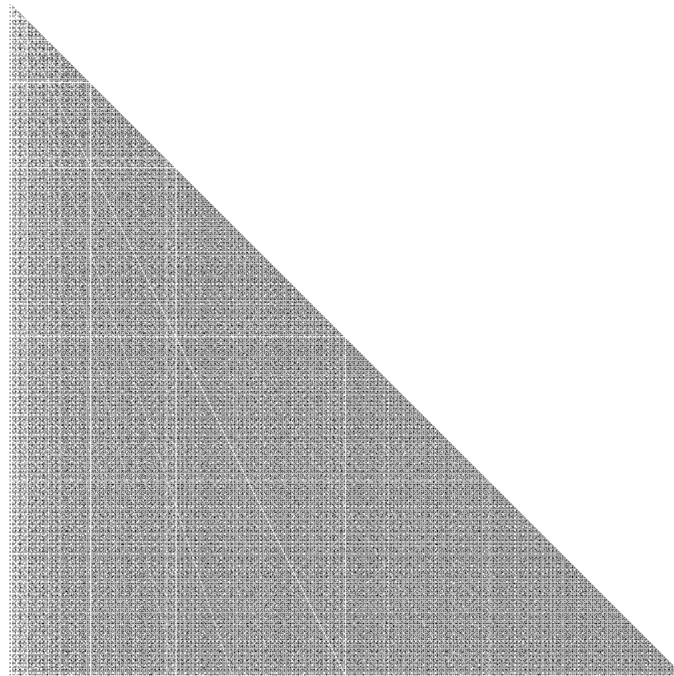

Figure 3: We plot k in black if k ① n and k < n and n ≤ 2^{10}, else white.

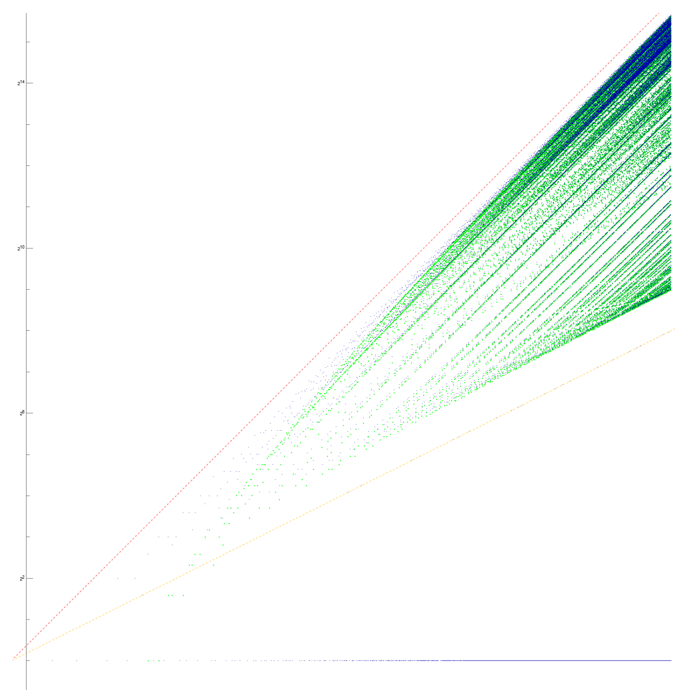

Figure 4: Log-log scatterplot of A360480(n) for n = 1 ... 2^{15}, ignoring 0s, showing squarefree composite n in green, n neither squarefree nor prime power in blue, with products of composite prime powers in large light blue and primorials in magenta.

