Relationship of complementary divisors $\textsc{rad}(k)$ and $k/\textsc{rad}(k)$.

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Abstract.

We examine sequences A341645 and A341646 by Wiseman that concern the relation of complementary divisors $\textsc{rad}(k)$ and $k/\textsc{rad}(k)$, where $\textsc{rad}(k)$ is the squarefree kernel of k. Specifically, we identify a block of A341645, new sequence A366250, that is a proper subset of A364702.

Introduction.

Definitions:

$p_1 = \textsc{lpf}(n) =$ A020639(n), least prime factor of n.
$p_2 =$ A119288(n), second least prime factor of n.
$q =$ A053669(n), least prime that does not divide n.
$\mathcal{P}(n) =$ A2110(n), primorial, product of smallest n primes.
$\varkappa = \textsc{rad}(n) =$ A7947(n), squarefree kernel of n.
$m = n/\textsc{rad}(n) =$ A3557(n), (squarefree) kernel ratio. [A]

Define a strictly superior divisor d, $d \mid n$ to be such that $d > n/d$.

We are concerned with finding squarefree d, recognizing that $\textsc{rad}(n)$ is the largest squarefree divisor of n.

Define the sequence of powerful numbers A1694, wherein k is a product of perfect powers of primes p^δ, $\delta > 1$, to be as follows:

$$\text{A1694} = \{\, k : p \mid k \rightarrow p^2 \mid k \,\}. \qquad \text{[B]}$$

It is clear that the sequence of perfect powers, A1597, is a proper subset of A1694, since multiplicities of prime power factors may or may not be coprime.

A proper subset common to both A1694 and A1597 is the set of perfect powers of primes, A246547, herein called multus numbers.

$$\text{A246547} = \{\, k : \Omega(k) > \omega(k) = 1 \,\}. \qquad \text{[C]}$$

Define the sequence of tantus numbers k, those neither squarefree nor prime powers, to be as follows:

$$\text{A126706} = \{\, k : \Omega(k) > \omega(k) > 1 \,\}. \qquad \text{[D]}$$

Define the sequence of plenus numbers k, powerful numbers that are not prime powers, to be as follows:

$$\begin{aligned}\text{A286708} &= \{\, k : \Omega(k) > \omega(k) > 1, p \mid k \rightarrow p^2 \mid k \,\} \\ &= \text{A1694} \setminus \text{A246547}.\end{aligned} \qquad \text{[E]}$$

Strictly superior squarefree divisors.

Gus Wiseman defined a divisor $d \mid n$ to be strictly superior if $d > n/d$. He wrote 2 sequences addressing strictly superior squarefree divisors, A341645 and its complement A341646 as follows:

$$\begin{aligned}\text{A341645} &= \{\, k : \textsc{rad}(k) \le k/\textsc{rad}(k) \,\} \\ &= \{\, k : \text{A7947}(k) \le \text{A3557}(k) \,\} \\ &= \{\, k : \varkappa \le m \,\}.\end{aligned} \qquad [1.1]$$

$$\begin{aligned}\text{A341646} &= \{\, k : \textsc{rad}(k) > k/\textsc{rad}(k) \,\} \\ &= \{\, k : \text{A7947}(k) > \text{A3557}(k) \,\} \\ &= \{\, k : \varkappa > m \,\}.\end{aligned} \qquad [1.2]$$

Smallest terms of A341645 appear below:

```
1, 4, 8, 9, 16, 25, 27, 32, 36, 48, 49, 54, 64, 72, 81,
96, 100, 108, 121, 125, 128, 144, 160, 162, 169, 192,
196, 200, 216, 224, 225, 243, 250, 256, 288, 289, 320,
324, 343, 361, 375, 384, 392, 400, 405, 432, 441, ...
```

Smallest terms of A341646 appear below:

```
2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20,
21, 22, 23, 24, 26, 28, 29, 30, 31, 33, 34, 35, 37, 38,
39, 40, 41, 42, 43, 44, 45, 46, 47, 50, 51, 52, 53, 55,
56, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, ...
```

Theorem 1. A1694 ⊂ A341645. Powerful numbers are contained in A341645.

