OEIS A367188.

A sequence of David James Sycamore · Windhoek, Namibia · 10 November 2023

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Abstract.

We examine a simple conditional sequence A367188 based on the difference of previous term and partial sums of primes. The sequence has a novelty condition that guarantees distinct terms.

Introduction.

Let us study a self-referential sequence A367188 which has the name "$a(1) = 1$, thereafter $a(n) = a(n-1) - \text{A7054}(n-1)$ if positive and novel, else $a(n) = a(n-1) + \text{A7054}(n-1)$."

The sequence begins as follows:

```
1, 3, 8, 18, 35, 7, 48, 106, 29, 129, 258, 98, 295, 57,
338, 10, 391, 831, 330, 898, 259, 971, 180, 1054, 91,
1151, 2312, 1048, 2419, 939, 2532, 812, 2663, 675, 2802,
526, 2953, 369, 3116, 202, 3289, 23, 3470, 7108, 3277,
7305, 3078, 7516, 2855, 7743, 2626, 7976, ...
```

We describe the first terms below:

$a(2) = 3 = 1 + 2$, since $1 - 2$ is not positive.
$a(3) = 8 = 3 + (2 + 3)$, since $3 - 5$ is not positive.
$a(4) = 18 = 8 + (2 + 3 + 5)$, since $8 - 10$ is not positive.
$a(5) = 35 = 13 + (2 + 3 + 5 + 7)$ since though positive,
$13 - (2 + 3 + 5 + 7) = 1 = a(1)$.
$a(6) = 7 = 35 - (2 + 3 + 5 + 7 + 11)$.
$a(7) = 48 = 7 + (2 + 3 + 5 + 7 + 11 + 13)$, etc.

In terms of algorithm, consider the following approach.
Set $\text{A367188}(1) = a(1) = 1$.
Define 2 functions to be as follows:

$$f(n) = a(n-1) - \text{A7054}(n-1) \text{ and} \quad [1.1]$$
$$g(n) = a(n-1) + \text{A7054}(n-1), \quad [1.2]$$

where A7054 is the sequence of partial sums of the primes.

Hence we may set sum $s = 2$ and at each step in A367188, we increment s by $p = \text{PRIME}(n)$. Therefore we may alternately define $f(n)$ and $g(n)$ as follows:

$$f(n) = a(n-1) - s, \quad [1.3]$$
$$g(n) = a(n-1) + s$$
$$\text{thereafter, } s \pm \text{PRIME}(n).$$

Then both nonzero positive $f(n)$ and $a(m) \ne f(n)$, $m < n$ imply $a(n) = f(n)$ else $a(n) = g(n)$.

We offer the following lemmas evident from sequence definition:

Lemma 1.1. $a(n-1) > s$ implies $a(n) = f(n) = a(n-1) - s$, unless $a(m) = f(n)$, $m < n$.

Lemma 1.2. $a(n-1) \le s$ implies $a(n) = g(n) = a(n-1) + s$.

The lemmas indeed are merely a way of rewriting definition. It follows from these lemmas that A7054 is key to understanding behavior of the sequence, hence we study A367188 in the context of A7054.

Lemma 1.3. The usual mode of the sequence is alternating application of $f(n)$ and $g(n)$.

Proof. Let us ignore the novelty constraint. Let $j = a(n-1)$ and let $k = a(n)$. Begin with $j \le s$, which implies $k = g(n) = j + s$. Set $j = k$. Now $j > s$ since $s = 2$ to start, and is a partial sum of primes; this implies $k = f(n) = j - s$. Setting $j = k$, we find ourselves at the start.

As n increases, p increases, and s increases by p. This implies that both $f(n)$ and $g(n)$ decrease each time they apply. Since j decreases each time $f(n)$ applies, yet p and thus s increase, $g(n)$ increases faster each application, which induces $f(n)$ to decrease toward 0 at each application.

Eventually, $\text{A7054}(n-1) < g(n) \le \text{A7054}(n)$, and $f(n) < 0$, for some transition point n, necessitating a repeated application of $g(n)$. Hence, we have the following relation:

$$\text{for } n \text{ such that } \text{A7054}(n-1) < g(n) \le \text{A7054}(n),$$
$$\text{we have } a(n-1) < a(n) < a(n+1).$$

Now we consider the forbidden condition $a(m) = f(n)$, $m < n$. This forces $k = g(n)$, increasing j in the next turn and $k = f(n)$. Suppose j repeatedly increases via novelty condition until it sets a record. This is tantamount to saying that j is unprecedented in the sequence, meaning that at some point, we return to the usual alternating application of functions. ■

Corollary 1.4. Parity of n where $a(n)$ is assigned a given function is preserved until $\text{A7054}(n-1) < g(n) \le \text{A7054}(n)$ or $a(m) = f(n)$, $m < n$.