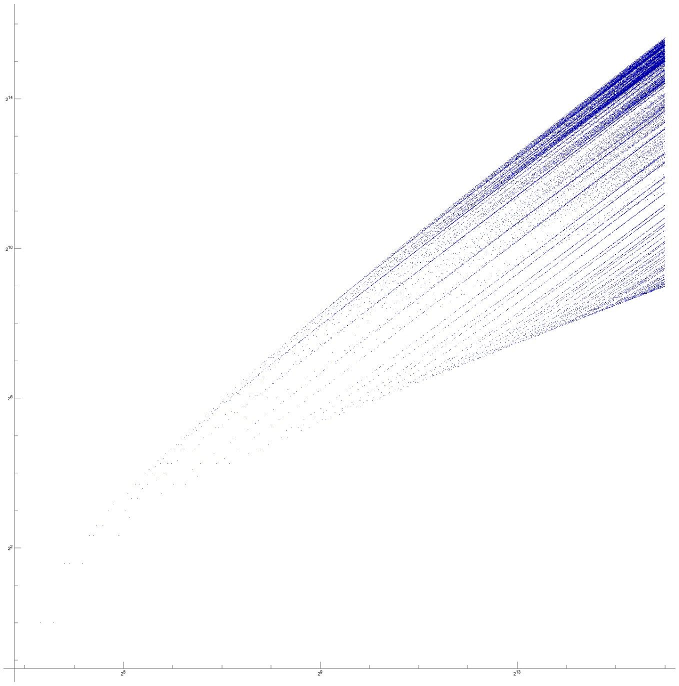

Figure 5: Log-log scatterplot of A360480*(n) for* $n = 1 \ldots 2^{15}$*, ignoring 0s, showing striations associated with* LPF*(n). This plot strongly resembles that of* A051953*(n) for sufficiently large n.*

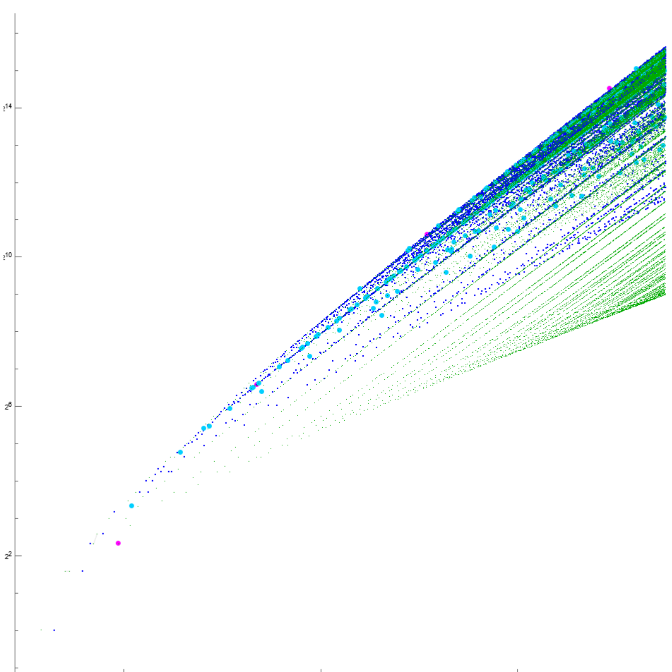

Figure 6: Log-log scatterplot of A360480*(n) for* $n = 1 \ldots 2^{15}$*, ignoring 0s, showing squarefree composite n in green, n neither squarefree nor prime power in blue, with products of composite prime powers in large light blue and primorials in magenta.*

It is clear that we might define a different way based on congruence relations, observing the following:

$$\begin{aligned} &\text{For } n \equiv 0 \pmod 6,\ Q(n) = ⅔\, n,\\ &\text{For } n \equiv \pm 1 \pmod 6,\ Q(n) = 0,\\ &\text{For } n \equiv \pm 2 \pmod 6,\ Q(n) = n/2,\\ &\text{For } n \equiv \pm 3 \pmod 6,\ Q(n) = n/3. \end{aligned} \qquad [3.5]$$

It is evident from scatterplot that A360840 that it is confined, once having “matured”, between ⅔ n and n. The upper bound is a consequence of the defintion of A360840 to be a counting function of a species of $k \le n$. We have not explored a reason for the apparent lower bound.

Regarding, we note the following:

$$\begin{aligned} \text{A051953}(n) &= \textstyle\sum \{ [(k, n) > 1] \wedge k \le n \} \\ &= \textstyle\sum \{ T(n, k) \wedge k \le n \} \\ &= \textstyle\sum \text{A349317}(n). \end{aligned} \qquad [3.6]$$

$$\begin{aligned} &\text{A051953}(n) > \text{A045763}(n) \ge \text{A360480}(n) \\ &\quad n - \phi(n) > \xi(n) \ge f_1(n) \\ &\quad n - \phi(n) > n - \phi(n) - \tau(n) + 1 \ge f_1(n) \end{aligned} \qquad [3.7]$$

The sum of A349298(n) is A051953$(n) = n - \phi(n)$. We find that, aside from prime powers, A360840 is a near image of A051953 and A045763. (See Figure 4.)

