Minimally Tantus Numbers.

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Abstract.

This work explores successors to squarefree numbers k in the list of numbers n such that $\text{RAD}(n) = k$, specifically, when k is not a prime power. We place these numbers in the context of other "tantus" numbers, i.e., those that are neither prime powers nor squarefree.

Introduction.

Let $\Omega(n) = \text{A1220}(n)$ be the number of prime factors of n with multiplicity and let $\omega(n) = \text{A1221}(n)$ be the number of distinct prime factors of n. Consider numbers $k \in \text{A024619}$, where A024619 is defined to be as follows:

$$\begin{aligned}\text{A024619} &= \{\, k : \Omega(k) \ge \omega(k) > 1 \,\}\\ &= \text{A120944} \cup \text{A126706}\\ &= \{\, k : \Omega(k) = \omega(k) > 1 \,\} \cup \{\, k : \Omega(k) > \omega(k) > 1 \,\} \quad [0.1]\end{aligned}$$

We define the following aspects of these numbers to be as follows:

Define tantus to be a number k such that $\Omega(k) > \omega(k) > 1$, i.e., $k \in$ A126706, a composite neither squarefree nor prime power.

Define varius to be a number k such that $\Omega(k) = \omega(k) > 1$, i.e., $k \in$ A120944, a squarefree composite.

Let p_1 be the smallest prime factor of k, i.e., $\text{LPF}(k) = \text{A020639}(k)$.

Let $\varkappa = \text{RAD}(k) = \text{A7947}(k)$, product of distinct prime factors of k.

Now we define the sequence $\boldsymbol{R}_\varkappa = \{\, k : \text{RAD}(k) \mid \varkappa \,\}$ to include any k that is a product restricted to primes that divide $\varkappa$. It is clear that 1, the empty product, is in $\boldsymbol{R}_\varkappa$, since 1 is the product of the null set of prime factors of $\varkappa$. If $\varkappa = 1$, then $\boldsymbol{R}_1 = \{1\}$, finite, while $\varkappa > 1$ implies countably infinite $\boldsymbol{R}_\varkappa$. The following formula generates $\boldsymbol{R}_\varkappa$:

$$\boldsymbol{R}_\varkappa = \bigotimes_{p \mid \varkappa} \{\, p^\varepsilon : \varepsilon \ge 0 \,\} \quad [0.2]$$

We call $\boldsymbol{R}_\varkappa$ the set of $\varkappa$-regular numbers [2, 3], while the set $\varkappa\boldsymbol{R}_\varkappa$ is the set of $\varkappa$-coregular numbers.

Examples:

$$\begin{aligned}\boldsymbol{R}_6 &= \bigotimes_{p \mid 6} \{p^\varepsilon : \varepsilon \ge 0\}\\ &= \{2^\delta : \delta \ge 0\} \otimes \{3^\varepsilon : \varepsilon \ge 0\}\\ &= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, \dots\}\\ &= \text{A3586}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{R}_{10} &= \bigotimes_{p \mid 10} \{p^\varepsilon : \varepsilon \ge 0\}\\ &= \{2^\delta : \delta \ge 0\} \otimes \{5^\varepsilon : \varepsilon \ge 0\}\\ &= \{1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, \dots\}\\ &= \text{A3592}.\end{aligned}$$

Then $\varkappa$-coregular numbers $r \in \varkappa\boldsymbol{R}_\varkappa$ are such that $\text{RAD}(r) = \varkappa$.

$$\begin{aligned}6\boldsymbol{R}_6 &= \bigotimes_{p \mid 6} \{p^\varepsilon : \varepsilon \ge 0\} \times 6\\ &= \{2^\delta : \delta \ge 0\} \otimes \{3^\varepsilon : \varepsilon \ge 0\} \times 6\\ &= \{6, 12, 18, 24, 36, 48, 54, 72, 96, 108, \dots\}\\ &= 6 \times \{\text{A3586}\}.\end{aligned}$$

It is evident that multiplication of $\boldsymbol{R}_\varkappa$ by $\varkappa$ guarantees $\varkappa \mid r$ for all r.

Minimally Tantus Numbers.

