On A369690 = MAX(A119288(n), A053669(n))

A sequence of Peter Munn

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Abstract.

This is a brief study of the mappings of the function MAX(p_2, q) across natural numbers, where p_2 is the second least prime factor of n (or 1 if n is a prime power), and q is the smallest prime nondivisor of n. We show that the function takes prime values for certain classes of number.

Introduction.

Let p_2 = A119288(n) be the second least prime factor of n or 1 if n is a prime power, i.e., $\omega(n) = 1$, and let q = A053669(n) be the smallest prime that does not divide n.

We define the following functions:

$\Omega(n)$ = A1220(n), number of prime factors of n with multiplicity ,
$\omega(n)$ = A1221(n), number of distinct prime factors of n,
$\varkappa$ = RAD(n) = A7947(n), squarefree kernel of n,
$\mathcal{P}(n)$ = A2110(n), product of the smallest n primes,
n/RAD(n) = A3557(n).

A126706 = { $k : \Omega(k) > \omega(k) > 1$ }, tantus numbers k neither squarefree nor prime powers.

We present 4 blocks resulting from the partition of tantus, known as the "constitutive quadrisection" of tantus numbers $k \in$ A126706 according to the magnitudes of A3557(k), A053669(k), and A119288(k). These sequences are defined below:

V74 = A361098 = { $n : \Omega(n) > \omega(n) > 1,\ p_2 < n/\varkappa,\ q < n/\varkappa$ }.
V75 = A364999 = { $n : \Omega(n) > \omega(n) > 1,\ n/\varkappa < p_2,\ n/\varkappa < q$ }.
V76 = A364998 = { $n : \Omega(n) > \omega(n) > 1,\ q < n/\varkappa < p_2$ }.
V77 = A364997 = { $n : \Omega(n) > \omega(n) > 1,\ p_2 \le n/\varkappa < q$ }. [1.1]

In [2] we find that A364999 represents even terms in A366825, minimally tantus numbers. The "panstitutive" sequence A361098, defined in [2], is remarkable in that it contains both A286708 and A131605, powerful tantus and tantus that are perfect powers, respectively [3].

In reflecting upon the definition of A361098, Peter Munn suggested the following sequence:

$$\text{V0223} = \text{MAX}(p_2, q) \mapsto \mathbb{N},$$

The sequence begins as follows:

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2, 3, 2, 3, 2, 5, 2, 3, 2, 5, 2, 5, 2, 7, 5, 3, 2, 5, 2,
5, 7, 11, 2, 5, 2, 13, 2, 7, 2, 7, 2, 3, 11, 17, 7, 5,
2, 19, 13, 5, 2, 5, 2, 11, 5, 23, 2, 5, 2, 5, 17, 13, 2,
5, 11, 7, 19, 29, 2, 7, 2, 31, 7, 3, 13, 5, 2, 17, ...
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Let a = V0223. The sequence is proposed in OEIS as A369690.

In the function MAX(p_2, q), we deal with primes related to n through divisibility and coprimality. It is easy to see the following:

$a(n) = p \rightarrow a(m\varkappa) = p$, RAD($m$) | $\varkappa$, $\varkappa$ = RAD(n).
$a(n) = p$ for certain $k \in$ { $k = m\varkappa$: RAD(m) | $\varkappa$ }. [1.2]

LEMMA A1. $a(n) = 2$ for n in A061345, where we define A061345 to be thus:

A061345 = { $p^\delta : p > 2, \delta \ge 0$ }
= ∪({1}, { $mp : p > 2$, RAD(m) | p }). [1.3]

This is to say, n is an odd prime power.

PROOF. Consequence of $q = 2$ since n is odd, but $p = 1$ since n is a prime power. ■

LEMMA A2. $a(n) = 3$ for n in A79, where we define A79 to be thus:

A79 = { $2^\delta : \delta \ge 0$ }
= ∪({1}, { $2m$: RAD(m) | 2 }). [1.4]

This is to say, n is a power of 2.

