The NextRegular(n) Function

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Abstract.

We introduce several related functions having to do with numbers k in an infinite sequence $R_\varkappa$ where $\mathrm{RAD}(k) \mid \varkappa$, where $\varkappa$ is squarefree. These functions furnish successors and predecessors to a given number in the sequence $R_\varkappa$. We also examine the sequence $\varkappa R_\varkappa$ wherein all numbers share the same squarefree kernel $\varkappa$. Some of these functions have long existed in the OEIS and others we have proposed recently. This work merely lays out the basics about each function.

Introduction.

Let $\mathrm{RAD}(n) = \mathrm{A}7947(n) = \varkappa$, the squarefree kernel of n, that is, the product of distinct prime divisors $p \mid n$.

Definition 1.0. Define k **regular** to n, integers, as k such that $\mathrm{RAD}(k) \mid \varkappa$. In other words, n-regular k is a product that does not involve any prime q coprime to n. It's clear from this definition that n-regularity ascribes to $\mathrm{RAD}(n) = \varkappa$, hence we define $R_\varkappa$ to be the sequence of n-regular k.

We may construct $R_\varkappa$ as follows:

$$R_\varkappa = \bigotimes_{p \mid \varkappa} \{p^\varepsilon : \varepsilon \geq 0\}. \quad [1.0]$$

As a tensor product of countably infinite sets, it is clear $R_\varkappa$ is also countably infinite. Sorting $R_\varkappa$ according to magnitude of its elements, we may assign an index and hence we have a countably infinite set.

Definition 1.1. Define k **strongly regular** to n as k such that $\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$. Alternatively, we may say that such strongly regular k and n are **coregular**. Since strongly n-regular k is a product (distinct from n) of all distinct prime divisors p such that $p \mid n$, we define $\varkappa R_\varkappa$ to be the set of strongly n-regular k. Multiplication by the common squarefree kernel $\varkappa$ guarantees the presence of all distinct prime factors of $\varkappa$.

For example, let $n = 12$. Then $\mathrm{RAD}(12) = \varkappa = 6$. Then the set R_6 is the tensor product of prime power ranges of 2 and 3, i.e.,

$$R_6 = \{2^\varepsilon : \varepsilon \geq 0\} \otimes \{3^\varepsilon : \varepsilon \geq 0\}. \quad [1.1]$$

This is A3586, which begins as follows:

```
1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54,
64, 72, 81, 96, 108, 128, 144, 162, 192, 216, 243, 256,
288, 324, 384, 432, 486, 512, 576, 648, 729, 768, 864,
972, 1024, 1152, ...
```

From this, we construct $\varkappa R_\varkappa = 6R_6$ which begins as follows:

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6, 12, 18, 24, 36, 48, 54, 72, 96, 108, 144, 162, 192,
216, 288, 324, 384, 432, 486, 576, 648, 768, 864, 972,
1152, 1296, 1458, 1536, 1728, 1944, 2304, 2592, 2916,
3072, 3456, 3888, 4374, 4608, 5184, 5832, 6144, ...
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Distinct numbers $k, n \in \varkappa R_\varkappa$ are coregular, meaning that they share the same squarefree kernel $\varkappa$.

For the empty product $\varkappa = 1$, $R_1 = \{1\}$, a finite set. There is only 1 natural number that is the product of zero primes and that is 1 itself.

There are two notable species of n-regular number; these are the n-**divisor** $d \mid n$, and the n-**semidivisor** $k \mid n^\varepsilon$, $\varepsilon > 1$. Alternatively we might call the n-semidivisor a nondivisor n-regular number.

Because the divisor constitutes a major focus of mathematical interest since antiquity, this distinction among n-regular numbers proves of interest.

Lemma 1.2. The number 1 is regular to all numbers, since it divides all numbers, and divisors are a finite subset of regular numbers. The number 1 is not divisible by any prime, hence no prime q coprime to $\varkappa$. Therefore, 1 is the smallest element in $R_\varkappa$.

Lemma 1.3. In the sorted sequence of $\varkappa$-coregular numbers $\varkappa R_\varkappa$, we see the squarefree number $\varkappa$ followed by $m\varkappa$ where m is $\varkappa$-regular, i.e., $\mathrm{RAD}(m) \mid \varkappa$. This lemma follows from Definition 1.1.

Lemma 1.4. Prime $\varkappa = p$ implies the sorted p-coregular sequence pR_p begins with prime p followed by composite prime powers $p^\varepsilon : \varepsilon > 1$.

$$R_p = \{p^\varepsilon : \varepsilon \geq 0\}. \quad [1.2]$$

$$pR_p = \{p^\varepsilon : \varepsilon \geq 1\}. \quad [1.3]$$

This is evident given the nature of [1.2] and [1.3].

Lemma 1.5. For squarefree composite $\varkappa \in \mathrm{A}120944$, the sorted $\varkappa$-coregular sequence $\varkappa R_\varkappa$ begins with $\varkappa$ followed by "tantus numbers" $m\varkappa \in \mathrm{A}126706$, which are neither squarefree nor prime powers. This is clear since we may divide $\varkappa R_\varkappa$ by $\varkappa$ to derive $R_\varkappa$, whose minimum is 1 via Lemma 1.2. Multiplying 1 by $\varkappa$, we have squarefree composite $\varkappa$ as the minimum of $\varkappa R_\varkappa$. ■

Lemma 1.5 suggests that the only nontantus number in $6R_6$ is 6 itself. This follows from the definition of $\varkappa$-coregular; all numbers in the sequence are distinct products of 6, and since there can only be one instance of $m\varkappa = 1 \times 6$, $\varkappa = 6$ is the only squarefree term in the sequence and its minimum.