Corollary 1.5. Suppress the novelty condition. Let $\mathcal{A}$ represent parity of n for $a(n) = g(n)$, let $\mathcal{B}$ represent parity of n for $f(n)$, and let $t(k)$ represent the k-th transition index such that both of the following are true:

$$\text{A7054}(t(k)-1) < g(t(k)) \le \text{A7054}(t(k)) \text{ and} \quad [1.4]$$
$$a(t(k)-1) < a(t(k)) < a(t(k)+1).$$

We may use $t(k)$ to delimit phases $\Phi(k) = a(t(k) \dots t(k-1))$ wherein $g(n)$ increases as n increases, and $f(n)$ decreases as n increases. Then phase $\Phi(k+m)$, m even, has $a(\mathcal{A}) = g(n)$ and $a(\mathcal{B}) = f(n)$, while for m odd, we have instead $a(\mathcal{B}) = g(n)$ and $a(\mathcal{A}) = f(n)$.

Corollary 1.6. Outside of $g(n)$ resulting from $a(m) = f(n)$, $m < n$, $a(n) < 2s$, i.e., $a(n) \le 2 \times \text{A7054}(n-1)$. Consequence of Lemma 1.2.

Effect of the Novelty Constraint.

We note that we cannot say that the novelty condition will not recur in runs that exceed 1, but we know that $g(n)$ implies that the run is finite, since j eventually sets a record. If the condition is repeated, then we will see index parity effects and singleton phases in the sequence akin to those that start the sequence. Suppose we have a singleton phase $\Phi(k)$ beginning with $a(t(k))$ such that there have already been phases $\Phi(m)$, $m < k$, with lengths $\ell > 1$. It is clear that we preserve Lemma 1.3 and Corollaries 1.4 and 1.5.

However, we cannot say that $\text{A367188}(n) \le 2 \times \text{A7054}(n-1)$ per Corollary 1.6.

We might define a sequence S20231112 unconstrained by novelty. This sequence begins as follows:

```
1, 3, 8, 18, 1, 29, 70, 12, 89, 189, 60, 220, 23, 261,
542, 214, 595, 155, 656, 88, 727, 15, 806, 1680, 717,
1777, 616, 1880, 509, 1989, 396, 2116, 265, 2253, 126,
2402, 4829, 2245, 4992, 2078, 5165, 1899, 5346, 1708,
5539, 1511, 5738, 1300, 5961, 1073, 6190, 840, ...
```

The above sequence does not suffer defects (late singleton phases) introduced by the novelty condition, hence, it strictly exhibits Lemma 1.3 and Corollary 1.6. Terms in S20231112 are not distinct.

Regarding A367188, examining the occasion of $a(m) = f(n)$, in 2^{28} terms, we only have 2 collisions, namely $f(5)$ and $f(1407)$. Since this constraint minimally affects the sequence (but indeed produces a completely different sequence from the novelty-unrestrained sequence S20231112), we remark that generally, we have nonzero positive $f(n)$ implies $a(n) = f(n)$ else $a(n) = g(n)$. The principal effect of the novelty constraint in A367188 is that the sequence might be a permutation of natural numbers whereas S20231112 is definitely not such.

Scatterplot Features.

We show a log log scatterplot in Figure 1.

Consider transition points $t(k)$ that initialize phases $\phi(k)$ described by Corollary 1.5. We have $a(\mathcal{A}) = g(n)$ and $a(\mathcal{B}) = f(n)$, and $g(n) - s$ greatest for $n = t(k)$, and $s - f(n)$ greatest for $n = t(k+1) - 1$. We see that the trajectory of $g(n)$ in $\phi(k)$ increases until $a(t(k+1))$, and the trajectory of $f(n)$ in $\phi(k+1)$ appears to join that of $g(n)$ in $\phi(k)$ at $a(t(k+1) + 1)$ to fall toward 0 as n increases.

Corollary 1.5 shows that the index parity of $g(n)$ in $\phi(k)$ becomes that of $f(n)$ in $\phi(k+1)$. Therefore, the scatterplot shows conjoined trajectories $T(k)$ of $g(n)$ in $\phi(k)$ and $f(n)$ in $\phi(k+1)$ where n is of same parity. Hence, $T(k)$ consists of $a(\mathcal{A})$, while $a(\mathcal{B})$ comprises $T(k+1)$ under circumstances apart from singleton phase defects imparted by more than 1 instance of failure of $f(n)$ via novel condition.

