Constitutive State Counting Functions

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Abstract.

This work conceives of counting functions based on the constitutive relationship between k and n, $k \le n$. These functions address the cototient as Euler's totient function has addressed the totient.

Introduction.

Let's examine counting functions based on constitutive states. We consider a reference domain $k \in (1 \ldots n)$ such that k and n have some constitutive state.

We have the following state classes:

Coprimality (⓪)
Symmetric Semicoprimality (①)
Lean Divisorship (②④)
Mixed Neutrality (③⑦)
Equality (⑤)
Mixed Regularity (⑥⑧)
Symmetric Semidivisorship (⑨) [1.0]

These states are explained in depth in [2], this paper will not re-introduce concepts that appear in [2] for the sake of brevity.

The Euler totient function below is the most well-known such counting function:

$$\begin{aligned} \text{A10}(n) &= \phi(n) \\ &= \lvert \{ k < n : k \text{ ⓪ } n \} \rvert \\ &= \prod_{p \mid n} (1 - 1/p). \end{aligned} \quad [1.1]$$

The divisor counting function $\tau(n)$ pertains to the cardinality of several related species in the reference domain:

$$\begin{aligned} \text{A5}(n) &= \tau(n) \\ &= \lvert \{ k < n : k \text{ ④⑤⑥ } n \} \rvert. \\ &= \lvert \boldsymbol{D}_n = \bigotimes_{p^\delta \mid n} \{ p^\varepsilon : 0 \le \varepsilon \le \delta \} \rvert \\ &= \prod_{p^\varepsilon \mid n} (\varepsilon + 1). \end{aligned} \quad [1.2]$$

In the case of these counting functions, we enjoy handy formulae based on prime power decomposition. For instance, $\tau(144)$ is the cardinality of numbers $d \in \boldsymbol{D}_n$, within the box shown in Table 1; the formula follows from the tensor product (see A275055 in [1]).

Table 1.

1	2	4	8	16	32	64	128
3	6	12	24	48	96	192	384
9	18	36	72	**144**	288	576	1152
27	54	108	216	432	864	1728	3456
81	162	324	648	1296	2592	5184	10368

Table 1 can be extended infinitely to include the set of numbers k regular to n, which are shared by any number having squarefree kernel $\text{RAD}(n) = \varkappa$:

$$\boldsymbol{R}_\varkappa = \bigotimes_{p \mid \varkappa} \{ p^\varepsilon : \varepsilon \ge 0 \}. \quad [1.3]$$

Related to $\tau(n)$ is a regular counting function $\text{RCF}(n) = \text{A010846}(n)$, defined as follows:

$$\begin{aligned} \text{A010846}(n) &= \text{RCF}(n) \\ &= \lvert \{ k \le n : k \text{ ④⑤⑥⑦⑨ } n \} \rvert \\ &= \lvert \boldsymbol{r}_n = \{ k \le n : k \in \boldsymbol{R}_\varkappa \} \rvert \end{aligned} \quad [1.4]$$

This function has no simple formula. Regarding $\text{RCF}(144)$, it includes numbers in Table 1 that appear in the irregular-shaped "wings" to the right of and under the box of divisors of 144, hence, while $\tau(144) = 15$, $\text{RCF}(144) = 23$. In other words, the function $\text{RCF}(n)$ is a tensor product discretely bounded by n. Because of this, we can write an algorithm and efficiently calculate $\text{RCF}(n)$ for reasonably sized n.

Another significant extant counting function is the neutral counting function $\xi(n)$ defined below:

$$\begin{aligned} \text{A045763}(n) &= \xi(n) \\ &= \lvert \{ k < n : k \text{ ①③⑦⑨ } n \} \rvert \\ &= \lvert \{ k < n : k \nmid n \land (k, n) > 1 \} \rvert \\ &= n - \phi(n) - \tau(n) + 1 \end{aligned} \quad [1.5]$$

It is easy to see $\xi(n) = 0$ for prime n, but also for $n = 4$.

Given [2], we discern 2 species of numbers counted by $\xi(n)$; these are the semidivisors and semitotatives, neither divisors nor totatives of n, hence neutral. Therefore we may define a semidivisor and semitotative counting function. The former relates to both $\text{RCF}(n)$ and $\xi(n)$ and is defined as follows:

$$\begin{aligned} \text{A243822}(n) &= \xi_d(n) \\ &= \lvert \{ k < n : k \text{ ⑦⑨ } n \} \rvert \\ &= \lvert \boldsymbol{\eth}_n = \boldsymbol{r}_n \setminus \boldsymbol{D}_n \rvert \\ &= \xi(n) - \xi_t(n) \end{aligned} \quad [1.6]$$

It is clear from the third line, for example, that $\xi_d(144)$ counts products that appear in the "wings" of Table 1.

$$\begin{aligned} \text{A243823}(n) &= \xi_t(n) \\ &= \lvert \{ k < n : k \text{ ①③ } n \} \rvert \\ &= \xi(n) - \xi_d(n) \end{aligned} \quad [1.7]$$

Single-State Counting Functions.

The latter six classes in [1.0] occur in the proper cototient, which we define as follows:

$$\{ k < n : k \sqcup n \}, \text{ i.e.,}$$
$$\{ k < n : (k, n) > 1 \}. \quad [2.1]$$

We make this distinction so as to certify that we do not mean k that are congruent to some number in the cototient of n. We may find among k and n, $k \sqcup n$, neither restrained to the proper cototient, most any cototient constitutive state. For example, 6 ① 10 implies 10 ① 6 on account of symmetry. Some states are constrained to the proper cototient, such as those that have $k \mid n$. Two states do not occur in the proper cototient because they involve $n \mid k$, hence $n < k$. These are ② = ◊|, meaning $k \lozenge n \land n \mid k$, k semicoprime to n, for example, 10 ② 5, and ⑧ = ¦|, meaning $k \mathrel{¦} n \land n \mid k$, k semidivides n, e.g., 20 ⑧ 10.

States ①③⑦⑨ comprise neutrality and pertain to A133995 and A045763. State ⑦ pertains to multus factors of nonmultus $p^\varepsilon = k < n : \varepsilon > 1 \land p^\varepsilon \mid n$. State ⑦ also pertains to semicoprime multiples of n outside the proper cototient.

States ②④⑤⑥⑧ comprise divisorship, but in the proper cototient, we only have ④⑤⑥. State ② appears outside the proper cototient, and ⑧ represents regular multiples of n.

States ④⑤⑥⑦⑧⑨ comprise regularity; we have already shown ⑧ to lie outside the proper cototient.