Corollary 1.6. Squarefree n-regular k implies $k \mid n$.

Herinafter we construe the sets $R_\varkappa$ and $\varkappa R_\varkappa$ as being ordered according to magnitude, that is, beginning with the minimum.

The $\varkappa$-Regular Successor Function.

Define $f(n) = k$ such that $k > n$ and $\mathrm{RAD}(k) \mid \mathrm{RAD}(n) = \varkappa$ to be the $\varkappa$-regular successor function.

Suppose n is the i-th element of $R_\varkappa$. Then $f(n) = R_\varkappa(i+1)$.

Prime p in R_p follows 1 and is succeeded by p^2, given [1.2], and generally, the successor to p^ε in R_p is $p^{(\varepsilon+1)}$.

The successor to 1 in $R_\varkappa$ is prime $p = \mathrm{LPF}(\varkappa) = \mathrm{A}020639(\varkappa)$. Generally, the successor function presents a problem similar to that explored in Mintz [2]. For squarefree semiprimes pq, where $q = \mathrm{NEXTPRIME}(p)$, we have the following sequence:

$$R_{pq} = \{1, p, q, p^2, pq, \ldots, q^2, \ldots\}. \quad [2.1]$$

We can imagine an even squarefree semiprime $2q$, where q is an immense prime, and then see many powers of 2 appear before q and between q and $2q$, etc.

$$R_{2q} = \{1, 2, 2^2, 2^3, \ldots, q, \ldots, 2q, \ldots, q^2, \ldots\}. \quad [2.2]$$

Let $A(1)$ remain undefined; for $n > 1$, $A(n) = ⊠k$ such that $k > n$ and $\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$. (The symbol $⊠k$ means k is the smallest such.) This sequence is Sigrist's A289280 which begins as follows:

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4, 9, 8, 25, 8, 49, 16, 27, 16, 121, 16, 169, 16, 25,
32, 289, 24, 361, 25, 27, 32, 529, 27, 125, 32, 81, 32,
841, 32, 961, 64, 81, 64, 49, 48, 1369, 64, 81, 50,
1681, 48, 1849, 64, 75, 64, 2209, 54, 343, 64, 81, 64,
2809, 64, 121, 64, 81, 64, 3481, 64, ...
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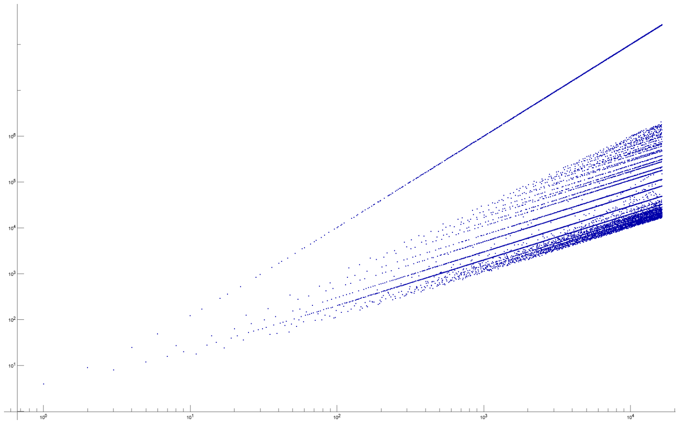

Figure 1: Log log scatterplot of A065642$(1\ldots 2^{14})$.

We can use a naive greedy approach to arrive at answers. Let's attempt a more efficient method based on theorems.

Theorem 2.1. $A(n) \le n^2$.
Proof. Aside from $A(1) = 1$, $A(n) > n$ by definition. For $n = p$ prime, via [1.2] $A(p) = p^2 = n^2$. Generally, $A(p^\varepsilon) = p^{(\varepsilon+1)}$, and we see through induction on ε that $p^{(\varepsilon+1)} < p^{(2\varepsilon)}$ for $\varepsilon > 1$.

For composite n, we have composite squarefree $\varkappa$ such that $\omega(\varkappa) > 1$. Our approach involves attempting to find n-regular k such that $n < k < n^2$. Since $\textsc{rad}(n) = \varkappa$, we are not concerned with $n < \varkappa$ in $\boldsymbol{R}_\varkappa$.

Let $p = \textsc{lpf}(\varkappa)$ and $q = \textsc{gpf}(\varkappa) =$ A6530$(\varkappa)$. We know the following:

$$\log_p n > \log_q n > \log_n n. \qquad [2.3]$$

Hence, between n and n^2 in $\boldsymbol{R}_\varkappa$, for $n \ge \varkappa$, there exists at least 1 prime power associated with each of p and q. Therefore, $A(n) \le n^2$. ∎

Theorem 2.1 implies that, given a means to generate $\boldsymbol{R}_\varkappa$ via [1.0], we need only generate $\{n, \ldots, n^2\}$ and the answer is the second term in that subsequence. In other words, $n = \boldsymbol{R}_\varkappa(i)$ implies $A(n) = \boldsymbol{R}_\varkappa(i+1)$.

Code [c6] generates the sequence efficiently. The scatterplot appears in Figure 2. Records derive from $A(p) = p^2$. Horizontal quasi-linear features mostly derive from powers of small primes. The lower bound is comprised by $A(n) = n + 2$, where n is even and $n + 2$ is a power of 2, as $n + 1$ is coprime to n, hence, never n-regular.