Proof. Powerful number k is such that all prime power factors $p^\delta \mid k$ have multiplicity $\delta > 1$ Therefore, $\textsc{rad}(k) \le k/\textsc{rad}(k)$, since the least powerful case for any squarefree $\varkappa$ is $\varkappa^2$. ∎

Corollary 1.1. A052485 ⊂ A341646. Weak numbers (those that are not powerful) are contained in A341646.

Corollary 1.2. A341645 contains multus A246547 and plenus A286708, since A1694 \ A246547 = A286708.

Corollary 1.3. A1597 ⊂ A341645. Perfect powers are contained in A341645, since A1597 ⊂ A1694. Hence these are not in A341646.

Corollary 1.4. A052486 ⊂ A341645. Achilles numbers are contained in A341645, since A052486 = A1694 \ A1597. This implies A052486 and A341646 do not meet.

Theorem 2. Outside of empty product, squarefree numbers are such that $\varkappa > m$.

$$\text{A5117} \setminus \{1\} \subset \text{A341646} \qquad [1.3]$$

Proof. For $k = 1$, $\textsc{rad}(1) = (1/\textsc{rad}(1)) = 1$, a special case. For squarefree k, $\textsc{rad}(k) = \varkappa = k$, hence $m = k/\textsc{rad}(k) = 1$ and $\varkappa > m$. ∎

Corollary 2.1. Empty product 1 is the only squarefree number in A341645, the rest of the squarefree numbers are in A341646.

Corollary 1.2 and Theorem 2 stimulate interest in that subset of A341645 that is not powerful. For this reason, we define A366250 to be as follows:

$$\text{A366250} = \text{A341645} \setminus \text{A001694} \qquad [1.4]$$

Through Corollary 1.2, we find interest in the following sequences, since they contain A366250. Via Corollary 2.1, we obtain the sequence $\mathbb{N} \setminus$ A5117 = A013929, numbers not squarefree.

Within nonsquarefree A013929, we eliminate powerful numbers and therefore obtain the following:

$$\begin{aligned}\text{A332785} &= \text{A013929} \setminus \text{A1694} \\ &= \text{A126706} \setminus \text{A286708}.\end{aligned} \qquad [1.5]$$

Lemma 3. Kernel ratio smaller than second least distinct prime factor, $m < p_2$, implies m is a prime power $p_1{}^\delta$ such that $\delta < \log p_2 / \log p$.

$$\begin{aligned}&k/\textsc{rad}(k) < p_2 \rightarrow k/\textsc{rad}(k) = \textsc{lpf}(k)^\delta,\ \delta < \log p_2 / \log p. \\ &k \in \text{A360767} \rightarrow k/\textsc{rad}(k) = \textsc{lpf}(k)^\delta,\ \delta < \log p_2 / \log p.\end{aligned} \qquad [1.6]$$

Proof. The inequality $m < p_2$ implies $p_2 \nmid m$, hence the only prime divisor common to m and k that remains available is $p_1 = \textsc{lpf}(k)$. Then some freedom remains to find a prime power $p_1{}^\delta < p_2$. ∎

Theorem 4. There is at least 1 strictly superior squarefree divisor d for $k \in$ A332785 such that $(m < p_2)$, i.e., A3557$(k) <$ A119288(k).
Proof. The inequality $m < p_2$ implies m is some power $p_1^{\delta} < p_2$, therefore, multiplicity is constrained thus: $\delta < \log p_2 \,/\, \log p$, as shown by Lemma 3. We may rewrite k instead as follows:

$$k = p_1^{(\delta+1)} \times p_2 \times Q,\ Q \geq p_2,\ p_1 \nmid Q. \qquad [1.7]$$

Then since $p_1^{\delta} < p_2$, we have $m < p_2 < \text{RAD}(k)$, since both $p_1 \mid \text{RAD}(k)$ and $p_2 \mid \text{RAD}(k)$. ■

Corollary 4.1: A341645 ∩ A360767 = ∅, i.e., weak tantus k have at least 1 strictly superior squarefree divisor, i.e., $\varkappa = \text{RAD}(k)$.