PROOF. Consequence of $q = 3$ since n is even, but $p = 1$ since n is a prime power. ■

LEMMA A3. $a(n)$ = PRIME(j) for $j > 2$ and $\mathcal{P}(j-1)$-coregular sequence

{ $k = m \times \mathcal{P}(j-1)$: RAD(m) | $\mathcal{P}(j-1)$ }. [1.5]

PROOF. For primorials $\mathcal{P}(i)$, $i > 1$, we show $p_2 = 3$, but $q > 3$. Hence, $a(\mathcal{P}(i))$ = PRIME(i+1), thus, $a(\mathcal{P}(j-1))$ = PRIME(j). This pertains to the $\mathcal{P}(j-1)$-coregular sequence via Lemma A1. ■

LEMMA A4. $a(n)$ = PRIME(j) for $j > 2$ and the following sequence:

{ $k = m \times \mathcal{P}(j-1) \times Q$: RAD(m) | $\mathcal{P}(j-1)$,
$\forall\, p_+ \mid Q,\ p_+ >$ PRIME(j) }. [1.6]

PROOF. Consequence of Lemma A3 and the fact that multiplication of $\mathcal{P}(j-1)$ by a product of primes $p_+ >$ PRIME(j) preserves $p_2 < q$, with $a(n) = q$ = PRIME(j). ■

Define number triangle $T(n, k)$ = PRIME(k) × PRIME(n), $k < n$, where the vectorized sequence is A339116, a permutation of squarefree semiprimes, A100484 \ {4}.

The triangle $T(n, k)$ begins as shown below:

n\k	1	2	3	4	5	6	7	8
2:	6;							
3:	10,	15;						
4:	14,	21,	35;					
5:	22,	33,	55,	77;				
6:	26,	39,	65,	91,	143;			
7:	34,	51,	85,	119,	187,	221;		
8:	38,	57,	95,	133,	209,	247,	323;	
9:	46,	69,	115,	161,	253,	299,	391,	437;

LEMMA A5:

$T(n, k)$ = PRIME(k) × PRIME(n), $n > 2$, $k < n$ implies $q < p_2$.

PROOF. Aside from $s = T(2, 1) = 6$, $q < p_2$, since even s implies $q = 3$ and odd s implies $q = 2$, but p_2 = PRIME(n) $> q$ in both cases. ■

This is tantamount to saying that A100484(n) > 6 implies $q < p_2$.

LEMMA A6. The inequality $q < p_2$ is preserved for the following infinite sequences:

{ $T(n, k) \times Q : n > 2,\ \forall\, p_+ \mid Q,\ p_+ \ge$ PRIME(n) } → $q < p_2$.

PROOF. Consequence of p_2 = PRIME(n) $< p_+$. The factor Q is a product of primes larger than p_2 = PRIME(n) and presents no impact on the fact that $q < p_2$. ■

LEMMA A7. $a(n) = \operatorname{prime}(j)$ for $j > 2$ and for the following k-coregular sequence:

$$\{\ k = m \times T(j,k) \times Q : \operatorname{rad}(m) \mid T(j,k), \forall\ p_+ \mid Q, p_+ \geq \operatorname{prime}(j)\ \}. \quad [1.7]$$

PROOF. Consequence of Lemmas A3, A4, and A5.
However, we explore the case of prime $m = p$ such that $p < \operatorname{prime}(j)$ and $p \neq \operatorname{prime}(k)$. Suppose $m > q$. Then $a(n) = p_2 = p$ and we have a number in the form of Lemma A5 instead for $j = \pi(p)$. It is clear that $m = 5$, or any product that yields $\mathcal{P}(i)$ for $i > 2$ gives us a number of the form shown by Lemma A4. ■

It is clear we may rewrite the formula shown in Lemma A7 thus:

$$\{\ k = m \times d \times Q : \operatorname{rad}(m) \mid d, \forall\ q \mid Q, q \geq \operatorname{prime}(j-1)\ \} \text{ for } d \mid \mathcal{P}(j) : \omega(d) = 2, \operatorname{prime}(j-1) \mid d. \quad [1.8]$$