It is clear that records consist of applications of $g(n)$.

Ancillary Sequences.

Define sequence S20231113 to consist of transition points $t(k)$ such that the condition shown by [1.4] subtends. The sequence S20231113 begins as follows:

```
2, 3, 4, 7, 10, 17, 26, 43, 70, 113, 186, 307, 508,
845, 1406, 2351, 3938, 6611, 11116, 18725, 31588,
53359, 90242, 152795, 258968, 439321, 745918, 1267489,
2155324, 3667519, 6244612, 10638815, 18134970, 30928729,
52773480, 90087511, 153849784, 262846613, ...
```

Define the following sequences to be as follows:

S20231110$(n) = $ A367188$(t(n)-1)$,

S20231111$(n) = $ A367188$(t(n))$,

S20231114$(n) = $ A367188$(t(n)+1)$, and

S20231109$(n) = $ A367188(n) – A7054(n).

First terms of sequence S20231110 :

```
1, 3, 8, 7, 29, 10, 91, 23, 63, 554, 637, 991, 2080,
1402, 18691, 6655, 22459, 23315, 96790, 43582, 354902,
34173, 549825, 773667, 735730, 6151042, 3007393,
7565982, 14382970, 3396373, 51439184, 57538257, ...
```

First terms of sequence S20231111 :

```
3, 8, 18, 48, 129, 391, 1151, 3470, 10601, 31671,
94693, 284072, 851990, 2557589, 7661707, 22976022,
68927905, 206789230, 620339638, 1861043557,
5583022406, 16749192448, 50247902141, ...
```

First terms of sequence S20231114 :

```
8, 18, 35, 106, 258, 831, 2312, 7108, 21488, 63405,
189858, 569180, 1705531, 5120323, 15316440, 45966288,
137870522, 413621416, 1240700295, 3722252621,
11166060793, 33499007956, 100496417246, ...
```

Finally, S20231109 begins as follows:

```
-1, -2, -2, 1, 7, -34, -10, 29, -71, 0, 98, -99, 57,
-224, 10, -371, -49, 330, -238, 259, -453, 180, -694,
91, -969, -10, 1048, -323, 939, -654, 812, -1039, 675,
-1452, 526, -1901, 369, -2378, 202, -2885, 23, ...
```

Open questions:

1. Are there any further cases of $f(n) = a(m)$, $m < n$?
2. Are there any singleton phases $\phi(k)$, $k > 2$? This is the same as asking whether there are any $n \in \{m, m+1\}$ such that $f(n) = a(m)$, $m < n$?

Conclusion.

The sequence A367188 can be explained largely through its relationship with the sequence of partial sums of the primes, A7054.

We have shown that A367188 exhibits alternating applications of $f(n) = a(n-1)$ – A7054$(n-1)$ and $g(n) = a(n-1)$ + A7054$(n-1)$ until we have reach a transition point $n = t(k)$. This transition point is such that the following is true:

$$\text{A7054}(n-1) < a(n) \le \text{A7054}(n).$$

The sequence can be partitioned into phases as follows:

$$\phi(k) = \mathrm{a}(t(k) \ldots\ t(k+1) - 1).$$

Within these phases, the parity of n is preserved across applications of a given function, $f(n)$ or $g(n)$, as these functions alternate. We have explained the appearance of scatterplot, and examined the scarce occasion of $f(n) = a(m)$, $m < n$. Lastly, we have defined 6 ancillary sequences that explore A367188 unconstrained to novelty, transition point indices and values, first and last terms of phases, and the difference A367188(n) – A7054(n).

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved February 2023.

Code:

[C1] Efficiently calculate $a(n)$:

```
nn = 2^20; c[_] := False; a[1] = j = 1;
c[1] = True; s = 2;
  Do[If[Or[# < 1, c[#]],
    Set[k, j + s],
    Set[k, #]] &[j - s];
    s += Prime[n];
    Set[{a[n], j, c[k]}, {k, k, True}], {n, 2, nn}];
 Array[a, nn]
```

[C2] Efficiently calculate S20231113:

```
nn = 2^20; c[_] := False; i = 1; j = 1;
c[1] = True; s = 2;
  Reap[Do[If[Or[# < 1, c[#]],
    Set[k, j + s],
    Set[k, #]] &[j - s];
    s += Prime[n]; If[i < j < k, Sow[n - 1]];
    Set[{i, j, c[k]}, {j, k, True}], {n, 2, nn}]
  ][[-1, 1]]
```

Concerns sequences:

A000040: Prime numbers.