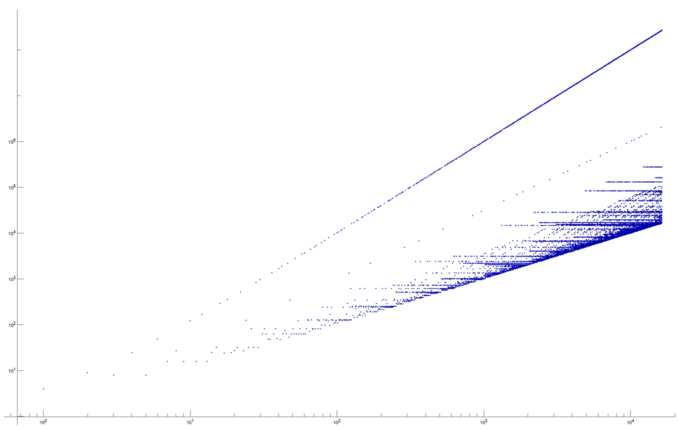

Figure 2: Log log scatterplot of A289280$(1\ldots 2^{14})$.

The $\varkappa$-Regular Predecessor Function.

We may modify the $\varkappa$-regular successor function $f(n)$ to work backward, perhaps by adding the latter parameter in $f(n, -1)$.

Let $B(1) = 1$; for $n > 1$, $B(n) = \boxtimes k$ such that $k < n$ and $\textsc{rad}(k) = \textsc{rad}(n) = \varkappa$. (The symbol $\boxtimes k$ means k is the largest such.) This is A079277 by Istvan Beck, which begins as follows:

```
1, 1, 2, 1, 4, 1, 4, 3, 8, 1, 9, 1, 8, 9, 8, 1, 16, 1,
16, 9, 16, 1, 18, 5, 16, 9, 16, 1, 27, 1, 16, 27, 32, 25,
32, 1, 32, 27, 32, 1, 36, 1, 32, 27, 32, 1, 36, 7, 40,
27, 32, 1, 48, 25, 49, 27, 32, 1, 54, 1, 32, 49, 32, 25,
64, 1, 64, 27, 64, 1, 64, 1, 64, 45, 64, 49, 72, 1, ...
```

Define row n of A162306 to be a sorted list of n-regular numbers k that do not exceed n. Then A079277 is the penultimate term in row n of A162306. We present two lemmas associated with A079277.

Lemma 3.1. For prime p, $B(p) = 1$, which follows from the construction of $\boldsymbol{R}_p$ in [1.1]. In the prime power range of p, the empty product 1 precedes p. Generally, the successor to p^ε in $\boldsymbol{R}_p$ is $p^{(\varepsilon-1)}$.

Lemma 3.2. For n with $\omega(n) > 1$, $B(n) \nmid n$.
Proof. We have to show that the largest proper divisor of n, $\textsc{d} = n/p$, is such that $\textsc{d} < k < n$, where $p = \textsc{lpf}(\varkappa) =$ A020639$(\varkappa)$.

$$\log_p n - \log_p n/p = 1 \qquad [3.2]$$

Since $\omega(n) > 1$, $\log_p n$ is not an integer, therefore there is some perfect power $k = p^\varepsilon$, $\varepsilon = \lfloor \log_p n \rfloor$ that interposes $\textsc{d}$ and n. (This is not to say that $B(n) = p^{\lfloor \log_p n \rfloor}$.) ∎

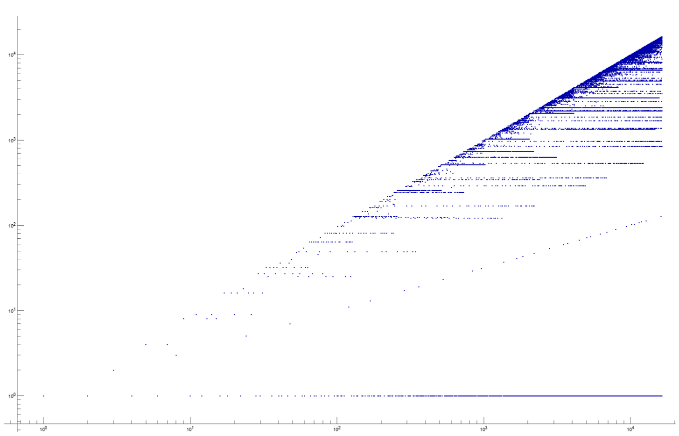

Figure 3: Log log scatterplot of A079277$(1\ldots 2^{14})$.

Hence we note that in $\boldsymbol{R}_\varkappa$, we have the subsequence {A079277(n), n, A289280(n)}. A couple generalizations:

1. For prime $n = p$, we have $\{1, p, p^2\}$ and generally, for n such that $\omega(n) = 1$, $\{p^{(\varepsilon-1)}, p^\varepsilon, p^{(\varepsilon+1)}\}$.
2. For squarefree composite $n = \varkappa$, we have $\{k, \varkappa, p\varkappa\}$, where $k \nmid \varkappa$ and $p = \textsc{lpf}(\varkappa)$.
3. The scatterplot of A079277 shown by Figure 3 resembles that of A289280 shown by Figure 2. Many of its features can be explained by reversing the approach. For instance, the upper bound is comprised by $B(n) = n - 2$, where n is even and $n - 2$ is a power of 2.

The $\varkappa$-Coregular Successor Function.