Therefore, there are two varieties of semidivisor $k < n$; these are described as follows, each with example:

symmetric	⑨	$k \mathbin{¦} n \wedge n \mathbin{¦} k$	12 ¦¦ 18	
mixed	⑦	$k \mathbin{¦} n \wedge n \lozenge k$	12 ¦◊ 30	[2.2]

There are two varieties of n-semitotative; these are as follows:

symmetric	①	$k \lozenge n \wedge n \lozenge k$	6 ◊◊ 14	
mixed	③	$k \lozenge n \wedge n \mathbin{¦} k$.	20 ◊¦ 25	[2.3]

Finally, there are 4 kinds of divisor:

coprime		$1 \mid n$		
symmetric	⑤	$d = n$		
nonplenary	④	$d \mid n \wedge n \lozenge d$.	2 |◊ 10	
plenary	⑥	$d \mid n \wedge n \mathbin{¦} d$.	6 |¦ 18	[2.4]

Let $\text{RAD}(m) = \text{A7947}(m)$ be the squarefree kernel of m. Regarding the latter 2 cases, plenary divisor d ⑥ n implies $\text{RAD}(d) = \text{RAD}(n) = \varkappa$, whereas nonplenary or "lean" d ④ n implies $\text{RAD}(d) \mid \text{RAD}(n)$, yet the converse, $\text{RAD}(n) \mid \text{RAD}(d)$, is false, hence $n \lozenge d$.

We don't see $k < n$ in states ②⑧; these represent semicoprime and regular multiples of n, respectively.

Constitutive Composition of the Cototient.

It's obvious that natural numbers $n \in \mathbb{N}$ have symmetric divisor state ⑤ (equality, $k = n$) in the proper cototient. The pattern for primes p regarding $\{ k : k < p \}$ is coprimality. The proper cototient only contains state ⑤. Let $f_1(n)$ be the symmetric divisor counting function. Since $f_1(n) = 1$ for $n \in \mathbb{N}$, we identify A27 as f_1.

Multus (A246547).

The cototient of multus n (composite prime powers, $n \in \text{A246547}$) contains plenary divisors ⑥ and mixed semitotatives ③ in addition to equality ⑤. The smallest multus number, $n = 4$, has no semitotatives, since $k < 4$ are prime powers.

Lemma 1.1: Divisors $d \mid n$, $d > 1$, for multus $n \in \text{A246547}$ are such that $\text{RAD}(d) = \text{RAD}(n)$.
Proof: By definition, $n = p^\varepsilon$, $\varepsilon > 1$, is such that $\omega(n) = 1$, i.e., p^ε has a single distinct prime divisor p. Hence $d \mid p^\varepsilon = p^\delta$, $0 \le \delta \le \varepsilon$, and it is clear, given $\text{RAD}(p) = p$, that $\text{RAD}(d) = \text{RAD}(n)$. ■

Corollary 1.2: There are no semidivisors $k < n$ for $n \in \text{A246547}$, a consequence of $k \mid p^\varepsilon = p^\delta$, $0 \le \delta \le \varepsilon$; all n-regular k such that $k < n$ divide n.

Lemma 1.3: Semitotatives $k \lozenge n$, $k < n$, for multus $n \in \text{A246547}$ are never symmetric.
Proof: An n-semitotative k is n-semicoprime, with $k < n$. The definition of n-semicoprime k requires a prime $p \mid k$ that does not divide n, yet $(k, n) > 1$. Symmetric semicoprimality has prime $q \mid n$ such that q does not divide k, yet $(k, n) > 1$. We have to show that n is k-semicoprime, however, such implies $\omega(n) > 1$. (See [2], Lemma 1.2) ■

Theorem 1: In the range $[1 \ldots n]$ for $n \in \text{A246547}$, we have k ⓪ n, k ③ n, k ⑤ n, and k ⑥ n.
Proof: The number k must have a constitutive relationship with n, meaning k is coprime to n, k divides n, or if both composite (see [2], Lemma 1.1), k must semidivide or be a semitotative of n. Recognizing this along with Lemmas 1.1 and 1.3, and Corollary 1.2, we show the proposition to be true. ■

Varius (A120944).

The cototient of varius $n \in \text{A120944}$ is restricted to nonplenary divisors ④, mixed semidivisors ⑦, and symmetric semitotatives ①, along with k ⑤ n. Squarefree semiprime 6 has no semitotatives, since all $k < 6$ are prime powers, hence it has no symmetric semitotatives.

Lemma 2.1: Divisors $d \mid n$, $d < n$, for varius $n \in \text{A120944}$ are such that $\text{RAD}(d) \ne \text{RAD}(n)$.
Proof: The number $n \in \text{A120944}$ is squarefree by definition, hence divisors $d \mid n$, $d < n$, are products of proper subsets of $\{ p : p \mid n \}$, and it is evident that for $d < n$, $\text{RAD}(d) \ne \text{RAD}(n)$. ■

Lemma 2.2: Varius n has n-regular k, $k < n$, such that $\text{RAD}(k) \ne \text{RAD}(n)$, implying mixed semidivisors $k < n$.
Proof: The proposition is true because nondivisor $k < n$ are tantus, i.e., k possesses multiplicity $\varepsilon > 1$ for at least one prime power factor $p^\varepsilon \mid k$, hence are regular multiples of n-divisors. Therefore all semidivisors are nonplenary or lean, i.e., $\text{RAD}(k) \ne \text{RAD}(n)$, implying that n has at least 1 prime factor q that does not divide tantus k. ■

Lemma 2.3: For varius n, all semitotatives are symmetric.
Proof: There are 2 kinds of semitotatives shown by [2.3]. Therefore we need to show that squarefree n does not semidivide k. The expression $n \mathbin{¦} k$ implies $\text{RAD}(n) \mid \text{RAD}(k)$, but n does not divide k. The latter is true since $k < n$, but the former implies $\text{RAD}(n) = \text{RAD}(k)$, that is, all primes $q \mid n$ also divide k, contradicting $k \lozenge n$. ■

Theorem 2: In the range $[1 \ldots n]$ for $n \in \text{A120944}$, we have k ⓪ n, k ① n, k ④ n, k ⑤ n, and k ⑦ n.
Proof: In a similar vein as Theorem 1, but along with Lemmas 2.1 and 2.3, and Corollary 2.2, we show the proposition to be true. ■

Tantus (A126706).

In addition to the states ①④⑦, tantus n enjoy plenary divisors ⑥ that include the squarefree kernel $\text{RAD}(n) = \varkappa$.

A tantus number is such that among distinct prime divisors $p \mid n$, there exists a prime power factor $p^\varepsilon \mid n$ such that $\varepsilon > 1$. In other words, $\varkappa \mid n$ and $\varkappa < n$.

We know natural numbers have trivial divisors since $1 \times n = n$. We also know that natural numbers have trivial totatives $t \equiv \pm 1 \pmod{n}$. For $n \le 2$, these are conflated into $t = 1$. We can use the neutral counting function $\xi(n)$ to show that, outside of primes and $n \le 4$, all n have some neutral k.

Therefore we need to show which species of semidivisor, semitotative, and divisor occurs in the cototient of tantus numbers.

Lemma 3.1: Divisors $d \mid n$, $1 < d < n$, for tantus $n \in \text{A126706}$ are such that $\text{RAD}(d) \le \text{RAD}(n)$.
Proof: This proposition merely states that $d \mid n$, $d < n$ are unrestricted to lean divisors. Therefore we must show that there exists at least 1 divisor $d \mid n$ such that $\text{RAD}(d) = \text{RAD}(n) = \varkappa$. It is clear, given the definition of tantus, that $\varkappa \mid n$ and $\varkappa < n$. ■

We examine n-semidivisor k for either variety, symmetric or mixed neutral, shown in [2.2].

