Powers of Superprimorials.

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Abstract.

This paper presents several basic theorems pertaining to Chernoff numbers, also known as superprimorials. The theorems attempt to find intersections in the Chernoff sequence A6939 with other constitutive classes of number.

Introduction.

Let $\Omega(n)$ = A1220(n) be the number of prime factors of n with multiplicity and let $\omega(n)$ = A1221(n) be the number of distinct prime factors of n.

Define tantus to be a number k such that $\Omega(k) > \omega(k) > 1$.

Let p_1 be the smallest prime factor of tantus k.

Let p_2 be the second smallest prime factor of tantus k.

Let prime q be the smallest prime that does not divide k.

Let $\varkappa = \mathrm{RAD}(k)$, the product of distinct prime factors of k.

Define strong tantus k to be the following:

$$\text{A360768} = \{\, k : \Omega(k) > \omega(k) > 1 \wedge k/\varkappa \geq p_2 \,\}$$

Define thick tantus k to be the following:

$$\text{A360765} = \{\, k : \Omega(k) > \omega(k) > 1 \wedge k/\varkappa > q \,\}$$

Define panstitutive k to be the following:

$$\text{A361098} = \text{A360765} \cap \text{A360768}.$$

Define Chernoff numbers $\text{ч}(n)$ = A6939(n), also known as superprimorials, to be as follows:

$$\text{ч}(n) = \{\, k = \prod_{i=1}^{j} \mathrm{PRIME}(i)^{(j-i+1)} = \prod_{i=1}^{j} \mathcal{P}(i) \,\}.$$

This sequence gets large quickly and begins as follows:

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1, 2, 12, 360, 75600, 174636000, 5244319080000,
2677277333530800000, 25968760179275365452000000,
5793445238736255798985527240000000,
37481813439427687898244906452608585200000000, ...
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This sequence contains the empty product and the prime 2. All the rest of the numbers are nonplenus tantus $k \in$ A332785.

$$\begin{aligned}\text{A332785} &= \{k : \Omega(k) > \omega(k) > 1,\ \exists\ p \mid k \wedge p^2 \nmid k\,\}\\ &= \text{A126706} \setminus \text{A286708}.\end{aligned}$$

Theorem 1. $\text{ч}(n)$, $n > 1$, implies $\text{ч}(n) \in$ A332785.

Proof. By definition, $\mathrm{GPF}(\text{ч}(n))^2 \nmid \text{ч}(n)$, hence $\text{ч}(n)$, $n > 1$ is not in A286708. For $n > 1$, $\text{ч}(n)$ is such that $\omega(\text{ч}(n)) > 1$, i.e., $\text{ч}(n)$ is in A024619, not a prime power. Furthermore, $\text{ч}(n)$, $n > 1$, is not squarefree (in A013929) since the multiplicities of prime power factors of are strictly decreasing as prime index increases. Therefore, $\text{ч}(n)$, $n > 1$, appears in the sequence of numbers in A126706 that are not in A286708. ■

Theorem 2. $\text{ч}(n)$, $n > 2$, implies $\text{ч}(n) \in$ A361098.

Proof. $\text{ч}(1) = 12 = 2^2 \times 3$, and since $k/\varkappa < p_2 \rightarrow 12/6 < 3$ and since $k/\varkappa < q \rightarrow 12/6 < 5$, 12 is thin and weak tantus (in A364999).

$\text{ч}(2) = 360 = 2^3 \times 3^2 \times 5$; $360/30 = 12$, and since $k/\varkappa \geq p_2 \rightarrow 12 \geq 3$ and since $k/\varkappa > q \rightarrow 12 > 7$, 360 is panstitutive (in A361098). Through induction on n, we can see the following:

1.) $k/\varkappa = \text{ч}(n-1)$ is always greater than $p_2 = 3$,

2.) $k/\varkappa = \text{ч}(n-1)$ is always greater than prime(n+2).

Therefore $\text{ч}(n)$, $n > 2$, implies $\text{ч}(n) \in$ A361098. ■

Corollary 3: 1 is the sole empty product, 2 the only prime, 12 the only minimally tantus Chernoff number $\text{ч}(n) \in$ A366825. Almost all Chernoff numbers are panstitutive (i.e., in A361098).

Corollary 4: $\text{ч}(n)$ is of the form $m \times$ A053669(m), $m \in$ A1694.

If we have superprimorials, then we should have their powers: this shall be a new sequence A368507.

$$\text{ч}(n)^\delta = \{\, k = (\prod_{i=1}^{j} \mathrm{PRIME}(i)^{(j-i+1)})^\delta = (\prod_{i=1}^{j} \mathcal{P}(i))^\delta \,\}.$$

This sequence begins as follows:

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1, 2, 4, 8, 12, 16, 32, 64, 128, 144, 256, 360, 512,
1024, 1728, 2048, 4096, 8192, 16384, 20736, 32768,
65536, 75600, 129600, 131072, 248832, 262144, 524288,
1048576, 2097152, 2985984, 4194304, 8388608, 16777216,
33554432, 35831808, 46656000, 67108864, 134217728, ...
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Let A364930 = A286708 $\cap$ A025487.

It is clear that the following sequences comprise A368507:

$$\begin{aligned}&\text{A79} = \{2^\delta : \delta \geq 0\},\\ &\{\ 12 \in \text{A364999}\ \},\\ &\{\ \text{ч}(n) \in \text{A361098},\ n > 2\ \},\\ &\{\ \text{ч}(n)^\delta \in \text{A364930},\ n > 2,\ \delta \geq 2\ \}.\end{aligned}$$

Let A368908 = $\{\ \text{ч}(n)^\delta : n > 2, \delta \geq 2\ \}$. This new sequence begins as shown below:

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144, 1728, 20736, 129600, 248832, 2985984, 35831808,
46656000, 429981696, 5159780352, 5715360000,
16796160000, 61917364224, 743008370688, 6046617600000,
8916100448256, 106993205379072, 432081216000000,
1283918464548864, 2176782336000000, ...
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Let A120944 = $\{\ k : \Omega(k) = \omega(k) > 1\ \}$ (the varius numbers).

Theorem 5. For $k \in$ A120944, $(k^\delta, \delta > 1)$ implies $k^\delta \in$ A286708.

Proof. k is squarefree and composite, implying more than one prime power factor $p^\varepsilon \mid k$, $\varepsilon = 1$. Raising k to exponent $\delta > 1$ implies prime power factors $p^\delta \mid k^\delta$, $\delta > 1$. Since all exponents of prime power factors of k are the same and exceed 1, since $\omega(k) > 1$, $k \in$ A303606, A303606 $\subset$ A286708. ■

Corollary 6. A368908 $\subseteq$ A286708, and since 36 $\in$ A286708, it is plain to see that A368908 $\subset$ A286708. Specifically, A368908 $\subset$ A364930, since A364930 = A286708 $\cap$ A025487.

Conclusion.

We have shown that A6939 is comprised of the empty product, the prime 2, the minimally tantus number 12, and the rest of the infinite sequence consists of panstitutive numbers $k \in$ A361098, where A361098 $\subset$ A332785. Furthermore, we showed that A368507 is the union of A79, A6939, and A368908, where A368908 $\subset$ A364930 $\subset$ A286708.

Concerns Sequences:

A79, A1220, A1221, A1694, A2110, A5117, A6939, A013929, A024619, A053669, A120944, A126706, A286708, A303606, A332785, A360765, A360768, A361098, A364999, A366825, A368507, A368508.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved December 2023.