Lemma 5. Thin tantus numbers are even.
Consider $k \in$ A126706 such that $m < q$. Such k is even. This is to say that only even k are such that A3557$(k) <$ A053669(k), that is, numbers $k \in$ A363082 are even.
Proof. Suppose k is odd, which implies $q = 2$. Then it is impossible to find $m : m \mid \text{RAD}(k)$, $m > 1$ to satisfy $m < q$ has us attempt to find some integer $1 < m < 2$, a contradiction. ■

Theorem 6. There is at least 1 strictly superior divisor d of numbers $k \in$ A332785 such that $m < q$, i.e., A3557$(k) <$ A053669(k).
Proof. We need to show $m < \text{RAD}(k)$, but if we see that $q < \text{RAD}(k)$, then we are assured of the former. Through Lemma 5, we know that k is even. We proceed by trying to maximize q, which implies primorial kernel $\varkappa = \mathcal{P}(j)$, i.e., $\text{RAD}(k) \in$ A2110, thus $k \in$ A059932.

Case $\varkappa = \mathcal{P}(j)$.

Consider $\varkappa = \mathcal{P}(2) = 6$, hence $q = \text{PRIME}(3) = 5$. Clearly it is true that $m < \text{RAD}(k)$ for numbers with a primorial kernel $\varkappa = 6$.

Now suppose $\varkappa = \mathcal{P}(3) = 30$, hence $q = \text{PRIME}(4) = 7$. It is also easy to see $m < \text{RAD}(k)$ remains true. Furthermore, via induction on j, we see that $k \in$ A059932 is such that $m < \text{RAD}(k)$.

Case $\varkappa = \mathcal{P}(j) \times Q(r, s)$.

Let $Q(r, s)$ be the product of r consecutive primes beginning with $\text{PRIME}(j + s + 1)$.

$$Q(r, s) = \prod_{i=1}^{r} \text{PRIME}(i + j + s - 1). \qquad [1.8]$$

Suppose $\text{RAD}(k) = \mathcal{P}(1) \times Q(1, 1) = 2 \times 5 = 10$. Then $q = \text{prime}(j{+}1) = 3$.

It is clear through induction on j, r, or s that $m < \text{RAD}(k)$ in all cases. The case of a wide gap in distinct prime divisors (large s) in an even number only increases $\text{RAD}(k)$, and that the optimum case for $m > \text{RAD}(k)$ pertains to $\varkappa = \mathcal{P}(j)$.

Therefore, $m < q < \text{RAD}(k)$ for $k \in$ A332785 such that $m < q$. ■

Corollary 6.1: A341645 ∩ A363082 = ∅, i.e., thin tantus k have at least 1 strictly superior squarefree divisor, i.e., $\text{RAD}(k)$.

Now we are interested in "panstitutive" numbers $k \in$ A361098, that is, those k that are neither prime powers nor squarefree such that neither p_2 nor q exceed m. Since powerful tantus A286708 ⊂ A341645, we are specifically interested in the following sequence:

$$\text{A364702} = \text{A361098} \setminus \text{A286708}. \qquad [1.9]$$

We know that A341645 intersects A364702 because 48 is the minimum element of A364702, and $\text{RAD}(48) < 48/\text{RAD}(48)$, i.e., $6 < 8$.

Therefore we define the following sequence to be as follows:

$$\text{A366250} = \text{A341645} \setminus \text{A1694}.$$
$$\text{A366250} \subset \text{A364702}. \qquad [1.10]$$

This sequence was conceived by Peter Munn on 4 February 2024.

