On A113901 = $\Omega(n)\omega(n)$ and A363920 = $n^{\text{A113901}(n)}$.

Sequences A113901 by Hilliard and A363920 by Luschny.

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Abstract.

We examine sequences related to the product of the elementary number-theoretical counting functions $\omega(n)$ and $\Omega(n)$. We show that A113901$(n) < n$. Moreover, we see that A363920(n) is a sequence bereft of squarefree composites and numbers in A332785, comprised of fixed points A8578, and composite $n \to$ squareful $m \in$ A1694.

Introduction.

Define the following functions:

$$\begin{aligned}\omega(n) &= \text{A1221}(n)\\ &= \left|\,\{\, p : p \mid n \,\}\,\right| \text{ where } p \text{ is prime.}\end{aligned} \qquad [1.1]$$

$$\begin{aligned}\Omega(n) &= \text{A1222}(n)\\ &= \sum_{p^\varepsilon \mid n} \varepsilon, \text{ where } \varepsilon \text{ is maximal such that } p^\varepsilon \mid n.\end{aligned} \qquad [1.2]$$

A prime power is a number $n = p^\varepsilon$, $\varepsilon \geq 0$, hence the sequence of prime powers is A961 = $\{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, \dots\}$.

The number 1 represents the product of no primes at all; it is the empty product, hence $\omega(n) = \Omega(n) = 0$. Therefore we are interested in the following set of prime powers:

$$\begin{aligned}\text{A246655} &= \text{A961} \setminus \{1\}\\ &= \{\, p^\varepsilon : \varepsilon \geq 1 \,\}.\end{aligned} \qquad [1.4]$$

Hereinafter we then mean $n \in$ A246655 when we write "prime power". Using this definition, it follows that prime power p^ε, $\varepsilon \geq 1$, implies $\omega(p^\varepsilon) = 1$ and $\Omega(n) = \varepsilon$.

A squarefree number n is a product of k distinct primes. It is clear that squarefree n implies $\omega(n) = \Omega(n) = k \geq 0$. The sequence of squarefree numbers is A5117 = $\{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, \dots\}$.

Sequence A113901.

Define function $f(x)$ to be as follows:

$$f(x) = \omega(x) \times \Omega(x). \qquad [2.0]$$

Then we generate the sequence A113901 = $f \mapsto \mathbb{N}$, which begins as shown below:

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0, 1, 1, 2, 1, 4, 1, 3, 2, 4, 1, 6, 1, 4, 4, 4, 1, 6, 1,
6, 4, 4, 1, 8, 2, 4, 3, 6, 1, 9, 1, 5, 4, 4, 4, 8, 1, 4,
4, 8, 1, 9, 1, 6, 6, 4, 1, 10, 2, 6, 4, 6, 1, 8, 4, 8,
4, 4, 1, 12, 1, 4, 6, 6, 4, 9, 1, 6, 4, 9, 1, 10, ...
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Lemma 2.1: Prime power p^ε, $\varepsilon \geq 0$ implies A113901$(p^\varepsilon) = \varepsilon$.

Proof:
$$\begin{aligned}f(p^\varepsilon) &= \omega(p^\varepsilon) \times \Omega(p^\varepsilon)\\ &= 1 \times \varepsilon\\ &= \varepsilon. \ \blacksquare\end{aligned}$$

Corollary 2.2: $f(1) = 0$, since $1 \times \varepsilon = 1 \times 0 = 0$.

Corollary 2.3: Prime p implies $f(p) = 1$.

Corollary 2.4: Prime square p^2 implies $f(p^2) = 2$.

Lemma 2.5: Squarefree n, product of k distinct primes p, implies A113901$(n) = k^2$.

Proof:
$$\begin{aligned}f(n) &= \omega(n) \times \Omega(n) : \Omega(n) = \omega(n) = k\\ &= k \times k\\ &= k^2. \ \blacksquare\end{aligned}$$

Corollary 2.6: Squarefree semiprime $n \in$ A6881 implies $f(n) = 4$.

Lemma 2.7: Number $n \in$ A126706, implies A113901$(n) > k^2$.

Proof: Let $k = \omega(n)$ and let $j = \Omega(n) - \omega(n) \geq 1$.
$$\begin{aligned}f(n) &= \omega(n) \times \Omega(n)\\ &= k\,(j + k)\end{aligned}$$
Since $j > 1$, $k\,(j + k) > k^2$. ■

Theorem 2.8: $f(n) < n$.