In [5], we studied the $\varkappa$-coregular successor function. Given a number k, we find the smallest $n > k$ such that $\text{RAD}(n) = \text{RAD}(k) = \varkappa$. This function appears in OEIS as Zumkeller's A065642.

Lemma 1.5 in [5] shows that, for squarefree composite $\varkappa$, $\varkappa\boldsymbol{R}_\varkappa$ consists of the minimum $\varkappa$ itself as sole varius number, while the rest of $\varkappa\boldsymbol{R}_\varkappa$ is tantus.

Hence here we speak of a "minimally tantus" number, that is, the smallest tantus element of $\varkappa\boldsymbol{R}_\varkappa$ achieved by the mappings:

$$\text{A065642} \mapsto \text{A120944} \quad [1.0]$$

We define the sequence A366807 to be as follows:

$$\begin{aligned}\text{A366807}(n) &= \text{A065642}(\text{A120944}(n))\\ &= \text{LPF}(\text{A120944}(n)) \times \text{A120944}(n). \quad [1.1]\end{aligned}$$

The first terms of this sequence are the following:

12, 20, 28, 45, 63, 44, 52, 60, 99, 68, 175, 76, 117, 84, 92, 153, 275, 171, 116, 124, 325, 132, 207, 140, 148, 539, 156, 164, 425, 172, 261, 637, 279, 188, 475, 204, 315, 212, 220, 333, 228, 575, 236, 833, 244, ...

Lemma 1.1. $\text{A366807}(n) = \text{LPF}(\text{A120944}(n)) \times \text{A120944}(n)$.

Proof. The smallest prime factor of squarefree composite $\varkappa$ is p_1 and it is clear that follows 1 in $\boldsymbol{R}_\varkappa$, because no number comes between 1 and p_1 in the list of divisors of $\varkappa$. Since the $\varkappa$-coregular successor of $\varkappa$ itself is by definition the number that follows $1 \times \varkappa$ in $\varkappa\boldsymbol{R}_\varkappa$, it is evident that the successor is $p_1 \times \varkappa$, proving the proposition. ■

Corollary 1.2. A366807 is well-defined and countably infinite.

We define a second sequence A366825 to be the following:

$$\begin{aligned}\text{A366825} &= \{\, k = p_1\varkappa : \Omega(\varkappa) = \omega(\varkappa) > 1,\ p_1 = \text{LPF}(\varkappa) \,\}\\ &= \{\, k = p_1^2\, m : \Omega(m) = \omega(m) \ge 1,\ (p_1, m) = 1 \,\}. \quad [1.2]\end{aligned}$$

In other words, A366825 is the list of composites of the form $p^2 \times m$, where $m > 1$ is squarefree. This sequence begins as follows:

12, 20, 28, 44, 45, 52, 60, 63, 68, 76, 84, 92, 99, 116, 117, 124, 132, 140, 148, 153, 156, 164, 171, 172, 175, 188, 204, 207, 212, 220, 228, 236, 244, 260, 261, 268, 275, 276, 279, 284, 292, 308, 315, 316, 325, 332, ...

Lemma 1.3. A366825 is a sorted version of the countably infinite set of mappings A065642 ↦ A120944, i.e., A366807. This is also to say A366807 is a permutation of A366825.

As an aside, we propose a sequence A366786 defined to be the sequence of mappings A065642 ↦ A5117, where the latter sequence of squarefree numbers is defined below:

$$\text{A5117} = \{1\} \cup \{\varkappa : p^\varepsilon \mid \varkappa \rightarrow \varepsilon = 1\,\}. \quad [1.3]$$

The sequence A366786 begins as follows:

1, 4, 9, 25, 12, 49, 20, 121, 169, 28, 45, 289, 361, 63, 44, 529, 52, 841, 60, 961, 99, 68, 175, 1369, 76, 117, 1681, 84, 1849, 92, 2209, 153, 2809, 275, 171, 116, 3481, 3721, 124, 325, 132, 4489, 207, 140, 5041, ...

Theorem 1.4. $\text{A366786} = \bigcup(\{1\}, \text{A1248}, \text{A366825})$. This is to say that is the union of the empty product, squares of primes, and composite squarefree numbers multiplied by their smallest prime factors.