The first terms of A366250 are shown below:

```
48, 54, 96, 160, 162, 192, 224, 250, 320, 375, 384, 405,
448, 486, 567, 640, 686, 704, 768, 832, 896, 960, 1029,
1080, 1200, 1215, 1250, 1280, 1350, 1408, 1440, 1458,
1500, 1536, 1620, 1664, 1701, 1715, 1792, 1875, ...
```

A364702(2) = 50, but $\text{RAD}(50) > 50/\text{RAD}(50)$, i.e., $10 > 5$.

A341645 and A341646 as Partitions of $\varkappa$-Coregular Sequences.

We may regard A341645 and A341646 as containing finite and infinite $\varkappa$-coregular blocks, respectively, where $\varkappa > 1$ is squarefree.

Define the $\varkappa$-regular sequence $\boldsymbol{R}_\varkappa$, squarefree $\varkappa > 1$ to be as follows:

$$\boldsymbol{R}_\varkappa = \bigotimes_{p|\varkappa} \{\, p^{\varepsilon} : \varepsilon \geq 0 \,\}$$
$$= \{\, m : \text{RAD}(m) \mid \varkappa \,\}. \qquad [2.1]$$

Note that $\boldsymbol{R}_1 = \{1\} \subset$ A341645.
Then the $\varkappa$-coregular sequence $\varkappa\boldsymbol{R}_\varkappa$ as follows:

$$\varkappa\boldsymbol{R}_\varkappa = \{\, m \times \varkappa : \text{RAD}(m) \mid \varkappa \,\}. \qquad [2.2]$$

It is clear that these sequences are countably infinite.

Also evident is that the minimum element in $\varkappa\boldsymbol{R}_\varkappa$ is squarefree $\varkappa$, and all the rest of the terms in $\varkappa\boldsymbol{R}_\varkappa$ are nonsquarefree. Furthermore, $\omega(\varkappa) = 1$ implies the rest of the terms are multus, while $\omega(\varkappa) > 1$ implies the rest of the terms are tantus.

Sequences A341645 and A341646 have to do with the relationship of $\varkappa = \text{RAD}(k)$ and $m = k/\text{RAD}(k)$ and it is clear from definitions that m is $\varkappa$-regular, hence in $\boldsymbol{R}_\varkappa$. Therefore we expect a situation wherein $\varkappa\boldsymbol{R}_\varkappa$ is partitioned by magnitude, with a finite interval in A341646 including terms that do not exceed $\varkappa^2$, and the infinite balance in A341645.

We partition $\varkappa\boldsymbol{R}_\varkappa$ thus:

$$\varkappa\boldsymbol{R}_\varkappa \setminus S = T$$
$$\text{where } S = \{\, k = m \times \varkappa : \text{RAD}(m) \mid \varkappa,\ m < \varkappa \,\}$$
$$= \{\, k = m \times \varkappa : \text{RAD}(m) \mid \varkappa,\ k < \varkappa^2 \,\}$$
$$T = \{\, k = m \times \varkappa : \text{RAD}(m) \mid \varkappa,\ 1 < \varkappa < m \,\}$$
$$= \{\, k = m \times \varkappa : \text{RAD}(m) \mid \varkappa,\ k \geq \varkappa^2 \,\}. \qquad [2.3]$$

This leads us to the following tautological theorem:

Theorem 7. $S \subset$ A341646 and $T \subset$ A341645.

Example:

$6\boldsymbol{R}_6 = \{\, 6m : \text{RAD}(m) \mid 6 \,\} =$ A033845.
$S_6 = \{6, 12, 18, 24\}$ are such that $\text{RAD}(k) > k/\text{RAD}(k)$, while
$T_6 = \{36, 48, 54, \ldots\}$ are such that $\text{RAD}(k) \leq k/\text{RAD}(k)$.