It seems evident, but remains unproved, that the following is true:

$$\xi(n) > f_1(n) \text{ for } n \in \text{A024619} \qquad [3.8]$$

From Theorems 4 and 5 in [3], we see that composites outside n = 4 and n = 6 have at least 1 semitotative, and non-prime powers outside n = 6 have at least 1 semidivisor $k < n$. The following table summarizes the findings in [3] regarding the existence of semidivisors and semitotatives in the reference domain of n of various species.

Table 1.

SPECIES	$\xi(n)$ A045763(n)	$\xi_d(n)$ A243822(n)	$\xi_t(n)$ A243823(n)
PRIMES (A40)	—	—	—
$n = 4$	—	—	—
MULTUS (A246547)	> 0	—	> 0
$n = 6$	—	1	1
VARIUS (A120944)	> 0	> 0	> 1
TANTUS (A126706)	> 0	> 0	> 1

Theorem 3.1: $\xi(n) > f_1(n)$ for $n \in$ A024619. Numbers n that are not prime powers are such that symmetric semicoprime $k < n$ are not the only n-neutral k such that $k < n$.

Proof: Theorem 5 in [3] shows that there is at least 1 semidivisor $k < n$ for numbers that are not prime powers. Additionally, Lemma 1.2 shows that all the semitotatives of multus n are in state ③. ■

Hence we have proved [3.8] to be true.

What remains is to explore the following difference:

$$\text{A360543}(n) = \xi(n) - f_1(n) \qquad [3.9]$$
(especially for $n \in$ A024619).

This sequence begins as follows:

```
0, 0, 0, 0, 0, 1, 0, 1, 1, 2, 0, 2, 0, 2, 1, 4, 0, 4,
0, 2, 1, 3, 0, 3, 3, 3, 6, 2, 0, 10, 0, 11, 2, 4, 1, 6,
0, 4, 2, 4, 0, 11, 0, 3, 3, 4, 0, 7, 5, 7, 2, 3, 0, 10,
1, 4, 2, 4, 0, 14, 0, 4, 3, 26, 1, 14, 0, 4, 2, 12, 0,
10, 0, 5, 5, 4, 1, 15, 0, 7, 23, 5, 0, 16, 1, 5, 3, 4,
0, 20, 1, 4, 3, 5, 1, 15, 0, 10, 3, 10, 0, 17, 0, 4, 8,
5, 0, 17, 0, 13, 3, 7, 0, 18, 1, 4, 3, 5, 1, 20, ...
```

Excepting $n \in$ A961, the records appear to be highly regular in many cases, and 3-smooth in others. The ratio S20230302(n)/$\xi(n)$ appears to converge to ⅙ for these records. Therefore the following seems apparent, though remains to be proved:

$$f_1(n)/\xi(n) \text{ converges to } 5/6 \text{ for } n \in \text{A024619} \quad [3.10]$$

If true, then we venture to suggest that symmetric semicoprimality is the most common form of semitotative, as seems to be borne by Figure 3.

For large numbers, accepting for the moment [3.10], then we may further see the following for large $n \in$ A024619:

$$\text{A051953}(n) \approx \text{A045763}(n) \approx 6/5\ \text{A045763}(n). \quad [3.11]$$

This unproved statement suggests that symmetric semicoprimality (state ①), with possible exception of coprimality, is the most common constitutive state.

Relation of $f_1(n)$ with the Semitotative Counting Function.

In the interest of context, the following is the related counting function $f_3(n)$ of mixed-neutral semitotatives:

$$f_3(n) = \{\, k < n : k \text{ ③ } n \,\} = \{\, k < n : k \lozenge\!\!\!\!\!\,\mid n \,\} = \text{A243823}(n) - \text{A360480}(n). \quad [4.1]$$

Since there are precisely 2 kinds of semitotatives; symmetric (state ①) and mixed-neutral (state ③), we may write the following:

$$\xi_T(n) = f_1(n) + f_3(n)$$
$$\text{A243823}(n) = \text{A360480}(n) + \text{A360543}(n). \quad [4.2]$$

Conclusion.