Lemma 3.2: Tantus n has n-semidivisor k such that there is a prime $q \mid n$ that does not divide k.
Proof: So as to minimize p, we select $p = \text{LPF}(n)$ and set $k = p^\varepsilon$ where $p^\delta \mid n$ and $\varepsilon = \delta + 1$. It is clear $k \mathbin{¦}\lozenge\, n$ (i.e., k ⑦ n) since $\omega(k) = 1$ but $\omega(n) > 1$. Let $q = \text{PRIME}(\pi(p)+1)$ and let $n = p^\delta q$. Since $q > p$ it is clear that $p^\varepsilon < n$. Induction on p and q or δ shows that n-regular $p^\varepsilon < n$ and does not divide n. ■

We have proved in [3], Theorem 3.5, that for certain tantus numbers, we have symmetric semidivisors k ⑨ n such that $k < n$. Let prime $p = \textsc{lpf}(n)$ and q be the second smallest prime divisor of n. Let p^ε be the largest power of p such that $p^\varepsilon \mid n$. Let $\textsc{rad}(n) = \varkappa$, and let $n/\varkappa = m$. For all $n \in$ A126706 such that $n/\varkappa \geq q$, there exists at least 1 symmetric semidivisor k ¦¦ n such that $k < n$.

Let a "strong tantus" number n be neither prime power nor squarefree semiprime, such that $p^\varepsilon > p^{\lfloor \log_p q \rfloor}$, where $p = \textsc{lpf}(n)$ and q is the second smallest prime divisor of n. The sequence A360768 of strong tantus numbers that have symmetric semidivisors ⑨ in the proper cototient begins as follows:

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18, 24, 36, 48, 50, 54, 72, 75, 80, 90, 96, 98, 100,
108, 112, 120, 126, 135, 144, 147, 150, 160, 162, 168,
180, 189, 192, 196, 198, 200, 216, 224, 225, 234, 240,
242, 245, 250, 252, 264, 270, 288, 294, 300, 306, 312,
320, 324, 336, 338, 342, 350, 352, 360, 363, 375, ...
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Finally, we need to show that both species of semitotative appear for tantus n.

symmetric	①	$k \lozenge n \wedge n \lozenge k$	6 ◊◊ 14
mixed	③	$k \lozenge n \wedge n$ ¦ k.	20 ◊¦ 25

Lemma 3.3: There exists k such that $k < n$ and k ① n for all $n \in$ A126706.

Proof: Set $p = \textsc{lpf}(n)$ and set $q =$ A053669(n), the smallest prime that is coprime to n. It is clear pq ① n for all $n \in$ A126706 by definition of "semicoprime". Is $pq < n$? The smallest tantus number is $n = 12$; for such we have $pq = 2 \times 5 = 10$; $10 < 12$. If we set $p > 2$, supposing $p^2 \mid n$ in order to minimize n, then $q = 2$, and we are only making larger n. It becomes clear that, so as to maximize q but keeping $p = 2$ and $p^2 \mid n$, we require $n = 2P(i) = 2 \times$ A2110(i), $i > 1$, but with induction on i, it is clear that $pq < 2P(i)$. ■

We note that for $n = 12$, 10 is the sole semitotative; there are none of the mixed neutral variety k ③ n. We do see that for $n = 45$, we have $k = 30$, thus k ③ n.

Lemma 3.4: There exists k such that $k < n$ and k ③ n for odd $n \in$ A126706.

Proof: Break the expression k ③ n into components $k \lozenge n$ and n ¦ k. We may rewrite the latter component as $\textsc{rad}(n) \mid \textsc{rad}(k)$, where, per the former component, $\textsc{rad}(k) = q \times \textsc{rad}(n)$ and q coprime to n. If n is odd, then we can produce the state via $k = 2\varkappa$, where $\varkappa = \textsc{rad}(n)$. Since n is tantus, $n \geq p\varkappa$ such that $p \mid \varkappa$ and $p > 2$. Therefore it is clear that $2 < p\varkappa \leq n$. ■

Lemma 3.5: There exists k such that $k < n$ and k ③ n for even $n \in$ A126706 such that $\omega(k) > \omega(n)$ and $\textsc{rad}(n) \mid \textsc{rad}(k)$ for some $k < n$.

Proof: Define the set of k-regular numbers $\boldsymbol{R}_\varkappa$, where $\varkappa = \textsc{rad}(k)$, to be as follows:

$$\boldsymbol{R}_\varkappa = \bigotimes_{p \mid \varkappa} \{p^\varepsilon : \varepsilon \geq 0\}. \qquad [2.5]$$

All numbers $m \in \boldsymbol{R}_\varkappa$ are such that $\textsc{rad}(m) \mid \varkappa$. Therefore, n ¦ k implies $k, n \in \boldsymbol{R}_\varkappa$ and hence $\textsc{rad}(n) \mid \varkappa$. Since we restrict all numbers in $\boldsymbol{R}_\varkappa$ to primes $p \mid \varkappa$, it is sufficient merely to find $n > k$ in this sequence such that $\omega(n) < \omega(k)$.

We thus define A360765 such that A360765 ⊂ A126706 containing tantus numbers that have mixed semitotatives ③ that begins with the following numbers:

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36, 40, 45, 48, 50, 54, 56, 63, 72, 75, 80, 88, 96, 98,
99, 100, 104, 108, 112, 117, 135, 136, 144, 147, 152,
153, 160, 162, 171, 175, 176, 184, 189, 192, 196, 200,
207, 208, 216, 224, 225, 232, 240, 242, 245, 248, ...
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For $n \in$ A360765, there exists $k < n$ such that $k =$ A053669$(n) \times \varkappa$.

Theorem 3.6. Distinct $m, n \in$ A360765 such that both have same squarefree kernel $\varkappa$ implies that mixed semicoprime k pertains to both m and n, and symmetric semicoprime k pertains to both m and n.

Proof: Suppose we have 2 distinct numbers $m, n \in$ A360765 such that $\textsc{rad}(m) = \textsc{rad}(n) = \varkappa$ and $n < m$. It is clear that $\textsc{rad}(m) = \textsc{rad}(n) = \varkappa$ implies $\omega(m) = \omega(n) = \textsc{q}$. Therefore, if we have $k < n$ such that k ③ n, we know that $\varkappa \mid \textsc{rad}(k)$ (which itself implies cototient), and $\omega(k) > \textsc{q}$. Hence, if we have k ③ n, then we have k ③ m and vice versa. ■

From this theorem, we merely need to store $k < n$ such that k ③ n in $\boldsymbol{S}_\varkappa$ for a large n such that $\textsc{rad}(n) = \varkappa$, and count k such that $k < m$ for any counting function $f_3(m) < f_3(n)$ and $m \in \varkappa\boldsymbol{R}_\varkappa$.

The "Panstitutive" Numbers.

Summarizing the numbers k in the reference domain of tantus n, we have numbers k ⓪ n, k ① n, k ④ n, k ⑤ n, k ⑥ n, and k ⑦ n. Tantus numbers in the subset A360765 have mixed as well as symmetric semitotatives k ③ n. Strong tantus numbers (in A360768) have symmetric semidivisors k ⑨ n.