Corollary to this are the following:

$$\text{A1694} \subset \text{A341645},\ \text{A5117} \setminus \{1\} \subset \text{A341646}.$$

Therefore we may write the following:

$$\text{A341646} = \bigcup \{\, k = m \times \varkappa : \text{RAD}(m) \mid \varkappa,\ k < \varkappa^2 \,\}.$$
$$\text{A341645} = \bigcup \{\, k = m \times \varkappa : \text{RAD}(m) \mid \varkappa,\ k \geq \varkappa^2 \,\} \cup \{1\}. \qquad [2.4]$$

Eliminating perfect powers of primes from A341645, we derive the following formulas for A366250:

$$\text{A366250} = \bigcup \{\, k = m \times \varkappa : \omega(\varkappa) > 1,$$
$$\text{RAD}(m) \mid \varkappa,\ k \geq \varkappa^2 \,\} \cup \{1\}$$
$$= \{\, k \in \text{A364702} : k \geq \text{RAD}(k)^2 \,\}$$
$$= \{\, k : \Omega(k) > \omega(k) > 1,\ \exists\, p^{\delta} \mid k : \delta = 1,$$
$$\varkappa \leq m,\ q < m \,\}. \qquad [2.5]$$

The sequence A366250 contains $k \in$ A364702 that are at least as large as $\text{RAD}(k)^2$.

Figures A1 and A2 show terms in A366250 in context of A364702, while figures B1 and B2 show terms in A366250 in context of A341645. These demonstrate density patterns of A366250 within the above mentioned sequences.

Additional Thoughts.

We present an extra theorem and corollary relating to minimally tantus numbers $k \in$ A366825. Note that A366825 ⊂ A332785, but A366825 does not meet A364702. Therefore Theorems 4 and 6 confirm the following theorem.

Theorem 8. There is at least 1 strictly superior squarefree divisor d for minimally tantus $k \in$ A366825, where A366825 is defined as follows:

$$\text{A366825} = \{\, k = \text{LPF}(k)^2 \times \text{RAD}(k) : \omega(k) > 1 \,\}. \qquad [3.1]$$

Proof. $\text{RAD}(k) > \text{LPF}(k)^2$ by definition of k and squarefree kernel $\text{RAD}(k)$. ■

Corollary 8.1. A341645 ∩ A366825 = ∅ and A341646 ⊂ A366825, i.e., minimally tantus k have at least 1 strictly superior squarefree divisor, i.e., $\text{RAD}(k)$.

We note a similarity of the definitions in [2.4] with that of A055932.

$$\text{A055932} = \text{V0210} = \{\, k : \text{RAD}(k) = \text{A2110}(\omega(k)) \,\}$$
$$= \{\, k : \prod_{i=1}^{j} p_i^{\delta_i}, \delta_i > 0, j = \omega(k) \,\}. \qquad [3.2]$$

Theorem 9. $k \in$ A055932 implies $\{\varkappa \boldsymbol{R}_\varkappa\} \subset$ A055932.

Proof. Since A055932 is the sequence of numbers that have a primorial kernel $\mathcal{P}(j)$, $k \in$ A055932 implies $\text{RAD}(k) = \mathcal{P}(j)$ for $j = \omega(k)$. This in turn implies $k = m \times \mathcal{P}(j)$, $\text{RAD}(m) \mid \mathcal{P}(j)$. From the latter, clearly, any number $R = m \times \mathcal{P}(j)$, $\text{RAD}(m) \mid \mathcal{P}(j)$ has a primorial kernel $\mathcal{P}(j)$ and is an element of A055932. ■

Corollary 9.1. A055932 is the union of primorial-coregular sequences as described below:

$$\text{A055932} = \bigcup_{j \ge 0} \{\, \mathcal{P}(j) \boldsymbol{R}_{\mathcal{P}(j)} \,\}$$
$$= \bigcup_{j \ge 0} \{\, m \times \mathcal{P}(j) : \text{RAD}(m) \mid \mathcal{P}(j) \,\}. \qquad [3.3]$$

Therefore, A341646 and A341645 represent a partition of $\varkappa$-coregular sequences $\varkappa \boldsymbol{R}_\varkappa$, $\varkappa > 1$, into a finite interval $k < \varkappa^2$ and an infinite interval $k \ge \varkappa^2$, respectively, while A055932 and A080259, represent unions of $\mathcal{P}(j)$-coregular and non-primorial-coregular sequences, respectively.