Proof: Natural numbers $n \in \mathbb{N}$ are either squarefree or not squarefree; they are either prime powers or not prime powers. Therefore Lemmas 2.1 through 2.7 cover all natural numbers.
Lemma 2.1 shows $p^\varepsilon \to \varepsilon$, and since prime $p > 1$, $p^\varepsilon > \varepsilon$.
Lemma 2.5 shows $n \to \omega(n)^2$ for squarefree n, and it is clear that such $n > \omega(n)^2$. Let $\mathcal{P}(k)$ = A2110(k) be the product of k smallest primes; it follows that $\mathcal{P}(k) > k^2$ for $k > 0$.
Lemma 2.7 shows $f(n) > \omega(n)^2$, with $\omega(n)^2 = k^2$ for numbers neither squarefree nor prime powers, that is, $n \in$ A126706. Knowing that $\mathcal{P}(k) > k^2$, and that we can produce a minimal $n \in$ A126706 via $2{\times}\mathcal{P}(k)$ with $f(2{\times}\mathcal{P}(k)) = k(k{+}1)$; $2{\times}\mathcal{P}(k) > k(k{+}1)$, $k > 1$. It becomes clear that we are dealing with exponentiation on the left hand side versus multiplication on the right, and we always have $n > f(n)$ for $n \in$ A126706. ■

Conclusions involving A113901:

- $1 \to 0$.
- Prime $p \to 1$ and generally $p^\varepsilon \to \varepsilon$.
- $n \to \omega(n)^2$ for squarefree n.
- $n \to k\,(j + k)$ for $n \in$ A126706, $k = \omega(n)$ and $j = \Omega(n) - \omega(n) \geq 1$.
- $n > f(n)$ for $n \in \mathbb{N}$.

Sequence A363920.

Define A363920 = a to be the following:

$$\begin{aligned}a(n) &= n^{\Omega(n)\omega(n)}\\ &= n^{\text{A113901}(n)}.\end{aligned} \qquad [3.0]$$

The first terms of this sequence appear below:

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1, 2, 3, 16, 5, 1296, 7, 512, 81, 10000, 11, 2985984,
13, 38416, 50625, 65536, 17, 34012224, 19, 64000000,
194481, 234256, 23, 110075314176, 625, 456976, 19683,
481890304, 29, 19683000000000, 31, 33554432, ...
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(In this work we do not address the fact Peter Luschny defines $a(0) = 1$, hence the sequence offset is 0.)

Lemma 3.1: Prime power p^ε, $\varepsilon \geq 0$ implies A363920$(p^\varepsilon) = p^{\varepsilon^2}$.

Proof:
$$\begin{aligned}a(p^\varepsilon) &= p^{\omega(p^\varepsilon)\Omega(p^\varepsilon)}\\ &= (p^\varepsilon)^{(1\times\varepsilon)}\\ &= p^{\varepsilon^2}. \ \blacksquare\end{aligned}$$

Theorem 3.2: Number $n \in$ A8578 (i.e., $n = 1$ or $n = p$ prime) implies fixed point $a(n) = n$.

Proof: $a(1) = 1$ since $\Omega(n) = \omega(n) = 0$. Hence $a(1) = 1^0 = 1$
For prime p, $a(p) = p$ since $\Omega(p) = \omega(p) = 1$. Hence $a(p) = p^1 = p$.
Therefore the proposition is true. ■

Corollary 3.3: Prime square p^2 implies $a(p^2) = p^4$.

Lemma 3.4: Squarefree n, a product of k distinct primes p, implies $a(n) = n^{(k^2)}$.

Proof:
$$\begin{aligned} a(n) &= n^{\omega(n)\Omega(n)} : \Omega(n) = \omega(n) = k > 1 \\ &= n^{(k \times k)} \\ &= n^{k^2}. \blacksquare \end{aligned}$$

Corollary 3.5: Squarefree semiprime $n \in$ A6881 implies $a(n) = n^4$.

Corollary 3.6: Squarefree composite $n \in$ A120944 implies A363920(n) $\in$ A303606. We note A303606 $\subset$ A286708 $\subset$ A126706. Remark:

$$\text{A1694} = \{1\} \cup \text{A246547} \cup \text{A286708}. \qquad [3.6]$$

Since $j > 1$, $n^{k\,(j+k)} > n^{k^2}$. ■

Lemma 3.7: Number $n \in$ A126706, implies A363920(n) $> n^{k^2}$.

Proof: Let $k = \omega(n)$ and let $j = \Omega(n) - \omega(n) \geq 1$.
$$\begin{aligned} a(n) &= n^{\omega(n)\Omega(n)} \\ &= n^{k\,(j+k)} \end{aligned}$$

Since $j > 1$, $n^{k\,(j+k)} > n^{k^2}$. ■

Corollary 3.8: Number $n \in$ A126706 implies $a(n) \in$ A286708. We note A286708 $\subset$ A126706.

Theorem 3.9: For $n \in$ A2808, $a(n) > n$.