THEOREM A assembles Lemmas A1-A7.
$a(n) = 2$ for n that are powers of odd primes.
$a(n) = 3$ for $n > 1$ that are powers of 2.
$a(n) = \operatorname{prime}(j)$ for $j > 2$ and for both of the following:

$$\{\ k = m \times \mathcal{P}(j-1) \times Q : \operatorname{rad}(m) \mid \mathcal{P}(j-1), \forall\ p_+ \mid Q, p_+ > \operatorname{prime}(j)\ \}, \text{ and}$$

$$\{\ k = m \times d \times Q : \operatorname{rad}(m) \mid d, \forall\ q \mid Q, q \geq \operatorname{prime}(j-1)\ \} \text{ for } d \mid \mathcal{P}(j) : \omega(d) = 2, \operatorname{prime}(j-1) \mid d.$$

COROLLARY A8. $a(n) < 5$ for $n \in$ A961.

COROLLARY A9. $a(n) \geq 5$ for $n \in$ A024619.

COROLLARY A10. A3557$(n) > a(n) \geq 5$ for $n \in$ A361098.

CONCLUSION.

We have shown that V0223$(n) = 2$ for odd prime powers n, V0223$(n) = 3$ for n that are powers of 2, and exceeds 3 for numbers that are not prime powers. Among numbers that are not prime powers, we see that A3557$(n) > a(n) \geq 5$ for $n \in$ A361098.

CONCERNS SEQUENCES:

A000079, A000224, A000961, A002110, A003557, A006530, A006881, A007947, A024619, A053669, A061345, A100484, A119288, A126706, A339116, A361098, A364997, A364998, A364999, A369690. [1]

(V7, V9, V74, V75, V76, V77, V0103, V0109, V0111, V0119, V0220, V0222, V0223, V0401, V0402, V0612, V1002, V1003.)

REFERENCES:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved January 2024.

[2] Michael Thomas De Vlieger, Constitutive State Counting Functions, *Simple Sequence Analysis*, 20230226.

[3] Michael Thomas De Vlieger, Partitioning the Set of Tantus Numbers, *Simple Sequence Analysis*, 20240106.

CODE:

[C1] Generate 2^{14} terms of V0223:

```
v0223 = {2}~Join~Array[
 If[PrimePowerQ[#],
   q = 2; While[Divisible[#, q], q = NextPrime[q]]; q,
   q = 2; While[Divisible[#, q], q = NextPrime[q]];
     Max[FactorInteger[#][[2, 1]], q]] &, 2^14, 2] ];
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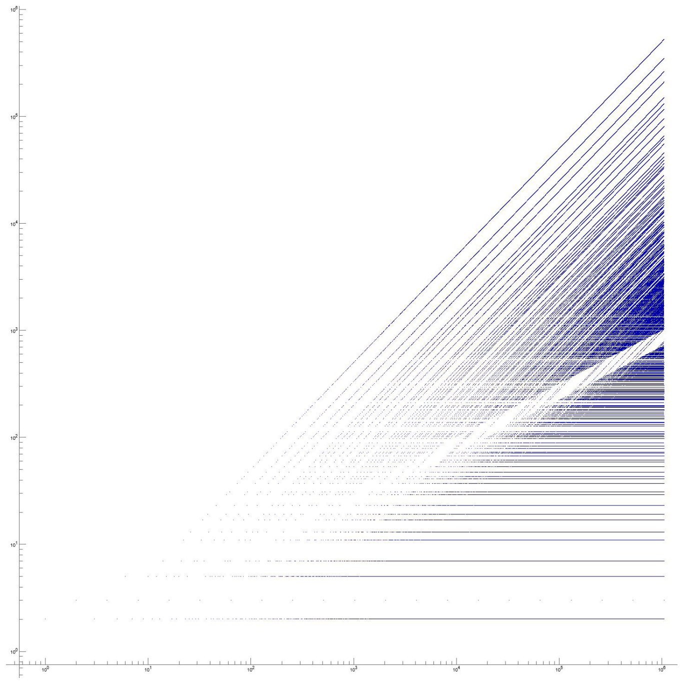

Figure 1: Log log scatterplot of V0223(n), n = 1 … 2^{20}.