Numbers $n \in$ (A360765 ∪ A360768) have both ③ and ⑨, and therefore all 8 possible constitutive states among $k \leq n$. Define the sequence of "panstitutive" numbers to be the following:

$$\text{S20230228} = (\text{A360765} \cup \text{A360768}). \qquad [2.6]$$

This sequence begins as follows:

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36, 48, 50, 54, 72, 75, 80, 96, 98, 100, 108, 112, 135,
144, 147, 160, 162, 189, 192, 196, 200, 216, 224, 225,
240, 242, 245, 250, 252, 270, 288, 294, 300, 320, 324,
336, 338, 350, 352, 360, 363, 375, 378, 384, 392, ...
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For $n = 36$ we have the following smallest k satisfying a given constitutive state with respect to n:

totatives	semitotatives	divisors	semidivisors
1 ⓪ 36	10 ① 36	2 ④ 36	8 ⑦ 36
	30 ③ 36	36 ⑤ 36	24 ⑨ 36
		6 ⑥ 36	

For the smallest odd term in S20230228, $n = 135$, we have the following smallest k satisfying a given constitutive state with n:

totatives	semitotatives	divisors	semidivisors
1 ⓪ 135	6 ① 135	3 ④ 135	25 ⑦ 135
	30 ③ 135	135 ⑤ 135	75 ⑨ 135
		15 ⑥ 135	

Now we show that plenus numbers $n \in$ A286708, products of at least 2 multus numbers (i.e., composite prime powers $n \in$ A246547) are panstitutive.

Lemma 3.7: A286708 ⊂ A360765. In other words, numbers n that are products of at least 2 composite prime powers p^ε, $\varepsilon > 1$ (i.e., $n \in$ A286708) have $\varkappa q < n$ where $\varkappa = \textsc{rad}(n) =$ A7947(n) and $q = \textsc{lpc}(n) =$ A053669(n).

Proof: The proposition is true since $q < \varkappa$, hence $\varkappa q < m\varkappa^2$, $m \geq 1$. ■

Lemma 3.8: A286708 ⊂ A360768. In other words, numbers n that are products of at least 2 composite prime powers p^ε, $\varepsilon > 1$ (i.e., $n \in$ A286708) have $n/\varkappa > q$ where squarefree kernel $\varkappa = \textsc{rad}(n) =$ A7947(n), and q is the second-least prime factor of n.

Proof: Numbers $n \in$ A286708 are such that $n = m\varkappa^2$ with $m \geq 1$ and $\omega(\varkappa) > 1$. Thus, $n/\varkappa = m\varkappa$, and since $\varkappa$ is the product of at least 2 primes including q, $q < \varkappa$. ■

Theorem 3.9: Plenus numbers $n \in$ A286708 are panstitutive, because they are both $n \in$ A360765 and $n \in$ A360768.

Figure 1: A map of constitutive states between k and n for k ≤ 24 and n ≤ 24. For pairs involving one 1, we show the state in gray and write state ⓪, and for k = 1 and n = 1, we write state ⑤.

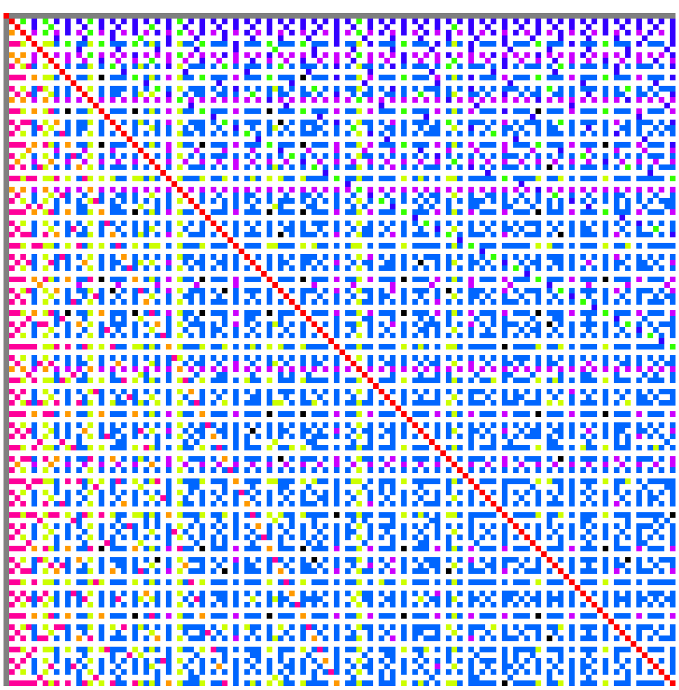

Figure 2: A map of constitutive states between k and n for k ≤ 120 and n ≤ 120 using the same color function as in Figure 1.

This said, there are panstitutive numbers like 48 and 50 that are not plenus, since their prime power factors are not always multus.

Several Constitutive State Counting Functions.

Recognizing the existence of the Euler totient function regarding the totient spurs us toward examination of counting functions based on constitutive states in the reference range. Some of these have been explored at the time of writing, while others have yet to be explored. Some may not merit inclusion in OEIS, but can be examined nonetheless in papers with datasets made available. Therefore, we define the following counting functions in the proper cototient. When we do not have an OEIS A-number for these sequences we use a provisional ISO-8601 style S-number.

Symmetric semiprime counting function f_1 (A360480).

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0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 3, 3, 0, 0, 3,
0, 5, 5, 6, 0, 6, 0, 8, 0, 9, 0, 5, 0, 0, 8, 11, 7, 10,
0, 13, 10, 13, 0, 12, 0, 16, 13, 17, 0, 16, 0, 18, 14,
20, 0, 19, 11, 21, 16, 23, 0, 19, 0, 25, 19, 0, 13, 25,
0, 27, 20, 27, 0, 27, 0, 30, 25, 31, 13, 32, 0, 32, 0,
34, 0, 33, 17, 36, 25, 37, 0, 35, 15, 39, 27, 40, 19,
38, 0, 41, 31, 42, 0, 46, 0, 45, 42, 46, 0, 44, 0, 50,
33, 48, 0, 53, 23, 51, 37, 52, 19, 53, ...
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$$\text{A360480}(n) = \big|\, \{\, k < n : (k, n) > 1 \land \\ (\text{RAD}(k) \mid \text{RAD}(n) \;\bar{\lor}\; \text{RAD}(n) \mid \text{RAD}(k)\,)\,\}\,\big| \qquad [3.1]$$

The definition of symmetric semicoprimality implies $\omega(n) \geq 2$. We have the following consequences:

A360480$(n) > 0$ for $n \in$ A024619.
A360480$(n) = 0$ for $n \in$ A961.
A360480$(6) = 0$ since $k < 6$ are prime powers.

The function is related closely with the cototient function A051953$(n) = n - \phi(n)$; its scatterplot resembles that of the cototient function. Almost all $k < n$ for composite n are in state ①. This function is covered in detail in [4].

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Mixed semicoprime counting function f_3 (A360543).