Proof: Natural numbers $n \in \mathbb{N}$ are either squarefree or not squarefree; they are either prime powers or not prime powers. Therefore Lemmas 3.1 through 3.8 cover all natural numbers. Theorem 3.2 shows that primes and $n = 1$ are fixed points, while Lemmas 3.4 and 3.7 show that $a(n) > n$ for $n \in$ A024619. Through Lemma 3.1, regarding $n = p^\varepsilon$ and setting $\varepsilon > 1$ to obtain $n \in$ A246547 results in $a(n) > n$. Since A024619 $\cup$ A246547 $=$ A2808, we verify the proposition. ■

Conclusions involving $a =$ A363920:

- $n \rightarrow n$ for $n \in$ A8578, that is, $n = 1$ or $n = p$ prime.
- $p^\varepsilon \rightarrow p^{\varepsilon^2}$, hence $n \in$ A246547 $\rightarrow m \in$ A246547, $m > n$.
- $n \rightarrow n^{k^2}$ for squarefree n.
- $n \in$ A120944 $\rightarrow m \in$ A303606; A303606 $\subset$ A286708 $\subset$ A126706.
- $pq \rightarrow (pq)^4$, primes $p < q$.
- $n \rightarrow n^{k\,(j+k)}$ for $n \in$ A126706, $k = \omega(n)$ and $j = \Omega(n) - \omega(n) \geq 1$.
- $n \in$ A126706 $\rightarrow m \in$ A286708; A286708 $\subset$ A126706.
- $a(n) > n$ for composite n; $n \rightarrow m \in$ A1694, with $m > n$.

Essentially, $n = 1$ or $n = p$ prime constitute fixed points, while composite prime power $n \in$ A246547 transforms into a number $m \in$ A246547 and $n \in$ A024619 transforms into a number $m \in$ A286708, $m > n$ in both latter cases. Therefore, A363920 is a sequence containing 1, the primes, and squareful numbers in A1694. The sequence lacks numbers in A120944 and A332785 = A126706 \ A286708.

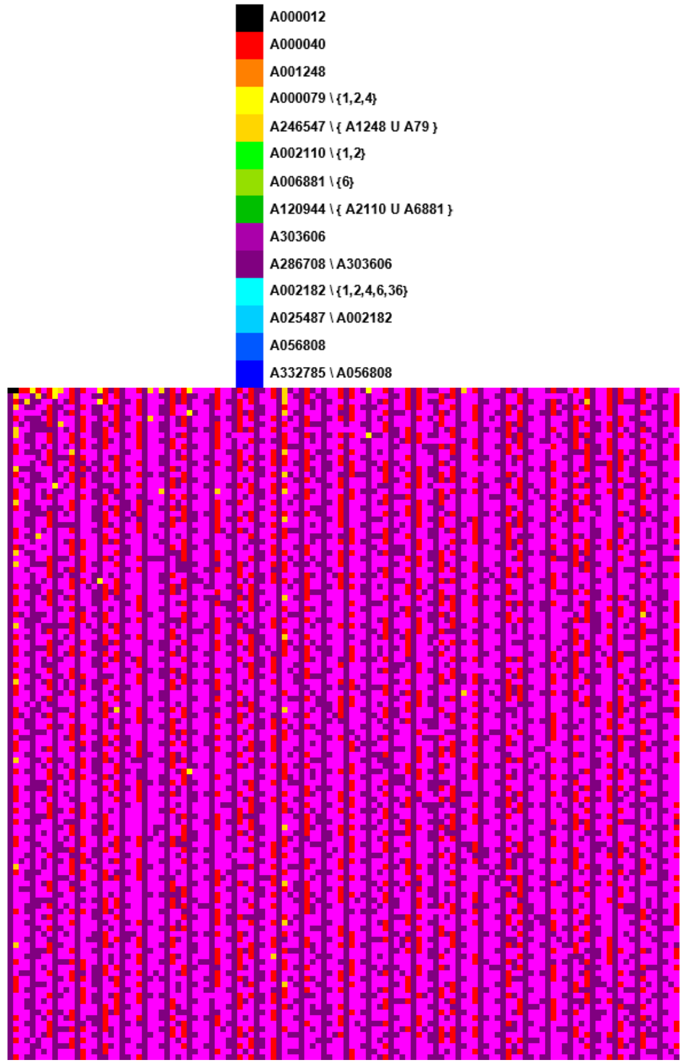

Figure 1: Plot the class of a(120m+k) at (x, y) = (k, m+1), where the class of a number is coded according to colors in the key above. This plot demonstrates that 1 and prime constitute fixed points, while squarefree composites transform to a number in A303606, and numbers neither squarefree nor composite transform to a number in A286708.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved July 2023.

Concerns sequences:

A000040: Prime numbers.
A001221: Number of distinct prime divisors of n, $\omega(n)$.
A001222: Number of prime divisors of n with multiplicity, $\Omega(n)$.
A001694: Squareful numbers, $\{a^2b^3 : a \geq 1, b \geq 1\}$.
A005117: Squarefree numbers.
A013929: Numbers that are not squarefree.
A024619: Numbers that are not prime powers.
A113901: $\omega(n) \times \Omega(n)$.
A120944: Squarefree composites.
A126706: Numbers neither prime power nor squarefree.
A246547: Composite prime powers $p^\varepsilon : \varepsilon \geq 1$.
A286708: Products of A246547.
A332785: A126706 \ A286708.
A363920: $n^{\Omega(n)\omega(n)} = n^{\text{A113901}(n)}$.

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