Definition 1.1 shows that we may derive a similar function $g(n) = k$ such that $k > n$ and $\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$ to be the $\varkappa$-coregular successor function. This function is of interest because of the quality noted in Lemma 1.3, that is, $\varkappa \boldsymbol{R}_\varkappa$ begins with squarefree $\varkappa$ followed by nonsquarefree $m\varkappa$, $m > 1$ and $m \in \boldsymbol{R}_\varkappa$.

Suppose n is the i-th element of $\varkappa \boldsymbol{R}_\varkappa$. Then $g(n) = \varkappa \boldsymbol{R}_\varkappa(i+1)$. Dividing by , we have $\boldsymbol{R}_\varkappa(i) = n/\varkappa$ and successor $\boldsymbol{R}_\varkappa(i+1)$.

The successor to p in $p\boldsymbol{R}_p$ is p^2, given [1.3], and generally, the successor to p^ε in $p\boldsymbol{R}_p$ is $p^{(\varepsilon+1)}$.

We find it not as simple for squarefree composite $\varkappa \in$ A120944.

The successor to $\varkappa$ in $\varkappa \boldsymbol{R}_\varkappa$ is $p\varkappa$, where $p = \mathrm{LPF}(\varkappa) =$ A020639$(\varkappa)$. The successor to $k > \varkappa$ in $\varkappa \boldsymbol{R}_\varkappa$ generally is not as easy to determine, and presents a problem similar to that explored in Mintz [2]. For squarefree semiprimes pq, where $q = \mathrm{NEXTPRIME}(p)$, we have the following sequence:

$$pq\boldsymbol{R}_{pq} = pq \times \{1, p, q, p^2, pq, \ldots, q^2, \ldots\}. \qquad [4.1]$$

We can imagine an even squarefree semiprime $2q$, where q is an immense prime, and then see 2

$$2q\boldsymbol{R}_{2q} = 2q \times \{1, 2, 2^2, 2^3, \ldots, q, \ldots, 2q, \ldots, q^2, \ldots\}. \quad [4.2]$$

It is clear that it is sufficient to find the successor n' to k' in $\boldsymbol{R}_\varkappa$, then taking $\varkappa n'$. Therefore there may be some predictability partly assisted by Mintz's approach in [2]. In aggregate, the problem of finding the successor to k in $\varkappa \boldsymbol{R}_\varkappa$ is akin to problems associated with the *abc* conjecture (which is outside the scope of this paper).

Let $a(1) = 1$; for $n > 1$, $a(n) = k$ such that $k > n$ and $\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$. This sequence is Zumkeller's A065642 which begins as follows:

```
1, 4, 9, 8, 25, 12, 49, 16, 27, 20, 121, 18, 169, 28,
45, 32, 289, 24, 361, 40, 63, 44, 529, 36, 125, 52, 81,
56, 841, 60, 961, 64, 99, 68, 175, 48, 1369, 76, 117,
50, 1681, 84, 1849, 88, 75, 92, 2209, 54, 343, 80, 153,
104, 2809, 72, 275, 98, 171, 116, 3481, 90, ...
```

Theorem 4.1. $a(n) \le n^2$.

Proof. Given the relation between $\varkappa$-regular $\boldsymbol{R}_\varkappa$ and $\varkappa$-coregular $\varkappa \boldsymbol{R}_\varkappa$ shown in Definition 1.1, the proposition follows from Theorem 2.1, via multiplication by $\varkappa$. ■

Code [C5] efficiently generates the sequence, whose scatterplot appears in Figure 1.

It is clear that infinite recursion of the coregular successor function g, begining with a squarefree number $\varkappa$, generates $\varkappa \boldsymbol{R}_\varkappa$. Therefore, suppose we begin with $a(6) = 12$, then take $a(12) = 18$, etc. It is clear from the definition of A065642 that we reconstruct $6\boldsymbol{R}_6 = 6 \times$ A3586.

Define sequence A360529 to be the mapping $g \mapsto$ A024619, the sequence of numbers that are not prime powers. This sequence begins as follows:

```
12, 20, 18, 28, 45, 24, 40, 63, 44, 36, 52, 56, 60, 99,
68, 175, 48, 76, 117, 50, 84, 88, 75, 92, 54, 80, 153,
104, 72, 275, 98, 171, 116, 90, 124, 147, 325, 132, 136,
207, 140, 96, 148, 135, 152, 539, 156, 100, 164, ...
```

Lemma 4.2. A360529$(n) <$ A024619$(n)^2$.

This is clear since we have eliminated prime powers from input.

Lemma 4.3. Squarefree composite A024619(n) implies tantus A360529(n) (i.e., A360529$(n) \in$ A126706).

Proof: Since $g(\varkappa) = p\varkappa$ for squarefree composite $\varkappa$, we have $p^2 \mid p\varkappa$ and hence a tantus number, that is, one that is neither squarefree nor a prime power. ■

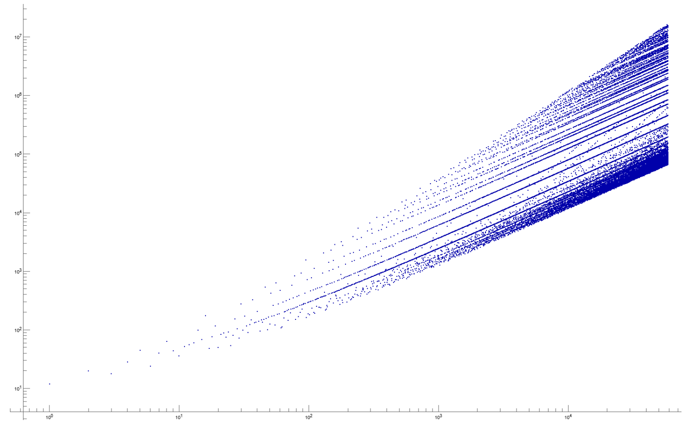

Figure 4: Log log scatterplot of A360529$(1 \ldots 2^{16})$.