Proof. We begin with the tautology $\text{A5117} = \bigcup(\{1\}, \text{A40}, \text{A120944})$. This is to say that squarefree numbers consist of the empty product 1, the primes, and composite squarefree numbers.

We have already shown in Theorem 1.1 that A366807, the sequence of mappings A065642 ↦ A120944, is a permutation of A366825. The sequence A366825 = $\{\, k = p_1 \varkappa \,\}$, where $\varkappa \in$ A120944.
Through a similar argument we can also recognize A065642 ↦ A40, the $\varkappa$-coregular successor function mapped across primes, to yield A1248, squares of primes. This, since

$$\boldsymbol{R}_p = \{\, p^\varepsilon : \varepsilon \geq 0 \,\}, \qquad [1.4\text{A}]$$

hence,

$$p\boldsymbol{R}_p = \{\, p^\varepsilon : \varepsilon \geq 1 \,\}, \qquad [1.4\text{B}]$$

Since p is the minimum of $\boldsymbol{R}_p$, the power range of p, it follows that p^2 succeeds p and thus A065642$(p) = p^2$.
The hitch regards $k = 1$. $\boldsymbol{R}_1 = \{1\}$ and there is no successor to 1 in that set. Hence, for concord with A5117 as it includes 1, we define A366786$(1) = 1$ instead of leaving it undefined. Having done this, we demonstrate the proposition. ■

Bisection of Minimally Tantus Numbers.

The definition of A366825 given by [1.2] implies both even and odd terms. Let sequence S include even terms in A366825 and sequence T odd terms in A366825. We then define S to be as follows:

$$\begin{aligned} S &= \{\, k = 2\varkappa : \Omega(\varkappa) = \omega(\varkappa) > 1,\ 2 \mid \varkappa \,\} \\ &= \{\, k = 2^2 \times m : \Omega(m) = \omega(m) \geq 1, \text{ odd } m \,\}. \end{aligned} \qquad [2.0]$$

Let p_2 be the second smallest prime factor of tantus k.
Let prime q be the smallest prime that does not divide k.
Define strong tantus k to be the following:

$$\text{A360768} = \text{V70} = \{\, k : \Omega(k) > \omega(k) > 1 \wedge k/\varkappa \geq p_2 \,\}. \qquad [2.1]$$

Therefore, weak tantus k consist of the following:

$$\text{A360767} = \text{V71} = \{\, k : \Omega(k) > \omega(k) > 1 \wedge k/\varkappa < p_2 \,\}. \qquad [2.2]$$

Define thick tantus k to be the following:

$$\text{A360765} = \text{V72} = \{\, k : \Omega(k) > \omega(k) > 1 \wedge k/\varkappa > q \,\}. \qquad [2.3]$$

Therefore, thin tantus k consist of the following:

$$\text{A363082} = \text{V73} = \{\, k : \Omega(k) > \omega(k) > 1 \wedge k/\varkappa < q \,\}. \qquad [2.4]$$

Intersections of these sequences were described in [4]. These departments of tantus numbers are listed below:

A364999: V75 = V71 ∩ V73, thin-weak tantus, [2.5]
A364998: V76 = V70 ∩ V73, thin-strong tantus,
A364997: V77 = V71 ∩ V72, thick-weak tantus,
A361098: V74 = V70 ∩ V72, thick-strong or "panstitutive" tantus.

Tantus numbers whose prime power factor exponents all exceed 1 are called "plenus numbers" A286708, the sequence of squareful numbers that are not prime powers.

$$\text{A1694} = \text{A246547} \cap \text{A286708}. \qquad [2.6]$$

Theorem 3.9 in [5] shows the following:

$$\text{A286708} \subseteq \text{A361098}. \qquad [2.7]$$

Here, we are concerned with similar relationships between the other departments of tantus numbers described in [2.5].