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0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 4, 0, 0,
0, 0, 0, 0, 0, 0, 3, 0, 6, 0, 0, 0, 0, 11, 0, 0, 0, 1,
0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 2, 5, 1, 0, 0, 0, 2,
0, 1, 0, 0, 0, 0, 0, 0, 1, 26, 0, 0, 0, 0, 0, 0, 0, 4,
0, 0, 2, 0, 0, 0, 0, 3, 23, 0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 0, 7, 0, 3, 1, 4, 0, 0, 0, 1, 0, 0, 0, 8,
0, 0, 0, 3, 0, 0, 0, 0, 1, 0, 0, 0, ...
```

Lemmas 3.4 and 3.5 prove the following:

$$\text{A360543}(n) = \\ \big|\, \{\, k < n : \text{RAD}(n) \mid \text{RAD}(k) \land \omega(k) > \omega(n)\,\}\,\big| \qquad [3.3]$$

Consequently, we find the following:

A360543$(n) = 0$ for $n \in$ A5117.
Let $\mathcal{M} = \{$A246547 $\cup$ A360765$\} \setminus \{4\}$.
A360543$(n) > 0$ for $n \in \mathcal{M}$.
For $n = p^{\varepsilon} \in$ A961 $: n > 1$, A360543$(p^{\varepsilon}) = p^{(\varepsilon-1)} - \varepsilon$.

Records seem to occur amid powers 2^{δ}, $\delta > 2$ and 3^{ε}, $\varepsilon > 1$, and may be related to A334151. This function is covered in [5].

~

Nonplenary divisor counting function f_4 (A183093$(n) - 1$).

```
0, 0, 0, 0, 0, 2, 0, 0, 0, 2, 0, 3, 0, 2, 2, 0, 0, 3,
0, 3, 2, 2, 0, 4, 0, 2, 0, 3, 0, 6, 0, 0, 2, 2, 2, 4,
0, 2, 2, 4, 0, 6, 0, 3, 3, 2, 0, 5, 0, 3, 2, 3, 0, 4,
2, 4, 2, 2, 0, 9, 0, 2, 3, 0, 2, 6, 0, 3, 2, 6, 0, 5,
0, 2, 3, 3, 2, 6, 0, 5, 0, 2, 0, 9, 2, 2, 2, 4, 0, 9,
2, 3, 2, 2, 2, 6, 0, 3, 3, 4, 0, 6, 0, 4, 6, 2, 0, 5,
0, 6, 2, 5, 0, 6, 2, 3, 3, 2, 2, 12, ...
```

This function counts divisors $d \mid n : \text{RAD}(d) \mid \text{RAD}(n) \wedge 1 < d < n$. We remark that A183093(n) > 1 for $n \in$ A013929.

Plenary divisor counting function f_6 (A183094(n) – 1) .

```
0, 0, 0, 1, 0, 0, 0, 2, 1, 0, 0, 1, 0, 0, 0, 3, 0, 1,
0, 1, 0, 0, 0, 2, 1, 0, 2, 1, 0, 0, 0, 4, 0, 0, 0, 3,
0, 0, 0, 2, 0, 0, 0, 1, 1, 0, 0, 3, 1, 1, 0, 1, 0, 2,
0, 2, 0, 0, 0, 1, 0, 0, 1, 5, 0, 0, 0, 1, 0, 0, 0, 5,
0, 0, 1, 1, 0, 0, 0, 3, 3, 0, 0, 1, 0, 0, 0, 2, 0, 1,
0, 1, 0, 0, 0, 4, 0, 1, 1, 3, 0, 0, 0, 2, 0, 0, 0, 5,
0, 0, 0, 3, 0, 0, 0, 1, 1, 0, 0, 2, ...
```

This function counts divisors $d \mid n : \text{RAD}(d) = \text{RAD}(n) \wedge d < n$. We remark that A183094(n) > 1 for $n \in$ A013929.

Mixed semidivisor counting function f_7 (A361235).

```
0, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 2, 0, 2, 1, 0, 0, 3,
0, 2, 1, 3, 0, 2, 0, 3, 0, 2, 0, 10, 0, 0, 2, 4, 1, 4,
0, 4, 2, 3, 0, 11, 0, 3, 2, 4, 0, 3, 0, 4, 2, 3, 0, 4,
1, 3, 2, 4, 0, 14, 0, 4, 2, 0, 1, 14, 0, 4, 2, 12, 0,
4, 0, 5, 2, 4, 1, 15, 0, 3, 0, 5, 0, 16, 1, 5, 3, 3, 0,
19, 1, 4, 3, 5, 1, 4, 0, 5, 2, 4, 0, 17, 0, 3, 8, 5, 0,
5, 0, 13, 3, 3, 0, 18, 1, 4, 2, 5, 1, 19, ...
```

Lemma 2.2 and the fact state ⑦ is the reverse of ③ implies the following:

$$\text{A361235}(n) = \left|\{\, k < n : \text{RAD}(k) \mid \text{RAD}(n) \wedge \omega(k) < \omega(n) \,\}\right| \qquad [3.7]$$

Consequently, we find the following:

A361235(n) > 0 for $n \in$ A013929.

A361235(n) = 0 for $n \in$ A961.

Observation: record setters for this sequence seem to agree with A293555, the sequence of record setters for ξ_d = A243822.

Symmetric semidivisor counting function f_9 (A355432).