There are 2 varieties of n-semitotatives k; these are the symmetric and mixed variety. The former concerns $k < n$ such that prime $p \mid k$ but $(p, n) = 1$, while prime $q \mid n$ but $(q, k) = 1$. The latter regards k and n in cototient such that $\omega(n) \mid \omega(k) > \omega(n)$, while $\text{RAD}(n) \mid \text{RAD}(k)$. Using constitutive states, these are k ① n and k ③ n, respectively. We have shown that these are the only possible constitutive varieties of semitotative. We generated counting functions $f_1(n)$ = A360480(n) relating to k ① n, and $f_3(n)$ = A360543(n) relating to k ③ n, both such that $k < n$. Hence, $\xi_T(n) = f_1(n) + f_3(n)$, or in terms of OEIS, A243823(n) = A360480(n) + A360543(n).

Though k ③ n pertains to composite prime powers $n > 4$ exclusively, while k ① n pertains to squarefree composite $n > 6$ exclusively, both appear for certain numbers $n \in \{$ A360765 ∩ A360768 $\}$, a subset of A126706. Outside of these, generally $n \in$ A126706 harbors only k ① n.

The function $f_3(n)$ = A360543(n) is focus of a forthcoming paper.

We estimate that for numbers n that are not prime powers, the number of k ① n approaches ⅚ of the cototient of n, but this remains something to ascertain. Given the evident dominance of k ① n over k ③ n, it is not surprising that the scatterplot of A360480 resembles those of A045763 or A051953. ⁂

Appendix.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved February 2023.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

[3] Michael Thomas De Vlieger, Constitutive State Counting Function, *Simple Sequence Analysis*, 20230226.

[4] Michael Thomas De Vlieger, The Symmetric Semidivisor Counting Function, *Simple Sequence Analysis*, 20230216.

Code:

[C0] Function $f(k, n)$ yields the constitutive state (Svitek number) between k and n.

```
conState[j_, k_] :=
 Which[j == k, 5, GCD[j, k] == 1, 0, True,
  1 + FromDigits[
    Map[Which[Mod[##] == 0, 1,
      PowerMod[#1, #2, #2] == 0, 2, True, 0] & @@ # &,
      Permutations[{k, j}]], 3]]
```

[C1] Calculate $R_{\varkappa}$ bounded by an arbitrary limit m (i.e., calculate A275280(n); flatten and take union to provide A162306)

```
regularsExtended[n_, m_ : 0] :=
  Block[{w , lim = If[m <= 0, n, m]},
   Sort@ ToExpression@
     Function[w,
       StringJoin[
         "Block[{n = ", ToString@ lim,
         "}, Flatten@ Table[",
         StringJoin@
           Riffle[Map[ToString@ #1 <> "^" <>
             ToString@ #2 & @@ # &, w], " * "],
         ", ", Most@ Flatten@ Map[{#, ", "} &, #],
         "]]" ] &@
       MapIndexed[
         Function[p,
           StringJoin["{", ToString@ Last@ p,
             ", 0, Log[",
             ToString@ First@ p, ", n/(",
             ToString@
               InputForm[
                 Times @@ Map[Power @@ # &,
                   Take[w, First@ #2 - 1]]],
             ")]}" ] ]@ w[[First@ #2]] &, w]]@
       Map[{#, ToExpression["p" <>
         ToString@ PrimePi@ #]} &, #[[All, 1]] ] &@
       FactorInteger@ n];
```

[C2] Generate tantus numbers (A126706):

```
a126706 = Block[{k}, k = 0;
   Reap[Monitor[Do[
       If[And[#2 > 1, #1 != #2] & @@
         {PrimeOmega[n], PrimeNu[n]},
        Sow[n]; Set[k, n] ],

       {n, 2^21}], n]][[-1, -1]]] (* Tantus *);
```