Theorem 2.1. Minimally tantus k is weak (i.e., A366825 ⊆ A360767).
Proof. A minimally tantus $k = p_1\varkappa$ (with $\varkappa = \text{RAD}(k)$). Hence the ratio $k/\varkappa < p_2$, since $k/\varkappa = p_1$, and $p_1 < p_2$ by definition. ■

Theorem 2.2. Even minimally tantus k is thin (i.e., $S \subseteq$ A363082) and odd minimally tantus k is thick (i.e., $T \subseteq$ A360765).
Proof. Setting $p_1 = 2$, the smallest prime, implies $p_1 < q$. Hence the ratio $k/\varkappa < q$, since $k/\varkappa = p_1$, and $p_1 < q$ by their definitions This is to say that $S \subseteq$ A363082. Setting $p_1 > 2$ implies both k and $\varkappa$ odd, which in turn suggests $q = 2$ and $q > p_1$. The ratio $k/\varkappa = p_1$, thus $k/\varkappa > q$, which is to say that $T \subseteq$ A360765. ■

Therefore we may partition A366825 by parity into sequences S and T, where $S \subseteq$ A364999 and $T \subseteq$ A364997.

Relation of the Sequence of Minimally Tantus with That of Weak-Thin Tantus.

Dr. Richard Mathar wrote a conjecture in the comments of A364999 suggesting the following:

$$\text{A364999} = \text{A081770} \setminus \{4\}, \qquad [3.0]$$

Let's examine A081770. Definition:

$$\text{A081770} = \{\, k : k/\varkappa = 2 \,\}. \qquad [3.1]$$

It is clear that this sequence includes $4 = 2^2$ since $4/2 = 2$. It is the sole prime power in the sequence since setting $p > 2$, we cannot obtain the quotient $k/\varkappa = 2$.

Lemma 3.0: The ratio $k/\varkappa = m$ implies $\text{RAD}(m) \mid \varkappa$.
Proof. The proposition requires squarefree kernel $r = \text{RAD}(m)$ such that no prime factor $q \mid r$ that does not also divide $\varkappa$. We recognize that any number $m \mid k$ also divides $m \mid \varkappa$, since by definition, since $\varkappa$ is the product of distinct prime factors of k. Further, divisibility is a special case of $\text{RAD}(m) \mid \varkappa$ wherein m also divides $\varkappa$.
Now suppose r does not divide $\varkappa$. This would imply $q \mid k$ for some prime q, contradicting the definition of $\varkappa$ to be the product of distinct prime factors of k. ■

Theorem 3.1: $\{\, k : k/\varkappa = 2 \,\} \subseteq$ A364999.
Proof. We observe $k/\varkappa = 2$ implies both $2 \mid k$ and $2 \mid \varkappa$ via Lemma 3.0. Therefore, both $p_2 > 2$ and $q > 2$, that is, both p_2 and q must be odd primes. This proves the proposition. ■

Now we attempt via induction to show that $k/\varkappa = m$ cannot exceed 2.
Theorem 3.2: $\{\, k : k/\varkappa > 2 \,\} \not\subseteq$ A364999.
Proof. Set $m = 3$. This implies both $3 \mid k$ and $3 \mid \varkappa$ via Lemma 3.0. If k is even, then so is $\varkappa$, hence $p_2 = 3$, contradicting the definition of A364999. Therefore k must be odd, with $3 < p_2$. But then $q = 2 < 3$, also contradicting the definition of V75. Through induction, we set m to a progressively larger and thus odd prime and find that $m = p_1$ so as to be smaller than p_2, yet $q = 2 < m$.
Now we turn to the case of composite m. Set $m = 4$, hence both k and $\varkappa$ are even, $p_1 = 2$, and though this implies $p_1 < p_2$, we have the following. Setting $p_2 = 3$ implies $m > p_2$. Setting $p_2 > 3$ implies $m > q$, since $q = 3$. Any further prime powers of 2 yield the same situation.
Odd prime power m yields the situation of $q < m$.
Non-prime power m merely supplies multiplicity.
Therefore it is clear that m cannot exceed 2 for $k \in$ A364999. ■

Hence we confirm [3.0] and expand the definition below:

$$\begin{aligned} \text{A364999} &= \text{A081770} \setminus \{4\} \\ &= 2 \times \{\text{A039956} \setminus \{2\}\} \\ &= 4 \times \{\text{A056911} \setminus \{1\}\} \\ &= 2^2 \times m,\ 2 \nmid m,\ \Omega(m) = \omega(m) > 1. \end{aligned} \qquad [3.2]$$

Furthermore, apart from 4, terms in A081770 are weak-thin tantus.