```
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 4,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2,
0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 4, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 4,
0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, ...
```

This function is covered at length in [3]. From that work we see the following:

$$\text{A361235}(n) = \left|\{\, k < n : \text{RAD}(k) = \text{RAD}(n) \wedge k \nmid n \,\}\right| \qquad [3.9]$$

A355432(n) = 0 for prime powers, squarefree numbers, and weak tantus numbers in A360767. A355432(n) > 0 for $n \in$ A360768.

Existence of Neutral Species in Reference Domain.

We state a few axioms for the purpose of the following proofs.

An n-semicoprime number k is such that there is a prime q such that $q \mid k$ yet q does not divide n, while $\text{RAD}(n) \mid \text{RAD}(k)$. That is, n-semicoprime k is such that $\text{RAD}(n) \mid \text{RAD}(k)$ yet $\omega(k) > \omega(n)$.

An n-semidivisor k is such that $\text{RAD}(k) \mid \text{RAD}(n)$ yet $k \nmid n$. That is, $k \mid n^{\varepsilon}$, $\varepsilon > 1$.

Semicoprimality and semidivisorship are two kinds of neutrality; there are no other kinds [2]. Since primes p must either divide or be coprime to another number, primes do not have neutrality in the domain $k < n$, hence no semidivisors nor semitotatives.

Prime power p^{ε} implies $p^{\delta} \mid p^{\varepsilon}$ for $\delta \le \varepsilon$, hence multus numbers $n \in$ A246547 have no semidivisors $k < n$.

Lemma 4.1: $S_{\varkappa}(1) = pq$ = A096014(n), where p = LPF(n) = A020639(n) and q = LPT(n) = A053669(n).
Proof: Consequence of definition of semicoprime.

Corollary 4.2: Prime $n = p \in$ A40 implies $n < S_{\varkappa}(1)$ since $p < pq$ and as a consequence of the definition of prime.

Corollary 4.3: $n \le 6$ implies $n < S_{\varkappa}(1)$ since the smallest semiprime A6881(1) = $pq = 2 \times 3 = 6$, therefore there are no semiprimes smaller than 6.

Lemma 4.4: Multus $p^{\varepsilon} = n \in$ A246547, $\varepsilon > 1$, and $n > 4$ implies A243823($\varkappa$) > 0.
Proof: Suppose even $n = p^{\varepsilon} = 2^{\varepsilon}$, hence $\varepsilon > 2$. Then $q = 3$ and $pq = 6$; clearly $6 < 2^3$. Through induction, it is clear that $6 \lozenge 2^{\varepsilon}$ for all $\varepsilon > 2$. Now suppose odd $n = p^{\varepsilon}$, hence $\varepsilon > 1$. Then $q = 2$ and $pq = 6$; clearly $6 < p^2$ for odd prime p. Through induction, it is clear that $6 \lozenge p^{\varepsilon}$ for all $\varepsilon > 1$ and odd p. ■

Lemma 4.5: Varius $\varkappa \in$ A120944 and $\varkappa > 6$ implies A243823($\varkappa$) > 0.
Proof: Our strategy is to create the smallest varius number $\varkappa$ so as to induce $S_{\varkappa}(1) > \varkappa$. The smallest varius number $\varkappa$ = A120944(1) = 6 = A6881(1), and there is no smaller squarefree semiprime, hence A243823(6) = 0. We turn to $\varkappa$ = A120944(2) = 10 = 2 × 5, hence q = 3, and it is clear 6 < 10.

Therefore we attempt to close the gap between prime factors so as to force q to be as large as possible in comparison to $\varkappa$. This implies the use of primorial $\varkappa = P(i)$ = A2110(i). For $\varkappa = P(i)$, $q = \text{PRIME}(i+1)$, and we have $pq = 2 \times \text{PRIME}(i+1)$. For $\varkappa = 30$, $pq = 2 \times 7 = 14$. For $\varkappa = 210$, $pq = 2 \times 11 = 22$.

Now we turn to odd half-primorials $\varkappa = P(i)/2$ = A2110(i)/2. For $\varkappa = P(i)/2$, $q = 2$, and $p = 3$, and it is clear that these numbers also have semitotative $S_{\varkappa}(1) < \varkappa$.

Hence it is clear that except for $\varkappa = 6$, varius $\varkappa$. ■

Lemma 4.6: Tantus $n \in$ A126706 implies A243823($\varkappa$) > 0.
Proof: We employ again a strategy that begins with the smallest possible tantus number n = A126706(1) = 12 = $2^2 \times 3$, while $pq = 2 \times 5$; 10 < 12. These numbers are larger than their squarefree kernels $\varkappa$ on account of multiplicity, therefore it is evident the approach we pursued in Lemma 4.5 may also be applied here with same result. ■

Theorem 4: $n \notin$ A193461 implies A243823(n) ≥ 1.
Proof: Consequence of Lemmas 4.1, 4.3, 4.4, and 4.5 .

Lemma 5.1: Smallest n-semidivisor $Đ_{\varkappa}(1) = p^{(\varepsilon+1)}$, where p = LPF(n).
Proof: Consequence of definition of semidivisor.

Corollary 5.2: Prime power $n \in$ A246655 implies $n < Đ_{\varkappa}(1)$.
(A246655 = $\{\, m > 1 : \omega(m) = 1 \}$)

Corollary 5.3: Prime power $n \in$ A961 implies A243822(n) = 0, since $p^{(\varepsilon+1)} > p^{\varepsilon}$.
(A961 = $\{\, m : \omega(m) = 1 \}$)

Lemma 5.4: Varius $\varkappa \in$ A120944 implies A243822($\varkappa$) > 0.
Proof: Our strategy is to create the smallest varius $\varkappa$ such that $Đ_{\varkappa}(1) > \varkappa$. It is clear that a product of 2 similar primes pq, $p \ne q$, would create a smaller $\varkappa$ than any product involving more than 2 distinct primes. The very smallest such squarefree semiprime pq, $p \ne q$, is A6881(1) = 6 = 2 × 3, and it is clear that $2^2 < 6$. We thus attempt to edge $Đ_{\varkappa}(1) = p^2$ so as to exceed pq. By definition of p as least prime factor of $n = pq$, $p < \sqrt{n} \wedge q > \sqrt{n}$, hence $p^2 < pq$. ■

Lemma 5.5: Tantus $n \in$ A126706 implies A243822$(n) > 0$.

Proof: Again, we pursue a strategy of creating the smallest tantus number n that might induce $Đ_{\varkappa}(1) = p^{(\varepsilon+1)}$ to exceed n. We select a tantus number $p^{\varepsilon}q$, $p < q$, $\varepsilon > 1$, since additional distinct prime divisors would merely increase n. In fact, we examine $p^2q = 2^2 \times 3 = 12 =$ A126706(1), the smallest tantus number, and see that $2^3 < 12$. Furthermore, since $q > p$, through induction, we see that $p^{(\varepsilon+1)} < p^{\varepsilon}q$ for all $p < q$ and $\varepsilon > 1$. ■

Theorem 5: $n \in$ A024619 implies A243822$(n) > 0$.

Proof: Consequence of Lemmas 5.1, 5.4, and 5.5.

Recalling [1.7], we may write the following:

$$\xi(n) = \xi_d(n) + \xi_t(n)$$
$$\text{A045763}(n) = \text{A243822}(n) + \text{A243823}(n) \qquad [4.1]$$

Therefore we may write the following:

For prime n, A243822(n) = A243823(n) = A045763(n) = 0. There are no neutral numbers $k < n$ for prime n. Hence there are neither semitotatives nor semidivisors less than n.

For multus $n \in$ A246547, that is, composite prime powers, A243822$(n) = 0$, but A243823$(n) > 0$, therefore A045763$(n) > 0$. The exception: A243823(4) = 0. Multus numbers have no semidivisors because all n-regular k such that $k < n$ are n-divisors. Multus numbers $n > 4$ have at least 1 semitotative k = A096014(n) such that $k < n$. Therefore, aside from $n = 4$, multus numbers have at least 1 n-neutral k such that $k < n$.

For varius $n \in$ A120944, that is, squarefree composites, A243822$(n) > 0$ and A243823$(n) > 0$, therefore A045763$(n) > 0$. The exception: A243823(6) = 0 since 6 is the smallest squarefree number. Varius numbers have at least 1 semidivisor $k < n$ and at least 1 semitotative, except $n = 6$ has no semitotatives.

Tantus $n \in$ A126706 has at least 1 semidivisor $k < n$ and at least 1 semitotative, hence at least 2 n-neutral k such that $k < n$.

Composites outside $n = 4$ and $n = 6$ have at least 1 semitotative, and non-prime powers outside $n = 6$ have at least 1 semidivisor $k < n$.

Table 2.

species	$\xi(n)$ A045763(n)	$\xi_d(n)$ A243822(n)	$\xi_t(n)$ A243823(n)
primes (A40)	—	—	—
$n = 4$	—	—	—
multus (A246547)	> 0	—	> 0
$n = 6$	—	1	1
varius (A120944)	> 0	> 0	> 1
tantus (A126706)	> 0	> 0	> 1

Sequences Concerning Constitutive State Counting Functions.

Table 3 summarizes sequences having to do with constitutive states. In the first column are listed various species. These are the divisor, semidivisor, regular, semicoprime, coprime, and neutral species, followed by certain constitutive states, all bounded by n.

Table 3.

	counting function	list	record setter	record
D_n	A5	A027750	A2182	A2183
$Đ_{\varkappa}$	A243822	A272618	A293555	A293556
$R_{\varkappa}$	A010846	A162306	A244052	A244053
$S_{\varkappa}$	A243823	A272619	A292867	A292868
$T_{\varkappa}$ ⓪	A10	A038566	A8578	A6093
$\Xi_{\varkappa}$	A045763	A133995	A300859	A300914
①	A360480			
③	A360543		A334151	
④	A183093$(n) - 1$			
⑤	A27	A27	A27	A27
⑥	A183094$(n) - 1$			
⑦	A361235			
⑨	A355432		A360589	

From the definitions of constitutive states and their presence in the proper cototient, we can write the following formulae:

A051953$(n) = n -$ A10(n).
= A183093(n) + A183094(n) + A361235(n) + A355432(n) + A360543(n) + A360480(n)

A045763$(n) = n -$ A10(n) − A5(n) + 1.
= A243822(n) + A243823(n).
= A361235(n) + A355432(n) + A360543(n) + A360480(n)

A010846(n) = A5(n) + A243822(n).
= A183093(n) + A183094(n) + A361235(n) + A355432(n)

A243822(n) = A010846(n) − A5(n).
= A045763(n) − A243823(n).

A243823(n) = A045763(n) − A243822(n).
= $n -$ A10(n) − A010846(n) + 1.

A243822(n) = A361235(n) + A355432(n). (⑦⑨)
A243823(n) = A360543(n) + A360480(n). (①③)

A361235(n) = A243822(n) − A355432(n). (⑦)
= A045763(n) − A243823(n) − A355432(n).
= A051953(n) − A5(n) − A243823(n) − A355432(n) + 1.
= A010846(n) − A5(n) − A355432(n).

A355432(n) = A243822(n) − A361235(n). (⑨)
= A045763(n) − A243823(n) − A361235(n).
= A051953(n) − A5(n) − A243823(n) − A361235(n) + 1.
= A010846(n) − A5(n) − A361235(n).

A360543(n) = A243823(n) − A360480(n) (③)
= A045763(n) − A243822(n) − A360480(n)
= $n -$ A10(n) − A010846(n) − A360480(n) + 1
= A051953(n) − A010846(n) − A360480(n) + 1

A360480(n) = A243823(n) − A360543(n) (①)
= A045763(n) − A243822(n) − A360543(n)
= $n -$ A10(n) − A010846(n) − A360543(n) + 1
= A051953(n) − A010846(n) − A360543(n) + 1

In this way we have demonstrated complete coverage of all the possible constitutive states for k in the proper cototient of n regarding constitutive states.

Conclusion.

Together with the totient, we have described a finer classification of numbers k the range $1 \dots n$ based on the multiplicative properties described in [2]. The cototient harbors as many as 7 kinds of multiplicative relationship between k and n, including symmetric and mixed semitotatives, symmetric and mixed semidivisors, lean and plenary divisors, and $k = n$. These represent constitutive states ①, ③, ⑨, ⑦, ④, ⑥, and ⑤, respectively.

We have shown which constitutive states appear in the reference range $[1 \dots n]$ of certain classes of natural numbers n, including composite prime powers (multus numbers $n \in$ A246547), squarefree composites (varius numbers $n \in$ A120944), and numbers n neither squarefree nor prime powers (tantus numbers $n \in$ A126706). In the last-mentioned sequence, we have determined 2 special cases of tantus number that harbors certain constitutive states among $k < n$. These are the sequences of the "strong" tantus numbers ($n \in$ A360768) and the "limbo-bar" tantus numbers ($n \in$ A360765), which harbor symmetric semidivisors $k < n$ and mixed semitotatives, respectively.

We have proposed several counting functions based on constitutive states between k and n, for $k < n$. We have shown that the symmetric semidivisor counting function A355432 and the mixed semidivisor counting function A361235 sum to the semidivisor counting function A243822. Likewise, we have shown that the symmetric semitotative counting function A360480 and the mixed semitotative counting function A360543 sum to the semitotative counting function A243823. In turn, A243822 and A243823 represent the partition of the neutral numbers that are counted in A045763. Furthermore, A243822 relates to A010846 and the divisor counting function. Regarding A045763 and its relation to A051953, the Euler totient function, and the divisor counting function, we have shown how our nuanced proposals frame into the larger picture of elementary number theory.

Appendix.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved February 2023.

[2] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

[3] Michael Thomas De Vlieger, The Symmetric Semidivisor Counting Function, *Simple Sequence Analysis*, 20230216.

[4] Michael Thomas De Vlieger, The Symmetric Semicoprime Counting Function, *Simple Sequence Analysis*, 20230222.

[5] Michael Thomas De Vlieger, The Semitotative Counting Function and Species, *Simple Sequence Analysis*, 20230225.

Code:

[C0] Function $f(k, n)$ yields the constitutive state (Svitek number) between k and n.