A007054: Partial sums of A40.

A367188: $a(n)$.

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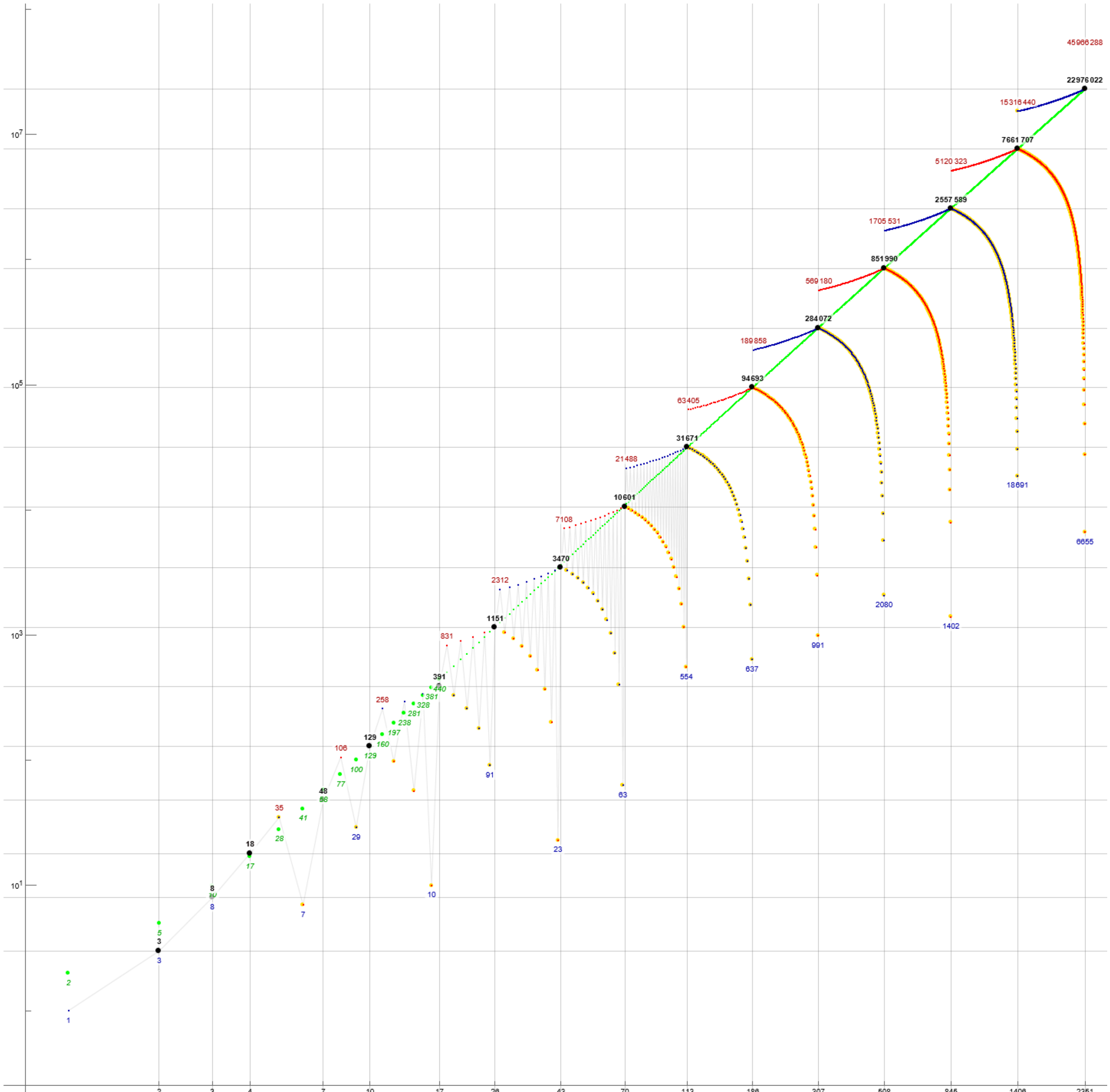

Figure 1: Log log scatterplot of A367188*(n) (red even n and blue for odd n) and* A7054*(n) (green) for n = 1 … 2351. We show a(t(k)) in black, and label a(t(k)+1) in red, local minima a(t(k)−1) in dark blue. We accentuate positive a(n−1)−*A7054*(n−1) in gold.*