Thereby we simplify the definition of thin-weak tantus as 4 times an odd squarefree number.

Now let us show that A364999 = S.

Theorem 3.3: A364999 is tantamount to the set S of even terms in A366825. This is to say, the thin-weak department of tantus numbers is the sequence of even minimally tantus numbers.

Proof: Let us rewrite the sequences A364999 and A366825.

$$\begin{aligned} \text{A364999} &= \{\ k = 2^2 \times m : 2 \nmid m, \Omega(m) = \omega(m) > 1\ \} \\ &= \{\ k = 2\varkappa : \varkappa = \text{RAD}(k), \omega(\varkappa) > 1\ \} \\ &= \{\ k = p_1 \times \varkappa : p_1 = \text{LPF}(\varkappa) = 2\ \}. \end{aligned} \quad [3.3]$$

$$\begin{aligned} \text{A366825} &= \{\ k : \omega(k) > 1 \land k/\varkappa = p_1\ \} \\ &= \{\ k = p_1 \times \varkappa : p_1 = \text{LPF}(\varkappa)\ \}. \end{aligned} \quad [3.4]$$

Setting $p_1 = 2$ in case of A366825, we have the following sequence:

$$S = \{\ k = 2\varkappa : \varkappa = \text{RAD}(k), \omega(\varkappa) > 1\ \}. \quad [3.5]$$

Hence the proposition is true: A364999 is the sequence of even minimally tantus numbers S. ■

Hence we may define the odd minimally tantus as follows:

$$T = \text{V0705} = \text{A366825} \setminus \text{A364999}. \quad [3.6]$$

The question then remains: does A364997 properly contain T, or are the sequences synonymous?

Lemma 3.4: $T \subseteq$ A364997.

Proof: Let $\varkappa = \text{RAD}(k)$ and let $p_1 = \text{LPF}(\varkappa)$. Knowing T is the sequence of odd terms in A366825, we define T to be as follows:

$$T = \{\ k = p_1 \times \varkappa : p_1 > 2 \to 2 \nmid k\ \}. \quad [3.7]$$

Least prime factor $p_1 > 2$ implies $q < p_1$, hence k is thick tantus, but certainly, $p_1 < p_2$, and since $m = k/\varkappa = p_1$, k is weak tantus, and indeed, in A364997, the sequence of thick-weak tantus. Therefore, we may say $T \subseteq$ A364997. ■

Lemma 3.5: Even terms exist in A364997.

Proof. Let $m = k/\varkappa$, and let us construct an even tantus k such that $q < k/\varkappa < p_2$. The definition of p_2 implies $p_1 \mid m$, in fact, $m = {p_1}^\delta$, $\delta > 1$. We set $m = {p_1}^\delta$ for $p_1 = 2$, $\delta = 1$, hence $m = 2$. This implies $p_2 > 3$ and $q = 3$, and we see that this $2\varkappa$ is instead in A364999. Setting $\delta = 2$ implies $p_2 > 3$ and thus $q = 3$, attaining $q < 2^2 < p_2$ and thus clearly $\{\ k = 4\varkappa,\ \Omega(k) > \omega(k) > 1\ \} \subset$ A364997, proving the proposition. ■

Lemma 3.6: $\{\ k = 2^\delta \times \varkappa,\ \Omega(k) > \omega(k) > 1, \delta > 1, 3 \nmid \varkappa\ \} \subset$ A364997.

Proof. Via induction on δ, we see that we maintain $q = 3$ while $p_2 > 2^\delta$, which in turn maintains $3 < 2^\delta < p_2$, proving the proposition. ■

Theorem 3.7: $T \subset$ A364997.

Proof: Lemmas 3.4 through 3.6 show that along with odd minimally tantus k, we have certain even tantus $k = 2^\delta \times \varkappa$, $\delta > 1$. Therefore, numbers like $40 = 2^3 \times 5$, $56 = 2^3 \times 7$, $88 = 2^3 \times 11$, and $104 = 2^3 \times 13$, that is, $\{\ k = 2^3 \times p_2 : p_2 > 3\ \} \subset$ A364997. ■

Lemma 3.8: The constraint $(k/\varkappa < p_2)$ restrains m to multus ${p_1}^\delta$.