```
conState[j_, k_] :=
 Which[j == k, 5, GCD[j, k] == 1, 0, True,
  1 + FromDigits[
    Map[Which[Mod[##] == 0, 1,
      PowerMod[#1, #2, #2] == 0, 2, True, 0] & @@ # &,
      Permutations[{k, j}]], 3]]
```

[C1] Calculate $\boldsymbol{R}_{\varkappa}$ bounded by an arbitrary limit m (i.e., calculate A275280(n); flatten and take union to provide A162306)

```
regularsExtended[n_, m_ : 0] :=
  Block[{w , lim = If[m <= 0, n, m]},
   Sort@ ToExpression@
     Function[w,
       StringJoin[
         "Block[{n = ", ToString@ lim,
         "}, Flatten@ Table[",
         StringJoin@
           Riffle[Map[ToString@ #1 <> "^" <>
             ToString@ #2 & @@ # &, w], " * "],
         ", ", Most@ Flatten@ Map[{#, ", "} &, #],
         "]]" ] &@
       MapIndexed[
         Function[p,
           StringJoin["{", ToString@ Last@ p,
             ", 0, Log[",
             ToString@ First@ p, ", n/(",
             ToString@
               InputForm[
                 Times @@ Map[Power @@ # &,
                   Take[w, First@ #2 - 1]]],
             ")]}" ] ]@ w[[First@ #2]] &, w]]@
       Map[{#, ToExpression["p" <>
         ToString@ PrimePi@ #]} &, #[[All, 1]] ] &@
       FactorInteger@ n];
```

[C2] Generate the squarefree kernel of n (A7947):

```
rad[n_] := rad[n] = Times @@
  FactorInteger[n][[All, 1]];
```

[C3] Generate tantus numbers (A126706):

```
a126706 = Block[{k}, k = 0;
   Reap[Monitor[Do[
       If[And[#2 > 1, #1 != #2] & @@
         {PrimeOmega[n], PrimeNu[n]},
        Sow[n]; Set[k, n] ],
       {n, 2^21}], n]][[-1, -1]]] (* Tantus *);
```

[C4] Generate "strong tantus" numbers (A360768):

```
Select[a126706[[1 ;; 120]], #1/#2 >= #3 & @@
  {#1, Times @@ #2, #2[[2]]} & @@
  {#, FactorInteger[#][[All, 1]]} &]
```

[c5] Generate tantus numbers that have k ③ n (A360765):