[C3] Generate "strong tantus" numbers (A360768):

```
Select[a126706[[1 ;; 120]], #1/#2 >= #3 & @@
  {#1, Times @@ #2, #2[[2]]} & @@
  {#, FactorInteger[#][[All, 1]]} &]
```

[C4] Generate tantus numbers that have k ③ n (A360765):

```
nn = 2^20},
 rad[n_] := rad[n] = Times @@
  FactorInteger[n][[All, 1]];
 lcp[n_] := If[OddQ[n], 2,
   p = 2; While[Divisible[n, p], p = NextPrime[p]]; p];
 a = a126706[[1 ;; nn]];
 Monitor[ Reap[
   Do[n = a[[j]];
     If[rad[n]*lcp[n] < n, Sow[n]], {j, nn}]
   ][[-1, -1]], j] ]
```

[C5] Generate A360480, the k ① n counting function:

```
rad[x_] := rad[x] = Times @@
  FactorInteger[x][[All, 1]];
Table[k = rad[n];
  Count[Range[n],
    _?(Nor[CoprimeQ[#1, n], Divisible[#2, k],
       Divisible[k, #2]] & @@ {#, rad[#]} &)], {n, 88}]
```

[C6] Generate A360543, the k ③ n counting function:

```
nn = 120;
rad[n_] := rad[n] = Times @@
  FactorInteger[n][[All, 1]];
c = Select[Range[4, nn], CompositeQ];
s = Select[Select[Range[4, nn], Not @* SquareFreeQ],
  Function[{n, q, r},
    AnyTrue[TakeWhile[c, # <= n &],
      And[PrimeNu[#] > q,
          Divisible[rad[#], r]] &]] @@
          {#, PrimeNu[#], rad[#]} &];
Table[If[FreeQ[s, n], 0,
  Function[{q, r},
    Count[TakeWhile[
      c, # <= n &], _?(And[PrimeNu[#] > q,
         Divisible[rad[#], r]] &)]] @@
         {PrimeNu[n], rad[n]}], {n, nn}]
```

[C7] faster algorithm for A360543, the k ③ n counting function, given a dataset of A360765 and [C1]:

```
rad[n_] := rad[n] = Times @@
  FactorInteger[n][[All, 1]];
{{}, {}}~Join~Table[r = Rest@ regularsExtended[n];
  t = Rest@ Flatten@
    Outer[Plus, rad[n]*Range[0, n/rad[n] - 1],
      Select[Range[rad[n]], CoprimeQ[rad[n], #] &]];
  Union@ Flatten@
    Table[i j,
      {i, r[[1 ;; LengthWhile[r, n/t[[1]] > # &]]]},
      {j, t[[1 ;; LengthWhile[t, n/i > # &]]]}],
      {n, 3, 24}]
```

Concerns sequences:

A000005: Divisor counting function $\tau(n)$.
A000010: Euler totient function $\phi(n)$.
A000040: Prime numbers.
A000961: Prime powers.
A001221: Number of distinct prime divisors of n, $\omega(n)$.
A006881: Squarefree semiprimes.
A007947: Squarefree kernel of n; RAD(n).
A010846: Regular counting function.
A013929: Numbers that are not squarefree.
A024619: Numbers that are not prime powers.
A045763: Neutral counting function.
A051953: Cototient function: $n - \phi(n)$.
A120944: “Varius” numbers; squarefree composites.
A126706: “Tantus” numbers neither prime power nor squarefree.
A133995: Row n lists n-neutral k such that $k < n$.
A162306: Row n lists n-regular k such that $k \le n$.
A246547: “Multus” numbers; composite prime powers.
A272618: Row n lists n-semidivisors k such that $k < n$.
A272619: Row n lists n-semitotatives k such that $k < n$.
A355432: $a(n)$ = symmetric semidivisor counting function.
A360480: $a(n)$ = symmetric semicoprime counting function.
A360543: $a(n)$ = mixed semicoprime counting function.
A360765: $n \in$ A126706 : A7947$(n) \times$ A053669$(n) < n$.
A360767: Weakly tantus numbers.
A360768: Strongly tantus numbers.
A360769: Odd tantus numbers.
A361235: $a(n)$ = mixed semidivisor counting function..

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