Proof. Suppose not; suppose $\omega(m) \geq 1$. This implies that m is a composite product with some prime $p \geq p_2$ which in turn implies $p_2 < m$, contradicting the proposition. ■

Theorem 3.9: A364997 = $\{\ k = {p_1}^\delta \times \varkappa : \omega(\varkappa) > 1,\ \delta > [p_1 = 2]\ \}$. Consequence of previous lemmas and theorems.

Therefore we present additional definitions for A364999, A364997, and odd minimally tantus V0705:

$$\begin{aligned} \text{A364999} &= \{\ k = p_1 \times \varkappa : p_1 = \text{LPF}(\varkappa) = 2\ \}. \\ \text{A364997} &= \{\ k = {p_1}^\delta \times \varkappa : \omega(\varkappa) > 1, \delta > [p_1 = 2]\ \} \\ \text{V0705} &= \{\ k : \omega(k) > 1 \land k/\varkappa = p_1 > 2\ \} \subset \text{A364997}. \end{aligned}$$

Bifurcation of Tantus Numbers by p_2 and q.

Now we define partitions of tantus numbers according to the relation of p_2 and q:

$$\begin{aligned} \text{V0715} &= \{\ k : \Omega(k) > \omega(k) > 1 \land p_2 > q\ \}, \\ \text{V0716} &= \{\ k : \Omega(k) > \omega(k) > 1 \land p_2 < q\ \}. \end{aligned} \quad [4.0]$$

The sequence V0715 begins as follows:

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20, 28, 40, 44, 45, 50, 52, 56, 63, 68, 75, 76, 80, 88,
92, 98, 99, 100, 104, 112, 116, 117, 124, 135, 136, 140,
147, 148, 152, 153, 160, 164, 171, 172, 175, 176, 184,
188, 189, 196, 200, 207, 208, 212, 220, 224, 225, ...
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The sequence V0716 begins as follows:

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12, 18, 24, 36, 48, 54, 60, 72, 84, 90, 96, 108, 120,
126, 132, 144, 150, 156, 162, 168, 180, 192, 198, 204,
216, 228, 234, 240, 252, 264, 270, 276, 288, 294, 300,
306, 312, 324, 336, 342, 348, 360, 372, 378, 384, ...
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Theorem 4.1. Odd tantus k implies $p_2 > q$, i.e., A360769 ⊂ V0715.

Proof. If k is odd, $q = 2$, and since primes must be divisors or not and 2 is the smallest prime, it follows that $p_2 > q$. ■

Theorem 4.2. Tantus k with primorial $\varkappa$ implies $p_2 < q$, that is to say, {A126706 ∩ A055932} ⊂ V0716.

Proof. Suppose the converse: $p_2 > q$. This suggests that the second smallest distinct prime factor is larger than the smallest nondivisor prime q. Let $S = \{\ p : p \leq r, r \geq p_2\ \}$. Since squarefree kernel $\varkappa = \prod S$ and since q is coprime to k and thus also $\varkappa$ by definition, q must exceed r. Clearly, $p_2 \in S$, contradicting supposition. ■

Corollary 4.3. V0716 = $\{\ k = 6m : \Omega(k) > \omega(k) > 1\ \}$.

Corollary 4.4. V0715 = $\{\ k : \Omega(k) > \omega(k) > 1, m \bmod 6 \equiv 0\ \}$.

Conclusion.

This paper introduces the set of minimally tantus numbers A366825, which arise from the mappings A065642 ↦ A120944. We showed that this sequence is simply the product of squarefree composites with their least prime factor. This sequence A366825 is contained by weak-thin tantus A364999; we show that indeed, A364999 is the sequence of even terms in A366825. We confirm Mathar's conjecture that A364999 = A081770 \ {4}. Odd terms in A366825, which we assign the local identification V0705, are properly contained in A364997, since there are even terms in A364997.

Concerns Sequences:

A000040, A001220, A001221, A001694, A005117, A039956, A056911, A065642, A081770, A120944, A126706, A246547, A286708, A360765, A360767, A360768, A361098, A366786, A366807, A366825, V0705, V0715, V0716

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