```
lcp[n_] :=
 If[OddQ[n], 2,
   p = 2;
   While[Divisible[n, p], p = NextPrime[p]]; p];
nn = 120; a = a126706[[1 ;; nn]];
Reap[
  Do[n = a[[j]];
    If[rad[n]*lcp[n] < n, Sow[n]], {j, nn}]][[-1, -1]]
```

[c6] Generate A360480, the k ① n counting function:

```
Table[k = rad[n];
  Count[Range[n],
    _?(Nor[CoprimeQ[#1, n], Divisible[#2, k],
       Divisible[k, #2]] & @@ {#, rad[#]} &)], {n, 88}]
```

[c7] Generate A360543, the k ③ n counting function:

```
nn = 120;
c = Select[Range[4, nn], CompositeQ];
s = Select[Select[Range[4, nn], Not @* SquareFreeQ],
  Function[{n, q, r},
    AnyTrue[TakeWhile[c, # <= n &],
      And[PrimeNu[#] > q,
          Divisible[rad[#], r]] &]] @@
          {#, PrimeNu[#], rad[#]} &];
Table[If[FreeQ[s, n], 0,
  Function[{q, r},
    Count[TakeWhile[
      c, # <= n &], _?(And[PrimeNu[#] > q,
         Divisible[rad[#], r]] &)]] @@
         {PrimeNu[n], rad[n]}], {n, nn}]
```

[c7a] faster algorithm for A360543, the k ③ n counting function, given a dataset of A360765 and [c1]:

```
nn = 2^12;
Array[Set[s[#], a360765[[#]]] &, Length[a360765]];
next = 1;
Monitor[Table[Which[SquareFreeQ[n], 0,
   PrimePowerQ[n],
   #1^(#2 - 1) - #2 & @@ FactorInteger[n][[1]],
   n == s[next], next++;
   Function[{qq, rr}, k = 0; r = Rest@
     regularsExtended[n];
     t = Rest@ Flatten@
      Outer[Plus, rad[n]*Range[0, n/rad[n] - 1],
        Select[Range[rad[n]], CoprimeQ[rad[n], #] &]];
     Do[If[And[Divisible[#, rr], PrimeNu[#] > qq],
       k++] &[i j],
       {i, r[[1 ;; LengthWhile[r, n/t[[1]] > # &]]]},
       {j, t[[1 ;; LengthWhile[t, n/i > # &]]]}]] @@
       {PrimeNu[n], rad[n]}; k,
   True, 0], {n, nn}]
```

[c8] faster algorithm for A360543, the k ③ n counting function, given a dataset of A360765 and [c1]:

```
rad[n_] := rad[n] = Times @@
  FactorInteger[n][[All, 1]];
{{}, {}}~Join~Table[r = Rest@ regularsExtended[n];
  t = Rest@ Flatten@
    Outer[Plus, rad[n]*Range[0, n/rad[n] - 1],
      Select[Range[rad[n]], CoprimeQ[rad[n], #] &]];
  Union@ Flatten@
    Table[i j,
      {i, r[[1 ;; LengthWhile[r, n/t[[1]] > # &]]]},
      {j, t[[1 ;; LengthWhile[t, n/i > # &]]]}],
      {n, 3, 24}]
```

[c9] Generate A361235, the k ⑦ n counting function:

```
nn = 120;
c = Select[Range[4, nn], CompositeQ];
s = Select[Range[4, nn], Not @* PrimePowerQ];
Table[If[FreeQ[s, n], 0,
  Function[{q, r},
    Count[DeleteCases[
      TakeWhile[c, # <= n &], _?(Divisible[n, #] &)],
      _?(And[PrimeNu[#] < q,
         Divisible[r, rad[#]]] &)]] @@
         {PrimeNu[n], rad[n]}], {n, nn}]
```

[c10] Generate A355432 (needs [c1]), the k ⑨ n counting function:

```
A355432 = Block[{a, c, f, k, s, t, nn},
  nn = 2^20; c[_] = 0;
  f[n_] := f[n] = n regularsExtended[n, Floor[nn/n]];
  s = Select[Range[nn],
    And[CompositeQ[#], SquareFreeQ[#]] &];
  Monitor[
    Do[Set[t[ s[[i]] ], f@ s[[i]]], {i, Length[s]}],
  i];
  Monitor[
    Do[k = t[ s[[j]] ];
      Map[Function[m,
        Set[c[m],
          Count[TakeWhile[k, # <= m &],
          _?(Mod[m, #] != 0 &)]]], k], {j, Length[s]}],
  j];
  Array[c, nn] ];
```

[c11] Generate the k ④ n counting function ($\text{A183093}(n) - 1$):

```
{1}~Join~Table[Function[q,
  DivisorSum[n, 1 &, -1 + PrimeNu[#] < q &]]@
  PrimeNu[n], {n, 2, 120}]
```

[c12] Generate the k ⑥ n counting function ($\text{A183094}(n) - 1$):

```
Table[Function[q,
  DivisorSum[n, 1 &, -1 + PrimeNu[#] == q &]]@
  PrimeNu[n], {n, 120}]
```

Concerns sequences:

A000005: Divisor counting function $\tau(n)$.
A000010: Euler totient function $\phi(n)$.
A000040: Prime numbers.
A000961: Prime powers.
A001221: Number of distinct prime divisors of n, $\omega(n)$.
A007947: Squarefree kernel of n; $\text{RAD}(n)$.
A010846: Regular counting function.
A013929: Numbers that are not squarefree.
A024619: Numbers that are not prime powers.
A045763: Neutral counting function.
A051953: Cototient function: $n - \phi(n)$.
A053669: Smallest prime q that does not divide n.
A120944: "Varius" numbers; squarefree composites.
A126706: "Tantus" numbers neither prime power nor squarefree.
A183093: $\tau(n) - \text{A183094}(n) + 1$.
A183094: Number of powerful divisors $d \mid n : d > 1$.
A246547: "Multus" numbers; composite prime powers.
A275055: $\bigotimes_{p|n} \{ p^{\varepsilon} : 0 \le \varepsilon \le \delta \}$ where p^{δ} is the largest power of p that divides n.
A275280: $\{ k = \{ \bigotimes_{p \nmid k} \{ p^{\varepsilon} : \varepsilon \ge 0 \} \} \wedge k \le n \}$.
A355432: $a(n)$ = symmetric semidivisor counting function.
A360480: $a(n)$ = symmetric semicoprime counting function.
A360543: $a(n)$ = mixed semicoprime counting function.
A360765: $n \in \text{A126706} : \text{A7947}(n) \times \text{A053669}(n) < n$.
A360767: Weakly tantus numbers.
A360768: Strongly tantus numbers.
A360769: Odd tantus numbers.
A361235: $a(n)$ = mixed semidivisor counting function.

Document Revision Record:

2023 0226: Draft 1.
2023 0227: Draft 2.
2023 0301: Final.
2023 0302: Added Theorems 4 and 5, Table 2.
2023 0304: Added A360543, A360765, and A361235.
2023 0308: Added Theorems 3.6–3.9.
2023 0314: Added [5].