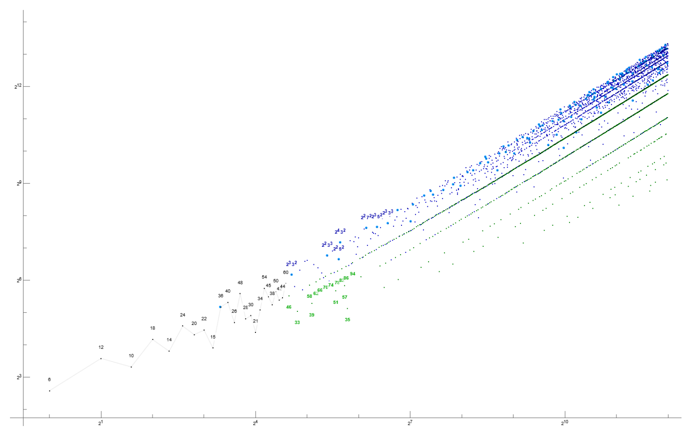

Figure 5: Log log scatterplot of A360719$(1 \ldots 2^{16})$, showing varius (squarefree composite) numbers in green and tantus (numbers neither squarefree nor prime powers) in blue. We highlight plenus numbers (tantus numbers that have multiplicity for all prime divisors) in light blue. The graph appears to feature striations according to LPF$(a(n))$ among varius numbers.

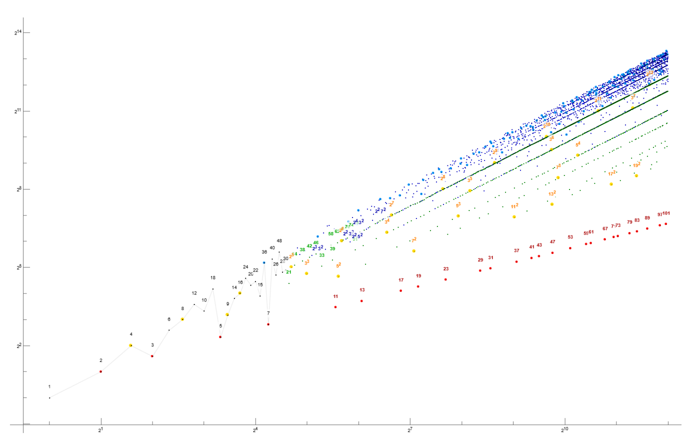

Figure 6: Log log scatterplot of A362041$(1 \ldots 2^{12})$ showing primes in red, multus numbers (composite prime powers) in gold, varius numbers (squarefree composites) in green, and tantus numbers (neither squarefree nor prime powers) in blue. We highlight those tantus numbers that have multiplicity for all prime divisors in light blue. There are striations among primes, varius, and multus numbers that have LPF$(a(n))$.

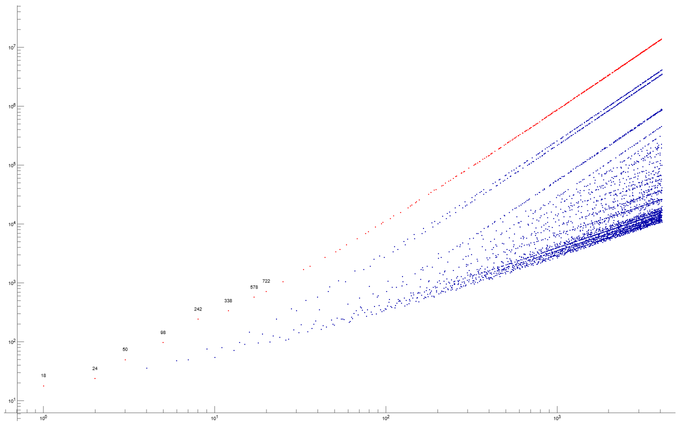

Figure 7: Log log scatterplot of A362432(1 ... 2^{12}), showing records in red.

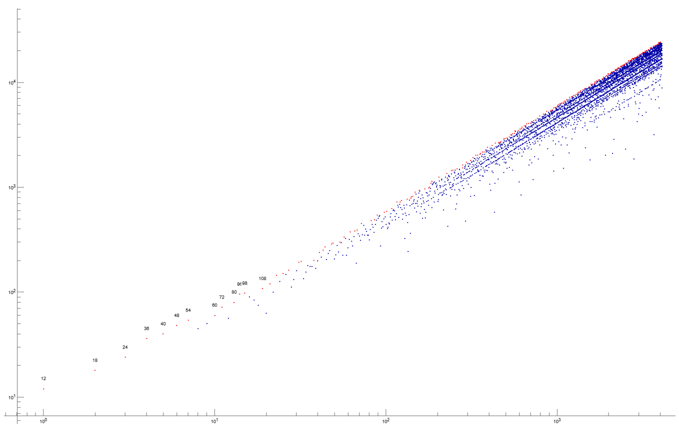

Figure 8: Log log scatterplot of A362844(1 ... 2^{12}), showing records in red.

