The Semitotative Counting Function and Species.

Michael Thomas De Vlieger · St. Louis, Missouri · 8 March 2023.

Abstract.

Consider $k, n \in \mathbb{N}$ and define n-semicoprime k to be such that sets of prime divisors of k and that of n meet, yet $p \mid k$ but does not divide n. It is clear that semicoprimality requires both k and n composite. We consider $k < n$, thus k a semitotative of n. We describe symmetric and nonsymmetric varieties of the semitotative. This paper expands on an earlier work regarding symmetric semitotatives.

Introduction.

Consider the cototient of n, that is, those $k < n$ such that $(k, n) > 1$. In other words, if the reduced residue set $\mathrm{RRS}(n)$ includes $k < n$ such that $(k, n) = 1$, then the cototient is defined as follows:

$$c(n) = \{1 \dots n\} \setminus \mathrm{RRS}(n). \quad [1.1]$$

$$\begin{aligned} \mathrm{A051953}(n) &= |\, c(n) \,| \quad [1.2] \\ &= n - \phi(n). \\ &= n - \mathrm{A10}(n). \end{aligned}$$

Clearly, $\mathrm{A051953}(n) = 1$ for $n = p$, prime.

Within $c(n)$, we have divisors $d \mid n$, therefore we define the neutral cototient, $\Xi(n)$, the set of k neither coprime to n nor divisors of n, as follows:

$$\Xi(n) = c(n) \setminus \{\, d : d \mid n \,\}. \quad [1.3]$$

$$\begin{aligned} \xi(n) &= |\, \Xi(n) \,| \quad [1.4] \\ &= |\, \mathrm{A133995}(n) \,| \\ &= n - \phi(n) - \tau(n) + 1. \\ &= n - \mathrm{A10}(n) - \mathrm{A5}(n) + 1. \\ &= \mathrm{A045763}(n). \end{aligned}$$

As consequence of neutrality, k and n are composite, since primes p either divide n or are coprime to n. Furthermore, for $n = p$, $\xi(n) = 0$.

We may distinguish 2 species of n-neutral k based on the squarefree kernel $\mathrm{RAD}(m) = \mathrm{A7947}(m)$. The case $\mathrm{RAD}(k) \mid \mathrm{RAD}(n)$ implies k is n-regular, meaning that $k \mid n^{\varepsilon}$, $\varepsilon \ge 0$, that is, all prime factors of k also divide n. The n-regular numbers k are a superset of divisors $d \mid n^{\varepsilon}$, $\varepsilon = 0 \dots 1$; for $k \le n$, these numbers are listed in row n of A162306.

$$\begin{aligned} \mathrm{A162306}(n) &= \{\, k \le n : \mathrm{RAD}(k) \mid \mathrm{RAD}(n) \,\} \quad [1.5] \\ &= \{\, k \le n : k \mid n^{\varepsilon}, \varepsilon \ge 0 \,\} \\ &= \{\, d : d \mid n \,\} \cup \{\, k < n : k \mid n^{\varepsilon}, \varepsilon > 1 \,\} \\ &= \mathrm{A027750}(n) \cup \mathrm{A272618}(n) \end{aligned}$$

$$\begin{aligned} \mathrm{A010846}(n) &= |\, \mathrm{A162306}(n) \,| \\ &= |\, \mathrm{A027750}(n) \,| + |\, \mathrm{A272618}(n) \,| \\ &= \mathrm{A5}(n) + \mathrm{A243822}(n) \\ &= \tau(n) + \xi_D(n). \quad [1.6] \end{aligned}$$

Nondivisor n-regular k are called semidivisors, and are 1 of the 2 species in the neutral cototient [2], [3]. These are listed in $\Xi_D(n)$, that is, row n of A272618. The semidivisor counting function $\xi_D(n) = \mathrm{A243822}(n)$.

The other species is n-semicoprime k, $k < n$, hence we have called this species a "semitotative" of n. These are listed in $\Xi_T(n)$, that is, row n of A272619. The semidivisor counting function $\xi_T(n) = \mathrm{A243823}(n)$.

$$\Xi_T(n) = \Xi(n) \setminus \Xi_D(n) \quad [1.7]$$

$$\mathrm{A272619}(n) = \mathrm{A133995}(n) \setminus \mathrm{A272618}(n)$$

We can define the sequence $\Xi_T(n)$ from first principles:

$$\Xi_T(n) = \{\, k : k < n \wedge (k, n) > 1 \wedge \mathrm{RAD}(k) \nmid \mathrm{RAD}(n) \,\} \quad [1.8]$$

$$\begin{aligned} \xi_T(n) &= |\, \Xi_T(n) \,| \quad [1.9] \\ &= |\, \mathrm{A133995}(n) \,| - |\, \mathrm{A272618}(n) \,| \\ &= \mathrm{A243823}(n). \end{aligned}$$

Semicoprimality.

Where coprimality between k and n represents disjunct sets of prime divisors of k and n and regularity represents one set a subset of the other, semicoprimality represents an inhabited symmetric difference. Hence we can have n-semicoprime k, yet k-regular n and vice versa, while coprimality is always symmetric.

Definition 1: When we have at least 1 prime p such that $p \mid k$ that does not divide n, and at least 1 prime q such that $q \mid n$ that does not divide k, we have "symmetric semicoprimality".

In [2] we present and explain the following symbols:

Table A.

$k \perp n$	k is coprime to n	$(k, n) = 1$	
$k \lozenge n$	k is semicoprime to n	$1 < (k, n) < \mathrm{MIN}$	$n/(k, n) \nparallel n$
$k \parallel n$	k is regular to n	$1 \le (k, n) \le \mathrm{MIN}$	$k \mid n^{\varepsilon} : \varepsilon \ge 0$
$k \mid n$	k divides n	$1 \le (k, n) = k$	$k \mid n^{\varepsilon} : \varepsilon = 0 \dots 1$
$k \,¦\, n$	k semidivides n	$1 < (k, n) < \mathrm{MIN}$	$k \mid n^{\varepsilon} : \varepsilon > 1$

Hence, writing $k \parallel\lozenge\, n$ signifies k regular to n, but n semicoprime to k, while $k \lozenge\lozenge\, n$ indicates symmetric semicoprimality.

The following table summarizes basic relations between k and n. Let $\mathrm{P}(m) = \{\, \text{prime } p : p \mid m \,\}$.

Table B.

Relation	Setwise	Kernelwise
$k \perp n$	$\mathrm{P}(n) \cap \mathrm{P}(k) = \varnothing$	$\mathrm{RAD}(k) \perp \mathrm{RAD}(n)$
$k \lozenge\lozenge\, n$	$\mathrm{P}(n) \ominus \mathrm{P}(k) \ne \varnothing$: $\mathrm{P}(k) \setminus \mathrm{P}(n) \ne \varnothing \wedge$ $\mathrm{P}(n) \setminus \mathrm{P}(k) \ne \varnothing$	$\mathrm{RAD}(k) \nmid \mathrm{RAD}(n) \wedge$ $\mathrm{RAD}(n) \nmid \mathrm{RAD}(k)$
$k \lozenge\parallel n$	$\mathrm{P}(n) \ominus \mathrm{P}(k) \ne \varnothing$: $\mathrm{P}(k) \setminus \mathrm{P}(n) \ne \varnothing \wedge$ $\mathrm{P}(n) \setminus \mathrm{P}(k) = \varnothing$	$\mathrm{RAD}(k) \nmid \mathrm{RAD}(n) \wedge$ $\mathrm{RAD}(n) \mid \mathrm{RAD}(k)$
$k \parallel\lozenge\, n$	$\mathrm{P}(n) \ominus \mathrm{P}(k) \ne \varnothing$: $\mathrm{P}(k) \setminus \mathrm{P}(n) = \varnothing \wedge$ $\mathrm{P}(n) \setminus \mathrm{P}(k) \ne \varnothing$	$\mathrm{RAD}(k) \mid \mathrm{RAD}(n) \wedge$ $\mathrm{RAD}(n) \nmid \mathrm{RAD}(k)$
$k \parallel\parallel n$	$\mathrm{P}(k) = \mathrm{P}(n)$	$\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$

Coprimality is always symmetric, as is the cototient. Within cototient, we have the following relations:

$\lozenge\lozenge$	$\lozenge\parallel$ or $\parallel\lozenge$	$\parallel\parallel$
Symmetric Semicoprimality	Mixed Cototient	Symmetric Regularity

In this work we are concerned only with symmetric semicoprimality ($\lozenge\lozenge$) and mixed semicoprimality ($\lozenge\parallel$). The other two relations are forms of regularity.