Table A

```
A362432
m = A126706(n)
k = A362432(n)
r = rad(m) = rad(k)
```

n	m	k	k/m	r
1	12	18	3/2	6
2	18	24	4/3	6
3	20	50	5/2	10
4	24	36	3/2	6
5	28	98	7/2	14
6	36	48	4/3	6
7	40	50	5/4	10
8	44	242	11/2	22
9	45	75	5/3	15
10	48	54	9/8	6
11	50	80	8/5	10
12	52	338	13/2	26
13	54	72	4/3	6
14	56	98	7/4	14
15	60	90	3/2	30
16	63	147	7/3	21
17	68	578	17/2	34
18	72	96	4/3	6
19	75	135	9/5	15
20	76	722	19/2	38
21	80	100	5/4	10
22	84	126	3/2	42
23	88	242	11/4	22
24	90	120	4/3	30
25	92	1058	23/2	46
26	96	108	9/8	6
27	98	112	8/7	14
28	99	363	11/3	33
29	100	160	8/5	10
30	104	338	13/4	26

```
A362844
m = A360768(n)
k = A362844(n)
r = rad(m) = rad(k)
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n	m	k	k/m	r
1	18	12	2/3	6
2	24	18	3/4	6
3	36	24	2/3	6
4	48	36	3/4	6
5	50	40	4/5	10
6	54	48	8/9	6
7	72	54	3/4	6
8	75	45	3/5	15
9	80	50	5/8	10
10	90	60	2/3	30
11	96	72	3/4	6
12	98	56	4/7	14
13	100	80	4/5	10
14	108	96	8/9	6
15	112	98	7/8	14
16	120	90	3/4	30
17	126	84	2/3	42
18	135	75	5/9	15
19	144	108	3/4	6
20	147	63	3/7	21
21	150	120	4/5	30
22	160	100	5/8	10
23	162	144	8/9	6
24	168	126	3/4	42
25	180	150	5/6	30
26	189	147	7/9	21
27	192	162	27/32	6
28	196	112	4/7	14
29	198	132	2/3	66
30	200	160	4/5	10

THEOREM 4.4. A360529 is a permutation of A126706.

This follows from the transformation of $\varkappa\boldsymbol{R}_\varkappa(i) \to \varkappa\boldsymbol{R}_\varkappa(i+1)$ across the domain A024619 = A120944 ∪ A126706. Since the smallest (and first) element of $\varkappa\boldsymbol{R}_\varkappa$ is squarefree composite $\varkappa$ itself, it is replaced by a tantus number, and that tantus number is replaced by its successor, etc. until we have completely remapped A126706. ■

Define the successor function $f(n)$ to be that function which gives the next term after n in $\varkappa\boldsymbol{R}_\varkappa$. Given the structure of $\varkappa\boldsymbol{R}_\varkappa$ with varius $\varkappa$, the successor function $f(n)$ yields tantus numbers. Therefore, we see that $f \mapsto$ A024619 yields a permutation of A126706.

The $\varkappa$-Coregular Predecessor Function.

We may likewise modify the $\varkappa$-coregular successor function to give the $\varkappa$-coregular predecessor, for instance, via $g(n, -1)$ in a way analogous to $f(n, -1)$ in the last section.

If we attempt to map $g(n, -1) \mapsto$ A024619, we find that there are no predecessors for squarefree composite $\varkappa \in$ A120944. It is sufficient thus only to map to $g(n, -1) \mapsto$ A126706, and avoid the obvious transformation $g(n, p^\varepsilon) \to p^{(\varepsilon-1)}$ with $\varepsilon > 2$.

Define sequence A360719 to be the mapping $g(n, -1) \mapsto$ A126706, the sequence of tantus numbers (i.e., those that are neither squarefree nor prime powers). This sequence begins as follows:

```
6, 12, 10, 18, 14, 24, 20, 22, 15, 36, 40, 26, 48, 28,
30, 21, 34, 54, 45, 38, 50, 42, 44, 60, 46, 72, 56, 33,
80, 52, 96, 98, 58, 39, 90, 62, 84, 66, 75, 68, 70, 108,
63, 74, 120, 76, 51, 78, 100, 144, 82, 126, 57, 86, ...
```

It is clear that this is a permutation of A024619 via arguments similar to Theorem 4.4. We can generate this sequence via Code [C9].

We know that squarefree numbers (both prime and composite) have no predecessor in $\varkappa\boldsymbol{R}_\varkappa$. Therefore, we find the mapping $g(n, -1) \mapsto$ A013929 of interest.

Define sequence A362041 to be the mapping $g(n, -1) \mapsto$ A013929, the sequence of numbers that are not prime powers. This sequence begins as follows:

```
2, 4, 3, 6, 8, 12, 10, 18, 5, 9, 14, 16, 24, 20, 22, 15,
36, 7, 40, 26, 48, 28, 30, 21, 32, 34, 54, 45, 38, 50,
27, 42, 44, 60, 46, 72, 56, 33, 80, 52, 96, 98, 58, 39,
90, 11, 62, 25, 84, 64, 66, 75, 68, 70, 108, 63, ...
```

We note that $g(p^2, -1) \to p$ and $g(q\varkappa, -1) \to \varkappa$ where in latter case $q = \mathrm{LPF}(\varkappa)$. Given arguments similar to Theorem 4.4, we see that, were we to append A362041(0) = 1, we have a permutation of natural numbers.

The scatterplot of this sequence shown by Figure 6 merits further study. It features striations associated with LPF(A362041(n)). Code [C10] efficiently generates A362041.

The Nondivisor $\varkappa$-Coregular Successor Function.

Within $\varkappa\boldsymbol{R}_\varkappa$, we want to find distinct k and n such that $k > n$ and $\mathrm{RAD}(k) = \mathrm{RAD}(n) = \varkappa$, yet $n \nmid k$, a relationship we abbreviate $k \between n$ (or equivalently, $n \between k$).

In other words, k and n are coregular exclusive of divisibility.

We had called this relation "symmetric semidivisibility", having explored it in depth in January 2023 [3]. Several sequences and papers arose addressing the case. Chief among the sequences were A360768 (strong tantus numbers), A355432 (the symmetric semidivisor counting function), and A360589 (highly symmetrically semidivisible numbers).

In this section we turn to $k < n$ such that $k \between n$, first regarding k, the successor of n in $\varkappa\boldsymbol{R}_\varkappa$.

Therefore we define the function $F(\varkappa)$ = ⧅k such that both $k > n$ and k ¦ n.

Let sequence A362432 constitute the mappings $F \mapsto$ A126706, since it is clear that $p \mid k$ for $k \in p\boldsymbol{R}_p$, $p^\varepsilon \mid k$ for $k \in p^\varepsilon \boldsymbol{R}_{p^\varepsilon}$, and $\varkappa \mid k$ for $k \in \varkappa\boldsymbol{R}_\varkappa$. This sequence begins as follows:

```
18, 24, 50, 36, 98, 48, 50, 242, 75, 54, 80, 338, 72,
98, 90, 147, 578, 96, 135, 722, 100, 126, 242, 120,
1058, 108, 112, 363, 160, 338, 144, 196, 1682, 507, 150,
1922, 168, 198, 225, 578, 350, 162, 189, 2738, 180, ...
```

For example, the regular successor $f(20) = 25$ and the coregular successor $g(20) = 40$, but since $20 \mid 40$, $F(20) = 50$. This sequence contains repeated terms; $F(20) = F(40) = 50$. Therefore it is not a permutation, say, of A360768. Table A on page 4 demonstrates the ratio A362432(n)/ A126706(n). There is structure in scatterplot that merits exploration.

The Nondivisor $\varkappa$-Coregular Predecessor Function.

We define the function $G(\varkappa)$ = ⧅k such that both $k < n$ and k ¦ n.

Let sequence A362844 be the mappings $F \mapsto$ A360768 (the strong tantus numbers, see [3]). This sequence begins as follows:

```
12, 18, 24, 36, 40, 48, 54, 45, 50, 60, 72, 56, 80, 96,
98, 90, 84, 75, 108, 63, 120, 100, 144, 126, 150, 147,
162, 112, 132, 160, 192, 196, 135, 156, 180, 176, 175,
200, 168, 198, 240, 216, 252, 270, 204, 234, 250, ...
```

This sequence is not a permutation of A126706, since 20 is missing. Table A on page 4 demonstrates the ratio A362844(n)/ A360768(n)/. Like the related successor function, this function scatterplot merits exploration that is outside the cursory scope of this work.

Conclusion.

This paper introduced functions given a number n whose squarefree kernel is $\varkappa$, that find the predecessor of and successor to n in the sequences of n-regular numbers in $\boldsymbol{R}_\varkappa$ and n-coregular numbers in $\varkappa\boldsymbol{R}_\varkappa$. Some of these functions were already available in OEIS and others were recently added. Additionally, we posed a couple sequences that restricted the domain to numbers that are not prime powers or those that are not squarefree so as to eliminate the more easily understandable output. The sequence $\boldsymbol{R}_\varkappa$ is of interest because of its association with the *abc* conjecture, and as focus for the work of Størmer and others which relate to A2071 and A2072.

Definition 1.1 constructs the sequence of $\varkappa$-coregular numbers via multiplication of n-regular numbers $\boldsymbol{R}_\varkappa$ by $\varkappa$. As a consequence it is clear that $\varkappa\boldsymbol{R}_\varkappa$ has a squarefree minimum and first term $\varkappa$, succeeded by nonsquarefree numbers. If $\varkappa$ is prime p, then all the rest of the terms in $p\boldsymbol{R}_p$ are powers p^ε, $\varepsilon > 0$, and indeed, $p\boldsymbol{R}_p = \{p^\varepsilon : \varepsilon \ge 1\}$ via [1.3]. If squarefree $\varkappa$ is composite (hence in A120944), then succeeding terms are tantus numbers (i.e., in A126706).

This work yielded 4 handy results regarding predecessors and successors in regular and coregular sequences. Theorem 2.1 shows that A289280$(n) \le n^2$; we further show A065642$(n) \le n^2$ in Theorem 4.1 as consequence of Definition 1.1. Lemmas 3.1 and 3.2 show that $\omega(n) > 1$ implies A079277$(n) \nmid n$. Theorem 4.4 shows A360529 to be a permutation of natural numbers.

This work is part of a series on nondivisor coregular numbers, also known as symmetric semidivisors.

Appendix.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved May 2023.

[2] Donald J. Mintz, 2,3 sequence as a binary mixture, *Fibonacci Quarterly*, Vol. 19, №. 4, Oct 1981, 351-360.

[3] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

Concerns sequences:

A007947: Squarefree kernel of n; RAD(n).
A013929: Numbers that are not squarefree.
A024619: Numbers that are not prime powers.
A065642: $\varkappa$-regular successor function.
A079277: $\varkappa$-regular predecessor function.
A120944: “Varius” numbers; squarefree composites.
A126706: “Tantus” numbers neither prime power nor squarefree.
A162306: Truncation of $\boldsymbol{R}_\varkappa$: row $n = \{ k \in \boldsymbol{R}_\varkappa : k \le n \}$, RAD$(n) = \varkappa$.
A360529: $\varkappa$-coregular successor function $g \mapsto$ A024619.
A360719: $\varkappa$-coregular predecessor function $\breve{g} \mapsto$ A126706.
A362041: $\varkappa$-coregular predecessor function $\breve{g} \mapsto$ A013929.
A362432: Nondivisor $\varkappa$-coregular successor $f \mapsto$ A126706.
A362844: Nondivisor $\varkappa$-coregular predecessor $f \mapsto$ A360768.

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This work is dedicated to my wife Laura Ann on the occasion of her birthday (15 May).

Code:

[c1] Calculate $R_\varkappa$ bounded by an arbitrary limit m (i.e., calculate A275280(n); flatten and take union to provide A162306)

```
regularsExtended[n_, m_ : 0] :=
  Block[{w , lim = If[m <= 0, n, m]},
   Sort@ ToExpression@
     Function[w,
       StringJoin[
         "Block[{n = ", ToString@ lim,
         "}, Flatten@ Table[",
         StringJoin@
           Riffle[Map[ToString@ #1 <> "^" <>
             ToString@ #2 & @@ # &, w], " * "],
         ", ", Most@ Flatten@ Map[{#, ", "} &, #],
         "]]" ] &@
       MapIndexed[
         Function[p,
           StringJoin["{", ToString@ Last@ p,
             ", 0, Log[",
             ToString@ First@ p, ", n/(",
             ToString@
               InputForm[
                 Times @@ Map[Power @@ # &,
                   Take[w, First@ #2 - 1]]],
             ")]}" ] ]@ w[[First@ #2]] &, w]]@
       Map[{#, ToExpression["p" <>
         ToString@ PrimePi@ #]} &, #[[All, 1]] ] &@
       FactorInteger@ n];
```

[c2] Generate tantus numbers (A126706):

```
a126706 = Block[{k}, k = 0;
   Reap[Monitor[Do[
       If[And[#2 > 1, #1 != #2] & @@
         {PrimeOmega[n], PrimeNu[n]},
        Sow[n]; Set[k, n] ],
       {n, 2^21}], n]][[-1, -1]]] (* Tantus *);
```

[c3] Generate "strong tantus" numbers (A360768):

```
Select[a126706[[1 ;; 120]], #1/#2 >= #3 & @@
  {#1, Times @@ #2, #2[[2]]} & @@
  {#, FactorInteger[#][[All, 1]]} &]
```

[c4] Generate A013929 and A024619:

```
a013929 = Select[Range[2^20], Not@*SquareFreeQ];
a024619 = Select[Range[2^20], Not@*PrimePowerQ];
```

[c5] Generate A065642:

```
rad[x_] := rad[x] = Times @@ FactorInteger[x][[All, 1]];
Table[Function[r,
   SelectFirst[regularsExtended[n, n^2],
     And[# > n, rad[#] == r] &]][rad[n]], {n, 2, 2^14}]
```

[c6] Generate A289280:

```
rad[x_] := rad[x] = Times @@ FactorInteger[x][[All, 1]];
Table[Function[r,
   SelectFirst[regularsExtended[n, n^2],
     And[# > n, Divisible[r, rad[#]] ] &]][rad[n]],
     {n, 2, 2^14}]
```

[c7] Generate A079277:

```
rad[x_] := rad[x] = Times @@ FactorInteger[x][[All, 1]];
Table[Function[r,
   SelectFirst[Reverse@ Most@ regulars[n],
     Divisible[r, rad[#]] &]][rad[n]], {n, 2, 2^14}]
```

[c8] Generate A360529:

```
rad[x_] := rad[x] = Times @@ FactorInteger[x][[All, 1]];
Table[Function[r,
  SelectFirst[regularsExtended[n, n^2],
    And[# > n, rad[#] == r] &]][rad[n]],
    {n, a024619[[1 ;; 2^10]]}]
```

[c9] Generate A360719:

```
rad[x_] := rad[x] = Times @@ FactorInteger[x][[All, 1]];
Table[m = a126706[[i]];
   Function[r, SelectFirst[Reverse@ Most@ regulars[m],
     rad[#] == r &]][rad[m]], {i, 2^10}], {i, m}] ]
```

[c10] Generate A362041:

```
rad[x_] := rad[x] = Times @@ FactorInteger[x][[All, 1]];
{1}~Join~Table[m = a013929[[i]];
   Function[r, SelectFirst[Reverse@ Most@ regulars[m],
     rad[#] == r &]][rad[m]], {i, 2^10}], {i, m}]
```

[c11] Generate A362432:

```
rad[x_] := rad[x] = Times @@ FactorInteger[x][[All, 1]];
Table[Function[r,
   SelectFirst[regularsExtended[n, n^2],
    And[# > n, rad[#] == r, ! Divisible[#, n]] &]]
    [rad[n]], {n, a126706[[1 ;; 2^10]]}]
```

[c12] Generate A362844:

```
rad[x_] := rad[x] = Times @@ FactorInteger[x][[All, 1]];
Table[Function[r,
   SelectFirst[Reverse@ Most@ regulars[n],
    And[r == rad[#], ! Divisible[n, #]] &]][rad[n]],
    {n, a360768[[1 ;; 2^10]]}]
```