The existence of 2 species of regular numbers (the divisor and the semidivisor) implies corresponding mixed cototient states [2]:

◊◊	◊∣ or ∣◊	◊¦ or ¦◊
Symmetric Semicoprimality	Lean Divisorship	Mixed Neutrality
①	②④	③⑦

We express symmetric semicoprimality symbolically via *k* ◊◊ *n*, i.e, *k* ① *n* per [2]. These are *k* and *n* in the semicoprime cototient absent divisorship between their squarefree kernels.

For $k < n$, it is clear that we cannot have state ②, that is *k* ◊∣ *n*, since that would require $k > n$, a contradiction. The mixed neutral state ⑦, i.e., *k* ¦◊ *n*, is not at issue, since it is a kind of semidivisor. The lean divisor state ④, *k* ∣◊ *n*, is also immaterial, since it is a kind of divisor. Therefore the category of lean divisorship can be ignored, but the test $n \nmid k$ is insufficient as a means to determine symmetric semicoprimality. This leaves us with ① (*k* ◊◊ *n*) or ③ (*k* ◊¦ *n*).

Thus, for our purposes, we are only interested in disambiguating states ① and ③, the former corresponding to symmetric semicoprimality and the latter to mixed or nonsymmetric semicoprimality. We are only interested in cases $k < n$.

Definition 2: When we have at least 1 prime *p* such that $p \mid k$ that does not divide *n*, yet all primes *q* that divide *n* also divide *k*, we have “nonsymmetric semicoprimality”.

Definition 3: An “*n*-semitotative” is *k* such that $k < n$ and *k* ◊ *n*. This term resonates with the term “totative” applied to a reduced residue $t < n$ such that $(t, n) = 1$. Therefore, we may call *k* ① *n* a symmetric semitotative, and *k* ③ *n* a nonsymmetric semitotative.

The set of semitotatives, $\Xi_T(n)$, potentially includes both *n*-semicoprime *k* for which *n* is *k*-regular, (i.e., $\mathrm{rad}(n) \mid \mathrm{rad}(k)$) and where *n* is *k*-semicoprime. Therefore the following is necessary to create a set S_1 of symmetric semitotatives:

$$\begin{aligned} S_1 &= \{\, k \in \text{A272619}(n) : \mathrm{rad}(n) \nmid k \,\} \\ &= \{\, k : k < n \wedge (k, n) > 1 \wedge \\ &\quad \mathrm{rad}(k) \nmid n \wedge \mathrm{rad}(n) \nmid k \,\} \end{aligned} \tag{2.1}$$

We also define a set S_3 of nonsymmetric semitotatives:

$$\begin{aligned} S_3 &= \{\, k \in \text{A272619}(n) : \mathrm{rad}(n) \mid k \,\} \\ &= \{\, k : k < n \wedge (k, n) > 1 \wedge \\ &\quad \omega(k) > \omega(n) \wedge \mathrm{rad}(n) \mid k \,\} \end{aligned} \tag{2.2}$$

The symmetric semicoprime counting function f_1 thus is defined as follows:

$$f_1(n) = \text{A360480}(n) = |\, S_1 \,| \tag{2.3}$$

The nonsymmetric semicoprime counting function thus is defined as follows:

$$f_3(n) = \text{A360543}(n) = |\, S_3 \,| \tag{2.4}$$

The sequence A272619 lists the following semidivisors $k < n$ for nonsquarefree $n = 8 \ldots 28$ (where 0 in OEIS represents a null row):

```
 8:  6;
 9:  6;
10:  6;
12: 10;
14:  6, 10, 12;
15:  6, 10, 12;
16:  6, 10, 12, 14;
18: 10, 14, 15;
20:  6, 12, 14, 15, 18;
21:  6, 12, 14, 15, 18;
22:  6, 10, 12, 14, 18, 20;
24: 10, 14, 15, 20, 21, 22;
25: 10, 15, 20;
26:  6, 10, 12, 14, 18, 20, 22, 24;
27:  6, 12, 15, 18, 21, 24;
28:  6, 10, 12, 18, 20, 21, 22, 24, 26; ...
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Distinguishing Species of Semitotatives.

We introduce means by which we may distinguish symmetric from nonsymmetric semitotatives.

We present some theorems from [2] and [3] having to do with semicoprimality and its relevant varieties. General proofs regarding semicoprimality precede proofs pertaining to the species of semitotatives and omega-multiplicity classes of *n* to which they pertain.

semicoprimality

Theorem 1.1: *n*-semicoprime *k* implies both *k* and *n* are composite.
Proof: The definition of *n*-semicoprime *k* implies $(k, n) > 1$ and $k \nmid n$. Primes *p* must either divide another number or be coprime to that number. Therefore, *n*-semicoprime *k* cannot be prime. Furthermore, *n* cannot be prime since all $k < n$ are coprime to *n*, but n-semicoprime *k* implies *k* and *n* are in cototient. ■

Theorem 1.2: *n*-semicoprime *k* implies *k* is not a prime power.
Proof: By definition, *n*-semicoprime *k* is such that *k* and *n* share at least 1 prime factor *p*, yet there is at least one prime factor *q* such that $q \mid k$ but $q \nmid n$. Therefore, at minimum, $k = pq$, $p \neq q$. ■

Corollary 1.3: Set $p = \mathrm{lpf}(n) = \text{A020639}(n)$, the least prime factor of *n*, and set $q = \text{A053669}(n)$, the smallest prime that is coprime to *n*. The number $k = pq$ is the smallest number semicoprime to *n*.

Corollary 1.4: For odd *n*, $k = 2p$ is the smallest semicoprime number, where $p = \mathrm{lpf}(n) = \text{A020639}(n)$.

Corollary 1.5: For prime $n = p$, *n*-semicoprime *k* is such that $k > p$.

nonsymmetric semicoprimality

Lemma 2.1: Numbers *k*, *n* such that *k* ◊¦ *n* imply both *k* and *n* are composite. In other words, if *k* or *n* are prime, *k* ◊¦ *n* is impossible.
Proof. We have shown *k* ◊ *n* implies both *k* and *n* are composite. We therefore show that this is true when *n* is a semidivisor of *k*. *n* ¦ *k* implies composite *n* since $1 < (k, n) < n$ by definition of semidivisor *n* ¦ *k* as nondivisor regular $n \mid k^\varepsilon : \varepsilon > 1$. Hence *k* and *n* are neutral in both directions, while a prime must either divide or be coprime to another number. Therefore both *k* and *n* are composite. ■

Lemma 2.2: Numbers *k*, *n* such that *k* ◊¦ *n* imply $\omega(k) > \omega(n)$.
Proof: *k* ◊ *n* implies that *k* is divisible by $Q > 1$ such that $(Q, n) = 1$, yet *n* is regular to *k*, meaning that *n* is a product of primes $p \mid k$ and no prime $q \nmid k$. Further, *n* does not divide *k*, yet $\mathrm{rad}(n) \mid k$, and it is clear that $\omega(k) > \omega(n)$. ■

Corollary 2.3: For *k*, *n* such that *k* ◊¦ *n*, *k* cannot be a prime power. Mixed neutrality and $n = p^\varepsilon$ implies *n* such that $p^{(\varepsilon - j)} \mid k \wedge j > 0$.

symmetric semicoprimality

Corollary 3.1: Symmetric semicoprimality implies both *k* and *n* are composite. Consequence of Theorem 1.1.

Lemma 3.2: Symmetric semicoprimality implies both $\omega(k)$ and $\omega(n)$ exceed 1. This is to say that both *k* and *n* are not prime powers.
Proof: A number *k* semicoprime to *n* is defined as $(k, n) > 1$ yet there exists at least 1 prime *q* such that $q \mid k$ but $q \nmid n$. Symmetric semicoprimality implies $|P \ominus Q| > 0$. Since *k* and *n* are at least divisible by some common prime *p*, and since each has at least 1 prime factor *q* not shared with the other, at least 2 prime factors are implied for both *k* and *n*. Hence both have at least 2 distinct prime divisors. ■

Corollary 3.3: Primes and multus numbers (composite prime powers) cannot be symmetrically semicoprime.

Semitotatives and Omega-Multiplicity Classes.

We now examine the three remaining omega-multiplicity classes regarding the existence of symmetric or nonsymmetric semitotatives. Let's recapitulate these classes that were described in [2]:

We divide natural numbers $n \in \mathbb{N}$ into 5 categories based upon prime decomposition of n. The number n is said to be squarefree iff $\omega(n) = \Omega(n)$. The number n is said to be prime iff $\omega(n) = \Omega(n) = 1$, and a prime power iff $\omega(n) = 1$. The empty product $n = 1$ occupies a category all to itself, therefore, we may hold that there are actually 4 nontrivial categories. We further distinguish numbers instead with $\textsc{m}(n)$ = the largest multiplicity in n, meaning the largest exponent ε such that any prime power $p^\varepsilon \mid n$.

Table D.

	$\textsc{m}(n) = 1$	$\textsc{m}(n) > 1$
$\omega(n) > 1$	**multus** 8, 27, 125 A246547	**tantus** 12, 75, 216 A126706
$\omega(n) = 1$	**prime** 2, 17, 101 A40	**varius** 6, 35, 210 A120944

Multus numbers are composite prime powers $n \in$ A246547, while varius numbers are squarefree composites $n \in$ A120944. Numbers that are neither squarefree nor prime powers are called tantus and appear in A126706. Numbers that are both squarefree and prime powers are prime.

We define a subset of tantus numbers for which all prime power factors $p^\varepsilon \mid n$ such that $\varepsilon > 1$. This is tantamount to the powerful numbers A1694 without prime powers A961, i.e., A1694 \ A961. We call these **plenus** ("full") numbers (A286708). Another way to think of plenus numbers is as a product of multus numbers, or varius numbers where each prime divisor is raised to some power $\varepsilon > 1$.

Multus

Lemma 4.1: Semitotatives $k \between n$, $k < n$, for multus $n \in$ A246547 are never symmetric.

Proof: An n-semitotative k is n-semicoprime, with $k < n$. The definition of n-semicoprime k requires a prime $p \mid k$ that does not divide n, yet $(k, n) > 1$. Symmetric semicoprimality has prime $q \mid n$ such that q does not divide k, yet $(k, n) > 1$. We have to show that n is k-semicoprime, however, such implies $\omega(n) > 1$. (See [2], Lemma 1.2) ■

Varius

Lemma 4.2: For varius $n > 6$, all semitotatives are symmetric.

Proof: There are 2 constititive species of semitotatives; symmetric k ① n and nonsymmetric k ③ n. Therefore to prove the proposition, we need to show that squarefree n does not semidivide k. The expression $n \mathrel{¦} k$ implies $\textsc{rad}(n) \mid \textsc{rad}(k)$, but $n \nmid k$. The latter is true since $k < n$, but the former implies $\textsc{rad}(n) = \textsc{rad}(k)$, that is, all primes $q \mid n$ also divide k, contradicting $k \between n$. We note $k < 6$ are prime powers, therefore, there are no semitotatives for $n = 6$. ■

Tantus

Lemma 4.3: For tantus n, there exists at least 1 symmetric semitotative k ① n.

Proof: Set $p = \textsc{lpf}(n) =$ A020639(n) and set $q =$ A053669(n), the smallest prime that is coprime to n. It is clear pq ① n for all $n \in$ A126706 by definition of "semicoprime". Now we attempt to show $pq < n$ for some $n \in$ A126706. For $n =$ A126706(1) = 12, we have $pq = 2 \times 5 = 10$; $10 < 12$. If we set $p > 2$, supposing $p^2 \mid n$ in order to minimize n so as to force pq to exceed n, then $q = 2$, and we are only making larger n. It becomes clear that to maximize q but retaining $p = 2$ and $p^2 \mid n$, we require $n = 2P(i) = 2 \times$ A2110(i), $i > 1$. Through induction on i, it is clear that $pq < 2P(i)$. ■

We note that for $n = 12$, 10 is the sole semitotative; there are none of the mixed neutral variety k ③ n. We do see that for $n = 45$, we have $k = 30$, thus k ③ n.

Lemma 4.4: There exists k such that $k < n$ and k ③ n for odd $n \in$ A126706. Odd tantus n have nonsymmetric semitotatives k.

Proof: Break the expression k ③ n into components $k \between n$ and $n \mathrel{¦} k$. We may rewrite the latter component as $\textsc{rad}(n) \mid \textsc{rad}(k)$, where, per the former component, $\textsc{rad}(k) = q \times \textsc{rad}(n)$ and q coprime to n. If n is odd, then we can produce the state via $k = 2\varkappa$, where $\varkappa = \textsc{rad}(n)$. Since n is tantus, $n \ge p\varkappa$ such that $p \mid \varkappa$ and $p > 2$. Therefore it is clear that $2 < p\varkappa \le n$. ■

Theorem 4.5: Certain even tantus numbers n have both symmetric (k ① n) and nonsymmetric semitotatives (k ③ n).

Proof: Define the set of k-regular numbers $\boldsymbol{R}_\varkappa$, where $\varkappa = \textsc{rad}(k)$, to be as follows:

$$\boldsymbol{R}_\varkappa = \bigotimes_{p \mid \varkappa} \{p^\varepsilon : \varepsilon \ge 0\}. \qquad [3.5]$$

All numbers $m \in \boldsymbol{R}_\varkappa$ are such that $\textsc{rad}(m) \mid \varkappa$. Therefore, $n \mathrel{¦} k$ implies $k, n \in \boldsymbol{R}_\varkappa$ and hence $\textsc{rad}(n) \mid \varkappa$. Since we restrict numbers in $\boldsymbol{R}_\varkappa$ to primes $p \mid \varkappa$, to construct k ③ n, $k < n$, it is sufficient merely to find $n \in \boldsymbol{R}_\varkappa$ such that $n > k$ and $\omega(n) < \omega(k)$.

We pursue a strategy akin to Theorem 1.1, setting $q =$ A053669(n) and resetting $\varkappa$ instead to $\textsc{rad}(n)$ to guarantee $\textsc{rad}(n) \mid k$. Therefore, $q\varkappa < n$ implies k ③ n and $k < n$. The smallest case is $k = 30$, $n = 36$.

It is clear that such tantus numbers n have k ① n and $k < n$, via Lemma 4.3. Hence, the proposition is true. ■

Heavy Tantus Numbers

We thus define the sequence of "heavy tantus" numbers A360765 ⊂ A126706 containing tantus numbers that have mixed semitotatives k ③ n and $k < n$ that begins with the following numbers:

```
36, 40, 45, 48, 50, 54, 56, 63, 72, 75, 80, 88, 96, 98,
99, 100, 104, 108, 112, 117, 135, 136, 144, 147, 152,
153, 160, 162, 171, 175, 176, 184, 189, 192, 196, 200,
207, 208, 216, 224, 225, 232, 240, 242, 245, 248, ...
```

It is clear that these numbers n comprise the only subclass that harbors both nonsymmetric and symmetric semitotatives $k < n$.

Theorem 4.6. Distinct $m, n \in$ A360765 such that both have same squarefree kernel $\varkappa$ implies that mixed semicoprime k pertains to both m and n, and symmetric semicoprime k pertains to both m and n.

Proof: Suppose we have 2 distinct numbers $m, n \in$ A360765 such that $\textsc{rad}(m) = \textsc{rad}(n) = \varkappa$ and $n < m$. It is clear that $\textsc{rad}(m) = \textsc{rad}(n) = \varkappa$ implies $\omega(m) = \omega(n) = \textsc{q}$. Therefore, if we have $k < n$ such that k ③ n, we know that $\varkappa \mid \textsc{rad}(k)$ (which itself implies cototient), and $\omega(k) > \textsc{q}$. Hence, if we have k ③ n, then we have k ③ m and vice versa. ■

Lemma 4.7: A286708 ⊂ A360765. In other words, plenus numbers n that are products of at least 2 composite prime powers p^ε such that $\varepsilon > 1$ (i.e., $n \in$ A286708) have $\varkappa q < n$ where $\varkappa = \textsc{rad}(n) =$ A7947(n) and $q = \textsc{lpc}(n) =$ A053669(n).

Proof: The proposition is true since $q < \varkappa$, hence $\varkappa q < m\varkappa^2$, $m \ge 1$. ■

Corollary 4.8: Powerful numbers $n > 1$ (i.e., $n \in$ A1694) have nonsymmetric semitotatives k ③ n. Consequence of Lemmas 4.3 and 4.7, and the following:

$$\text{A1694} = \text{A246547} \cup \text{A286708} \cup \{1\} \qquad [4.8]$$

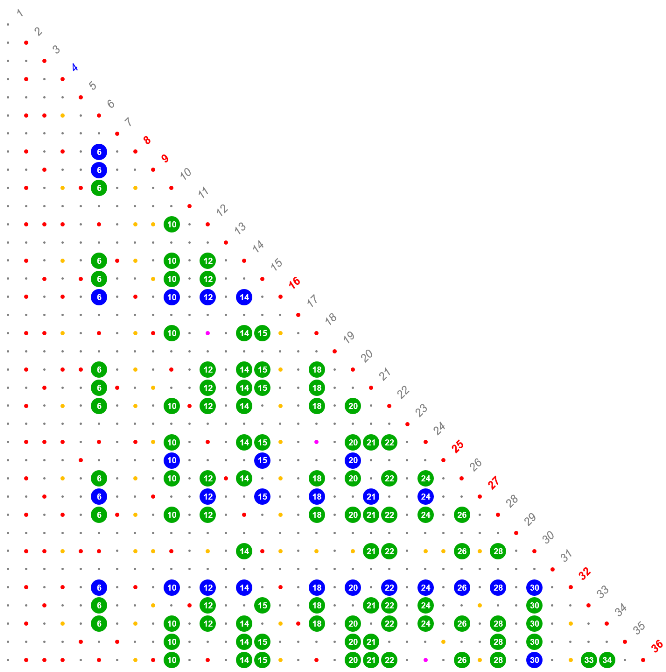

Figure 1: A map of constitutive states in the cototient between k and n for k ≤ 36 and n ≤ 36. Green circles are in state ① and blue represents state ③, while gray dots represent coprimality (state ⓪). Red dots represent divisor states ④⑤⑥, notably excepting k = 1. Yellow represents state ⑦. Finally, magenta represents symmetric semidivisibility, state ⑨, which requires rad(k) = rad(n).

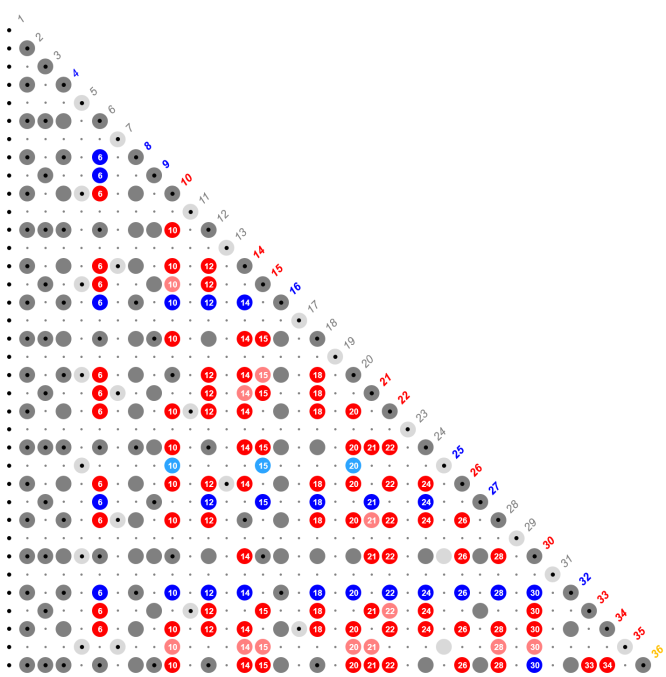

Figure 2: Relationship of symmetric n-semicoprime k to "quincunx" numbers and the cototient in general. Plot k and n for k ≤ 36 and n ≤ 36 at (k, −n). We show "quincunx" numbers Q(n, k) = [OR(2|k, 3|k, 2|n, 3|n)] in dark gray, T(n, k) = [(k, n) > 1] in light gray, k coprime to n with a gray dot, and k | n with a black dot. For k ① n : Q(n, k) = 1, we highlight in red, and for k ① n : T(n, k) = 1, we highlight in pink, in both cases labeling k in each row. For k ③ n : Q(n, k) = 1, we highlight in blue, and for k ③ n : T(n, k) = 1, we highlight in light blue, in both cases labeling k in each row.

Counting Functions f_1 and f_3.

The sequence S_1 lists symmetric semitotatives k ① n:

```
10:  6  .  .  .  .
11:  .  .  .  .  .  .
12:  .  .  .  . 10  .  .
13:  .  .  .  .  .  .  .  .
14:  6  .  .  . 10  . 12  .  .
15:  6  .  .  . 10  . 12  .  .  .
16:  .  .  .  .  .  .  .  .  .  .  .
17:  .  .  .  .  .  .  .  .  .  .  .  .
18:  .  .  .  . 10  .  .  . 14 15  .  .  .
19:  .  .  .  .  .  .  .  .  .  .  .  .  .  .
20:  6  .  .  .  .  . 12  . 14 15  .  . 18  .  .
21:  6  .  .  .  .  . 12  . 14 15  .  . 18  .  .  .
22:  6  .  .  . 10  . 12  . 14  .  .  . 18  . 20  .  .
23:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
24:  .  .  .  . 10  .  .  . 14 15  .  .  .  . 20 21 22  .  .
25:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
26:  6  .  .  . 10  . 12  . 14  .  .  . 18  . 20  . 22  . 24  .  .
```

The sequence S_3 lists symmetric semitotatives k ③ n:

```
 8:  6  .  .
 9:  6  .  .  .
10:  .  .  .  .  .
11:  .  .  .  .  .  .
12:  .  .  .  .  .  .  .
13:  .  .  .  .  .  .  .  .
14:  .  .  .  .  .  .  .  .  .
15:  .  .  .  .  .  .  .  .  .  .
16:  6  .  .  . 10  . 12  . 14  .  .
17:  .  .  .  .  .  .  .  .  .  .  .  .
18:  .  .  .  .  .  .  .  .  .  .  .  .  .
19:  .  .  .  .  .  .  .  .  .  .  .  .  .  .
20:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
21:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
22:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
23:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
24:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
25:  .  .  .  . 10  .  .  .  . 15  .  .  .  . 20  .  .  .  .  .
26:  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
```

Figure 1 overlays these 2 charts, with S_1 in green and S_3 in blue. Figure 3 shows instead in red, and shows that semitotatives dominate the cototient as n increases, but symmetric semitotatives predominate over nonsymmetric.

We defined a symmetric semitotative counting function $f_1(n) =$ A360480$(n) = |\, S_1 \,|$ in [2.3]. The definition of symmetric semicoprimality implies $\omega(n) \geq 2$ with the following consequences:

A360480$(n) > 0$ for $n \in$ A024619 via Lemmas 4.2 and 4.3.
A360480$(n) = 0$ for $n \in$ A961 via Lemma 4.1.
A360480$(6) = 0$ since $k < 6$ are prime powers.

Corollaries 3.1 and 3.3 and Lemma 3.2 show the following:

$$\text{A360480}(n) = \big|\, \{\, k < n : (k, n) > 1 \land (\text{RAD}(k) \mid n \mathbin{\bar\vee} \text{RAD}(n) \mid k)\, \}\,\big| \qquad [5.0]$$

The sequence A360480(n) begins as follows:

```
0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 3, 3, 0, 0, 3,
0, 5, 5, 6, 0, 6, 0, 8, 0, 9, 0, 5, 0, 0, 8, 11, 7, 10,
0, 13, 10, 13, 0, 12, 0, 16, 13, 17, 0, 16, 0, 18, 14,
20, 0, 19, 11, 21, 16, 23, 0, 19, 0, 25, 19, 0, 13, ...
```

Lemma 5.1: For non-prime-powers n (i.e., $n \in$ A024619), the following equation is true:

$$\begin{aligned}\text{A360480}(n) &= n - \phi(n) - \text{RCF}(n) + 1 \\ &= n - \text{A10}(n) - \text{A010846}(n) + 1 \qquad [5.1]\end{aligned}$$

Proof: Consequence of Lemmas 4.2 and 4.3 and the following:

A045763$(n) = n -$ A10$(n) -$ A5$(n) + 1 =$ A243822$(n) +$ A243823(n)
A243823$(n) = n -$ A10$(n) - ($A5$(n) +$ A243822$(n)) + 1$
A243823$(n) = n -$ A10$(n) -$ A010846$(n) + 1$.

Then, since all semitotatives of $n \in$ A024619 are symmetric, hence A243823$(n) =$ A360480(n), it is plain that the proposition is true. ■

Likewise, we defined a nonsymmetric semitotative counting function $f_3(n)$ = A360543(n) = $|\, S_3 \,|$ in [2.4]. The definition of symmetric semicoprimality implies $\omega(n) \geq 2$ with the following consequences:

The sequence A360543(n) begins as follows:

```
0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 4, 0, 0,
0, 0, 0, 0, 0, 0, 3, 0, 6, 0, 0, 0, 0, 11, 0, 0, 0, 1,
0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 2, 5, 1, 0, 0, 0, 2,
0, 1, 0, 0, 0, 0, 0, 0, 1, 26, 0, 0, 0, 0, 0, 0, 0, ...
```

Lemma 4.4 and Theorem 4.5 prove the following:

$$\text{A360543}(n) = \left|\, \{\, k < n : \text{RAD}(n) \mid k \wedge \omega(k) > \omega(n) \,\} \,\right| \qquad [5.2]$$

Consequently, we find the following:

A360543$(n) = 0$ for $n \in$ A5117 via Lemma 4.2.
Let $\mathcal{M} = \{$A246547 ∪ A360765$\} \setminus \{4\}$.
A360543$(n) > 0$ for $n \in \mathcal{M}$ via Lemmas 4.1, 4.4, and 4.5.

Theorem 5.2: For $n = p^\varepsilon \in$ A961 : $n > 4$, A360543$(p^\varepsilon) = p^{(\varepsilon-1)} - \varepsilon$.
Proof: Consider $k \in (1 \ldots p^\varepsilon)$, such that $(k, p^\varepsilon) > 1$, that is, any k such that k/p is an integer. This leaves us with $k = mp$, where $m \leq p^\varepsilon$. Through p^ε/p, we define the range of $m = 1 \ldots p^{(\varepsilon-1)}$. For $k = mp$, any prime power $p^\delta \mid p^\varepsilon$, $\delta \leq \varepsilon$, and there are $\tau(p^\varepsilon) - 1 = (\varepsilon+1) - 1 = \varepsilon$ of these. Hence we subtract $p^{(\varepsilon-1)} - \varepsilon$ to find $k < n$ such that k ③ n for n a prime power. The case of $n = 4$ yields A360543$(4) = 0$ since $k < 4$ are either coprime to 4 or divide 4. ■

Lemma 5.3: For composite prime powers n (i.e., $n \in$ A246547), the following equation is true:

$$\begin{aligned} \text{A360543}(n) &= n - \phi(n) - \tau(n) + 1 \\ &= n - \text{A10}(n) - \text{A5}(n) + 1 \qquad [5.3] \end{aligned}$$

Proof: Consequence of Lemma 4.2 and since prime powers do not have semidivisors $k < n$, the following:

$$\text{A045763}(n) = n - \text{A10}(n) - \text{A5}(n) + 1 = \text{A243823}(n)$$

Then, since all semitotatives of $n \in$ A246547 are nonsymmetric, hence A243823(n) = A360543(n), the proposition is true. ■

Record Setters in A360543.

Records seem to occur for n amid powers 2^δ, $\delta > 2$ and 3^ε, $\varepsilon > 1$, and may be related to A334151.

A3557 is the sequence defined as follows:

$$n = \prod p_i^{\varepsilon_i} \Rightarrow a(n) = \prod p_i^{(\varepsilon_i - 1)}, \text{ with } a(1) = 1, \\ \text{thus, } a(n) = n/\text{RAD}(n). \qquad [5.4]$$

Define A334151 to include the following k:

$$k/\text{RAD}(k) > j/\text{RAD}(j) \text{ for all } j < k. \qquad [5.5]$$

Hence, recordsetters in A3557 comprise A334151.

Lemma 5.4: $\varkappa \in$ A5117 implies $\varkappa/\text{RAD}(\varkappa) = 1$.
Proof: Prime p is such that $p = \text{RAD}(p)$, therefore $p/\text{RAD}(p) = 1$, and that, generally, squarefree $\varkappa \in$ A5117 is such that $\varkappa = \text{RAD}(\varkappa)$, and therefore $\varkappa/\text{RAD}(\varkappa) = 1$. Thus, we show only nonsquarefree numbers $k \in$ A013929 are such that $k/\text{RAD}(k) > 1$. ■

Let $\text{RAD}(j) = \varkappa$. We may express $k/\text{RAD}(k)$ as $k/\varkappa = m$, where m is an integer exceeding 1. Now the question, among multus and tantus numbers, is, which maximizes m first?

Note the following:

$$\text{For } k = p^\varepsilon, \varepsilon > 1, m = p^{(\varepsilon-1)}, \\ \text{For } k = p^\varepsilon q, \varepsilon > 1, m = p^{(\varepsilon-1)}, p^\varepsilon < p^\varepsilon q. \qquad [5.6]$$

Figure 3: For $n \leq 120$ and $k < n$, we plot k in red if k ① n, blue if k ③ n, light yellow if $k \mid n$, gray if $(k, n) > 1$, and white if $(k, n) = 1$. Since red and blue together represent k in A272619(n), the numbers k shown in gray are n-semidivisors that appear in A272618(n).

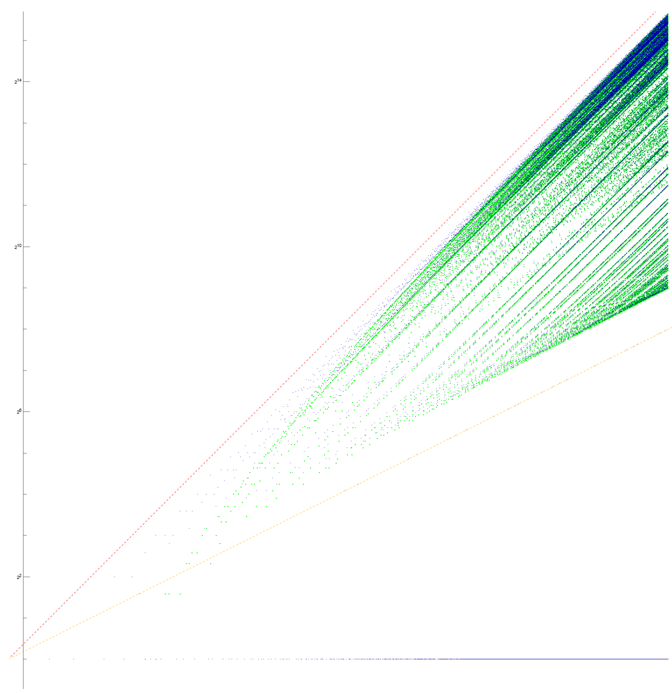

Figure 4: Log-log scatterplot of A360480(n) for $n = 1 \ldots 2^{15}$, ignoring 0s, showing squarefree composite n in green, n neither squarefree nor prime power in blue, with products of composite prime powers in large light blue and primorials in magenta.

LEMMA 5.5: $p^\varepsilon/\text{RAD}(p^\varepsilon) = p^\varepsilon q/\text{RAD}(p^\varepsilon q) = p^{(\varepsilon-1)}$, $p^\varepsilon < p^\varepsilon q$.

PROOF: Suppose we have 2 numbers with kernels p and pq, $p < q$, primes. So as to make the numbers the smallest they can be, we set $p = 2$, hence q is an odd prime. Since we know through Lemma that squarefree numbers have $m = 1$, we see $k = 2$ reaches 1 before any odd prime as k increases, additionally before any varius number. Now, so as to make for the smallest numbers with some multiplicity, we raise the least prime factor of p and pq to a power p^ε, to begin with, we set $\varepsilon = 2$ thus we compare p^2 and p^2q. Since we've squared the smallest prime factor in both cases, it is clear that the latter exceeds the former, though $m = p$, and it is clear by induction on ε that multus numbers win out over tantus. ■

This Lemma can be proved using the definition of A3557. We can simply ignore tantus as far as records are concerned.

LEMMA 5.6: $p^\varepsilon/\text{RAD}(p^\varepsilon) = p^\varepsilon q/\text{RAD}(p^\varepsilon q) = p^{(\varepsilon-1)}$, $p^\varepsilon < p^\varepsilon q$.

PROOF: Return to the definition A3557$(n) = \prod p_i^{(\varepsilon_i-1)}$ for $n > 1$. We perform the following:

$$\begin{aligned}\text{A053211} &= \text{A3557} \mapsto \text{A246547}\\ &= \text{A051953} \mapsto \text{A246547}.\\ &= \text{A051953}(p^\varepsilon).\\ &= p^{(\varepsilon-1)}.\end{aligned} \qquad [5.7]$$

Labos describes A053211 as the sequence of cototients of composite prime powers (multus numbers). This stands to reason, since the cototients of multus numbers are homogenously semicoprime. Since there is 1 prime divisor $p \mid p^\varepsilon$, and since $p^\delta \mid p^\varepsilon$, $0 \le \delta \le \varepsilon$, we create the cototient via $mp < p^\varepsilon$ such that $m \nmid p$. Hence A053211 is a permutation of A246655 = A961 \ {1}. Furthermore, we have shown that we can rewrite the formulation Gutkovskiy suggests via A3557 instead with A051953 as, when they regard $n \in$ A246547, they are equivalent. Therefore the record setters of A053211 comprise A334151. ■

Let $n = p^\varepsilon$, $\varepsilon > 0$; i.e., $n \in \{$ A961 \ {1} $\}$.

Let cototient(n) = A051953$(n) = n - \phi(n)$.

LEMMA 5.7: A051953$(p^\varepsilon) = p^{(\varepsilon-1)}$.

PROOF:
$$\begin{aligned}\text{A051953}(p^\varepsilon) &= p^\varepsilon - \phi(p^\varepsilon)\\ &= p^\varepsilon - p^\varepsilon \times (1-1/p)\\ &= p^\varepsilon - p^\varepsilon + p^\varepsilon/p\\ &= p^{(\varepsilon-1)}. \blacksquare\end{aligned} \qquad [5.8]$$

COROLLARY 5.8: A051953$(p^2) = p$.

COROLLARY 5.9: A051953$(p) = 1$.

Hence we can map $f(p^\varepsilon) = p^{(\varepsilon-1)}$ across A246547 to efficiently generate the sequence A053211. We can efficiently generate A246547 by taking the prime powers in A1694, using the construction

$$\text{A1694} = \{\, a^2 \times b^3 : a, b \ge 1 \,\}. \qquad [5.9]$$

We contemplate a proposition related to A334151 whose resolution lies outside the scope of this paper:

PROPOSITION A: A334151 is comprised of the empty product and powers of 2 and 3.

PROOF SKETCH: For $n \in$ A246547, we can write A3557$(n) = p^{(\varepsilon-1)} = p^\varepsilon/p$, hence, we have n/p. As p increases, decreases proportionately. We minimize decrease by minimizing p. The smallest prime $p = 2$, hence we should expect all $k \in$ A79 \ {2} to appear, since A3557(2) = A3557(1). Occasions of $p = 3$ appear on account of the similarity of 2 and 3 in magnitude, and the fact that A3557(p^ε) for $p = 3$ is the second-least reduced. Composite powers of 2 offer the largest value $p^{(\varepsilon-1)} > 1$ more frequently than any other prime.

The following theorem we propose, though it could use rigor:

THEOREM 5.10: $n \in \{$A334151 \ {4}$\}$ set records in A360543.

PROOF: The number $n = 1$ is a trivial record, A360543(1) = 0; $n = 4$ is missing since $k < 4$ are either divisors or coprime to 4.

From [1.4] and [5.7], for $n = p^\varepsilon \in$ A246547 and $n > 4$, we have the following:

$$\begin{aligned}\text{A045763}(n) = \xi(n) &= n - \phi(n) - \tau(n) + 1\\ &= \text{A051953}(p^\varepsilon) - \text{A5}(p^\varepsilon) + 1\\ &= p^{(\varepsilon-1)} - \varepsilon - 1 + 1\\ &= p^{(\varepsilon-1)} - \varepsilon.\end{aligned} \qquad [5.10]$$

Through Theorem 5.2, we have the following:

$$\text{A045763}(p^\varepsilon) = \text{A360543}(p^\varepsilon) = p^{(\varepsilon-1)} - \varepsilon. \qquad [5.11]$$

The sequence of record setters of A360543 would seem to depart from A334151 while $n = p^\varepsilon$ (with $\varepsilon > 0$) is small. Recognizing to be small compared to as n increases, we see that the sequence of record setters of A360543 and the sequence A334151 are the same, apart from 4 missing from the former. ■

PREEMINENCE OF THE SYMMETRIC SEMITOTATIVE.

We might remark on the "quincunx" pattern of semitotatives of n. The pattern arises given that of the cototient. Let us define the "quincunx" pattern as follows:

$$\begin{aligned}&\text{A349297}(n) = \{\, Q(n,k) : k \le n \,\},\\ &Q(n,k) = [2 \mid n \lor 2 \mid k \lor 3 \mid n \lor 3 \mid k].\end{aligned} \qquad [6.1]$$

In other words, we have all even or trine k for even or trine n, where trine signifies m mod 3 ≡ 0.

We use the name quincunx for the 5-die pattern " ⁙ " that forms part of the plot of A349297(n, k). The sequence A349297 stands at issue because it comprises a significant part of the cototient; symmetrically semicoprime $k < n$ occur in the nondivisor cototient. The cototient has the pattern described in A349317 as follows:

$$\begin{aligned}&\text{A349317}(n) = \{\, T(n,k) : k \le n \,\},\\ &T(n,k) = [\,(n,k) > 1\,].\end{aligned} \qquad [6.2]$$

We may write a sequence as follows:

$$\text{A349298}(n) = \{\, T(n,k) - Q(n,k) : k \le n \,\}. \qquad [6.3]$$

Let Q(n) represent the cardinality of A349297(n):

$$\text{Q}(n) = |\, \{\, Q(n,k) : k \le n \,\} \,| \qquad [6.4]$$

The first terms of Q(n), arranged mod 6, appear as follows:

```
0,  1,  1,  2, 0,  4,
0,  4,  3,  5, 0,  8,
0,  7,  5,  8, 0, 12,
0, 10,  7, 11, 0, 16,
0, 13,  9, 14, 0, 20,
0, 16, 11, 17, 0, 24, ...
```

It is clear that we might define a different way based on congruence relations, observing the following:

$$\begin{aligned}&\text{For } n \equiv 0 \pmod 6,\ \text{Q}(n) = \tfrac{2}{3}n,\\ &\text{For } n \equiv \pm1 \pmod 6,\ \text{Q}(n) = 0,\\ &\text{For } n \equiv \pm2 \pmod 6,\ \text{Q}(n) = n/2,\\ &\text{For } n \equiv \pm3 \pmod 6,\ \text{Q}(n) = n/3.\end{aligned} \qquad [6.5]$$

It is evident from scatterplot that A360840 that it is confined, once having "matured", between ⅔ n and n. The upper bound is a consequence of the defintion of A360840 to be a counting function of a species of $k \le n$. We have not explored a reason for the apparent lower bound.

Regarding the cototient, we note the following:

$$\begin{aligned}\text{A051953}(n) &= \Sigma\,\{\,[\,(k,n) > 1\,] \wedge k \le n\,\}\\ &= \Sigma\,\{\,T(n,k) \wedge k \le n\,\}\\ &= \Sigma\,\text{A349317}(n). \qquad [6.6]\end{aligned}$$

$$\begin{aligned}\text{A051953}(n) &> \text{A045763}(n) \ge \text{A360480}(n)\\ n - \phi(n) &> \xi(n) \ge f_1(n)\\ n - \phi(n) &> n - \phi(n) - \tau(n) + 1 \ge f_1(n) \qquad [6.7]\end{aligned}$$

The sum of A349298(n) is A051953(n) = $n - \phi(n)$. We find that, aside from prime powers, A360840 is a near image of A051953 and A045763. (See Figure 4.)

It seems evident, but remains unproved, that the following is true:

$$\xi(n) > f_1(n) \text{ for } n \in \text{A024619} \qquad [6.8]$$

From Theorems 4 and 5 in [3], we see that composites outside n = 4 and n = 6 have at least 1 semitotative, and non-prime powers outside n = 6 have at least 1 semidivisor $k < n$. The following table summarizes the findings in [3] regarding the existence of semidivisors and semitotatives in the reference domain of n of various species.

Table 1.

SPECIES	$\xi(n)$ A045763(n)	$\xi_d(n)$ A243822(n)	$\xi_t(n)$ A243823(n)
PRIMES (A40)	—	—	—
$n = 4$	—	—	—
MULTUS (A246547)	> 0	—	> 0
$n = 6$	—	1	1
VARIUS (A120944)	> 0	> 0	> 1
TANTUS (A126706)	> 0	> 0	> 1

Theorem 3.1: $\xi(n) > f_1(n)$ for $n \in$ A024619. Numbers n that are not prime powers are such that symmetric semicoprime $k < n$ are not the only n-neutral k such that $k < n$.

Proof: Theorem 5 in [3] shows that there is at least 1 semidivisor $k < n$ for numbers that are not prime powers. Additionally, Lemma 1.2 shows that all the semitotatives of multus n are in state ③. ■

Hence we have proved [6.8] to be true.

What remains is to explore the following difference:

$$\text{A360543}(n) = \xi(n) - f_1(n) \qquad [6.9]$$
(especially for $n \in$ A024619).

This sequence begins as follows:

```
0, 0, 0, 0, 0, 1, 0, 1, 1, 2, 0, 2, 0, 2, 1, 4, 0, 4,
0, 2, 1, 3, 0, 3, 3, 3, 6, 2, 0, 10, 0, 11, 2, 4, 1, 6,
0, 4, 2, 4, 0, 11, 0, 3, 3, 4, 0, 7, 5, 7, 2, 3, 0, 10,
1, 4, 2, 4, 0, 14, 0, 4, 3, 26, 1, 14, 0, 4, 2, 12, 0,
10, 0, 5, 5, 4, 1, 15, 0, 7, 23, 5, 0, 16, 1, 5, 3, 4,
0, 20, 1, 4, 3, 5, 1, 15, 0, 10, 3, 10, 0, 17, 0, 4, 8,
5, 0, 17, 0, 13, 3, 7, 0, 18, 1, 4, 3, 5, 1, 20, ...
```

Excepting $n \in$ A961, the records appear to be highly regular in many cases, and 3-smooth in others. The ratio S20230302(n)/$\xi(n)$ appears to converge to ⅙ for these records. Therefore the following seems apparent, though remains to be proved:

$$f_1(n)/\xi(n) \text{ converges to } 5/6 \text{ for } n \in \text{A024619} \qquad [6.10]$$

If true, then we venture to suggest that symmetric semicoprimality is the most common form of semitotative, as seems to be borne by Figure 3.

For large numbers, accepting for the moment [6.10], then we may further see the following for large $n \in$ A024619:

$$\text{A051953}(n) \approx \text{A045763}(n) \approx 6/5\ \text{A045763}(n). \qquad [6.11]$$

This unproved statement suggests that symmetric semicoprimality (state ①), with possible exception of coprimality, is the most common constitutive state.

Relation to the Semitotative Counting Function.

In the interest of context, the following is the related counting function $f_3(n)$ of mixed-neutral semitotatives:

$$\begin{aligned}f_3(n) &= \{\,k < n : k \text{ ③ } n\,\} = \{\,k < n : k \lozenge\,\vert\, n\,\}\\ &= \text{A243823}(n) - \text{A360480}(n). \qquad [7.1]\end{aligned}$$

Since there are precisely 2 kinds of semitotatives; symmetric (state ①) and mixed-neutral (state ③), we may write the following:

$$\begin{aligned}\xi_T(n) &= f_1(n) + f_3(n)\\ \text{A243823}(n) &= \text{A360480}(n) + \text{A360543}(n). \qquad [7.2]\end{aligned}$$

Conclusion.

There are 2 varieties of n-semitotatives k; these are the symmetric and mixed variety. The former concerns $k < n$ such that prime $p \mid k$ but $(p, n) = 1$, while prime $q \mid n$ but $(q, k) = 1$. The latter regards k and n in cototient such that $\omega(n) \mid \omega(k) > \omega(n)$, while RAD($n$) | RAD($k$). Using constitutive states, these are k ① n and k ③ n, respectively. We have shown that these are the only possible constitutive varieties of semitotative. We generated counting functions $f_1(n)$ = A360480(n) relating to k ① n, and $f_3(n)$ = A360543(n) relating to k ③ n, both such that $k < n$. Hence, $\xi_T(n) = f_1(n) + f_3(n)$, or in terms of OEIS, A243823(n) = A360480(n) + A360543(n).

Though k ③ n pertains to composite prime powers $n > 4$ exclusively, while k ① n pertains to squarefree composite $n > 6$ exclusively, both appear for certain numbers $n \in$ { A360765 ∩ A360768 }, a subset of A126706. Outside of these, generally $n \in$ A126706 harbors only k ① n.

We estimate that for numbers n that are not prime powers, the number of k ① n approaches ⅚ of the cototient of n, but this remains something to ascertain. Given the evident dominance of k ① n over k ③ n, it is not surprising that the scatterplot of A360480 resembles those of A045763 or A051953.

Appendix.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved February 2023.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

[3] Michael Thomas De Vlieger, Constitutive State Counting Functions, *Simple Sequence Analysis*, 20230226.

[4] Michael Thomas De Vlieger, The Symmetric Semidivisor Counting Function, *Simple Sequence Analysis*, 20230216.

Code:

[C0] Function $f(k, n)$ yields the constitutive state (Svitek number) between k and n.

```
conState[j_, k_] :=
 Which[j == k, 5, GCD[j, k] == 1, 0, True,
  1 + FromDigits[
    Map[Which[Mod[##] == 0, 1,
      PowerMod[#1, #2, #2] == 0, 2, True, 0] & @@ # &,
      Permutations[{k, j}]], 3]]
```

[C1] Calculate $\boldsymbol{R}_{\varkappa}$ bounded by an arbitrary limit m (i.e., calculate A275280(n); flatten and take union to provide A162306)

```
regularsExtended[n_, m_ : 0] :=
  Block[{w , lim = If[m <= 0, n, m]},
   Sort@ ToExpression@
     Function[w,
       StringJoin[
         "Block[{n = ", ToString@ lim,
         "}, Flatten@ Table[",
         StringJoin@
           Riffle[Map[ToString@ #1 <> "^" <>
             ToString@ #2 & @@ # &, w], " * "],
         ", ", Most@ Flatten@ Map[{#, ", "} &, #],
         "]]" ] &@
       MapIndexed[
         Function[p,
           StringJoin["{", ToString@ Last@ p,
             ", 0, Log[",
             ToString@ First@ p, ", n/(",
             ToString@
               InputForm[
                 Times @@ Map[Power @@ # &,
                   Take[w, First@ #2 - 1]]],
             ")]}" ] ]@ w[[First@ #2]] &, w]]@
       Map[{#, ToExpression["p" <>
         ToString@ PrimePi@ #]} &, #[[All, 1]] ] &@
       FactorInteger@ n];
```

[C2] Generate tantus numbers (A126706):

```
a126706 = Block[{k}, k = 0;
   Reap[Monitor[Do[
       If[And[#2 > 1, #1 != #2] & @@
         {PrimeOmega[n], PrimeNu[n]},
        Sow[n]; Set[k, n] ],
       {n, 2^21}], n]][[-1, -1]]] (* Tantus *);
```

[C3] Generate "strong tantus" numbers (A360768):

```
Select[a126706[[1 ;; 120]], #1/#2 >= #3 & @@
  {#1, Times @@ #2, #2[[2]]} & @@
  {#, FactorInteger[#][[All, 1]]} &]
```

[C4] Generate tantus numbers that have k ③ n (A360765):

```
nn = 2^20},
 rad[n_] := rad[n] = Times @@
  FactorInteger[n][[All, 1]];
 lcp[n_] := If[OddQ[n], 2,
   p = 2; While[Divisible[n, p], p = NextPrime[p]]; p];
 a = a126706[[1 ;; nn]];
 Monitor[ Reap[
   Do[n = a[[j]];
     If[rad[n]*lcp[n] < n, Sow[n]], {j, nn}]
   ][[-1, -1]], j] ]
```

[C5] Generate A360480, the k ① n counting function:

```
rad[x_] := rad[x] = Times @@
  FactorInteger[x][[All, 1]];
Table[k = rad[n];
  Count[Range[n],
    _?(Nor[CoprimeQ[#1, n], Divisible[#2, k],
       Divisible[k, #2]] & @@ {#, rad[#]} &)], {n, 88}]
```

[C6] Generate A360543, the k ③ n counting function:

```
nn = 120;
rad[n_] := rad[n] = Times @@
  FactorInteger[n][[All, 1]];
c = Select[Range[4, nn], CompositeQ];
s = Select[Select[Range[4, nn], Not @* SquareFreeQ],
  Function[{n, q, r},
    AnyTrue[TakeWhile[c, # <= n &],
      And[PrimeNu[#] > q,
          Divisible[rad[#], r]] &]] @@
          {#, PrimeNu[#], rad[#]} &];
Table[If[FreeQ[s, n], 0,
  Function[{q, r},
    Count[TakeWhile[
      c, # <= n &], _?(And[PrimeNu[#] > q,
         Divisible[rad[#], r]] &)]] @@
         {PrimeNu[n], rad[n]}], {n, nn}]
```

[C7] Faster algorithm for A360543, the k ③ n counting function, given a dataset of A360765 and [C1]:

```
rad[n_] := rad[n] = Times @@
  FactorInteger[n][[All, 1]];
{{}, {}}~Join~Table[r = Rest@ regularsExtended[n];
  t = Rest@ Flatten@
    Outer[Plus, rad[n]*Range[0, n/rad[n] - 1],
      Select[Range[rad[n]], CoprimeQ[rad[n], #] &]];
  Union@ Flatten@
    Table[i j,
      {i, r[[1 ;; LengthWhile[r, n/t[[1]] > # &]]]},
      {j, t[[1 ;; LengthWhile[t, n/i > # &]]]}],
      {n, 3, 24}]
```

[C8] Fast algorithm for A334151, a sequence of record setters in A3557, which is related to record setters for the k ③ n counting function:

```
pp = 4; nn = 2^29; j = 0;
c = e[_] = 1; r = Prime@ Range[pp];
Do[(e[#1]++; Set[{k, m}, {#1^#2, #1^(#2 - 1)}]) & @@
  First@ MinimalBy[Array[{#, e[#]} &[r[[#]]] &, pp],
Power @@ # &];
 If[m > j, Set[{a[c], j}, {k, m}]; c++];
 If[k > nn/2, Break[]], {n, Infinity}];
{1}~Join~Array[a, c - 2, 2]
```

Concerns sequences:

A000005: Divisor counting function $\tau(n)$.
A000010: Euler totient function $\phi(n)$.
A000040: Prime numbers.
A000961: Prime powers.
A001221: Number of distinct prime divisors of n, $\omega(n)$.
A003557: $n/\text{RAD}(n)$.
A006881: Squarefree semiprimes.
A007947: Squarefree kernel of n; $\text{RAD}(n)$.
A010846: Regular counting function.
A013929: Numbers that are not squarefree.
A024619: Numbers that are not prime powers.
A045763: Neutral counting function.
A051953: Cototient function: $n - \phi(n)$.
A053211: A3557 ↦ A246547 = A051953 ↦ A246547.
A120944: "Varius" numbers; squarefree composites.
A126706: "Tantus" numbers neither prime power nor squarefree.
A133995: Row n lists n-neutral k such that $k < n$.
A162306: Row n lists n-regular k such that $k \le n$.
A246547: "Multus" numbers; composite prime powers.
A272618: Row n lists n-semidivisors k such that $k < n$.
A272619: Row n lists n-semitotatives k such that $k < n$.
A334151: Record setters for A3557.
A355432: $a(n)$ = symmetric semidivisor counting function.
A360480: $a(n)$ = symmetric semicoprime counting function.
A360543: $a(n)$ = mixed semicoprime counting function.
A360765: $n \in$ A126706 : A7947$(n) \times$ A053669$(n) < n$.
A360767: Weakly tantus numbers.
A360768: Strongly tantus numbers.
A360769: Odd tantus numbers.
A361235: $a(n)$ = mixed semidivisor counting function..

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