Conclusion.

We have demonstrated that powerful numbers are contained by A341645, but aside from empty product, squarefree numbers do not appear in that sequence. This leaves the balance of non-powerful numbers in A341645 to appear in A332785 (carens tantus). Theorems 4 and 6 show that A341645 is comprised of all powerful numbers and some numbers in A364702.

This leads to the new sequence A366250 = A341645 \ A1694, a proper subset of A364702.

Regarding $\varkappa$-coregular sequences, we show that we may break the $\varkappa$-coregular sequence where $\varkappa > 1$ is squarefree, into a finite block that appears in A341646, while the balance appears in A341645 as shown by [2.4]. This leads to several formulas for A366250 in [2.5].

The sequence A366250 contains $k \in$ A364702 that are at least as large as $\text{RAD}(k)^2$. ⁂

Concerns Sequences:

A001597, A001694, A002110, A003557, A005117, A007947, A020639, A052485, A052486, A053669, A055932, A080259, A119288, A126706, A246547, A286708, A341645, A341646, A360765, A360767, A360768, A361098, A363082, A364702, A366250, A366825. [1]

Vinci Catalog:

(V2, V3, V4, V5, V7, V8, V70, V73, V74, V88, V0119, V0210, V0220, V0222, V0309, V0700, V0701, V0702, V0703, V1001, V1002, V3300, V3301, V3302.)

Acknowledgement

Peter Munn inspired examination of this problem through personal correspondence 4 February 2024.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved February 2024.

Code:

[C1] Generate powerful numbers A1694:

```
a1694 = With[{nn = 2^40},
   Union@ Flatten@
     Table[a^2*b^3, {b, nn^(1/3)}, {a, Sqrt[nn/b^3]}]];
```

[C2] Generate tantus numbers A126706:

```
a126706 = Block[{k}, k = 0;
  Reap[Monitor[Do[
    If[And[#2 > 1, #1 != #2] & @@
        {PrimeOmega[n], PrimeNu[n]},
      Sow[n]; Set[k, n] ],
    {n, 2^20}], n]][[-1, -1]]]
```

[C3] Generate tantus numbers A364702:

```
a364702 =
  Block[{q},
   Select[a126706,
    And[#1 >= #2, #1 > #3,
     ! AllTrue[#4, # > 1 &]] & @@
     {#1/(Times @@ #2[[All, 1]]),
        #2[[2, 1]], #3, #2[[All, -1]]} & @@
     {#, FactorInteger[#],
       If[OddQ[#], 2, q = 3;
         While[Divisible[#, q],
         q = NextPrime[q]]; q]} &]];
```

[C4] Generate A341645:

```
a341645 =
  With[{nn = 2^20},
   Union@ Join[TakeWhile[a1694, # <= nn &],
     Select[TakeWhile[a364702, # <= nn &],
      Function[n,
       Count[Divisors[n],
       _?(And[SquareFreeQ[#], # > n/#] &)] == 0] ] ] ];
```

[C5] Generate A366250:

```
Select[a364702[[1 ;; 2^10]],
 Function[n,
  Count[Divisors[n],
  _?(And[SquareFreeQ[#], # > n/#] &)] == 0]]

 (* or *)

Select[a364702[[1 ;; 2^16]],
  # >= Apply[Times, FactorInteger[#][[All, 1]]^2] &]
```

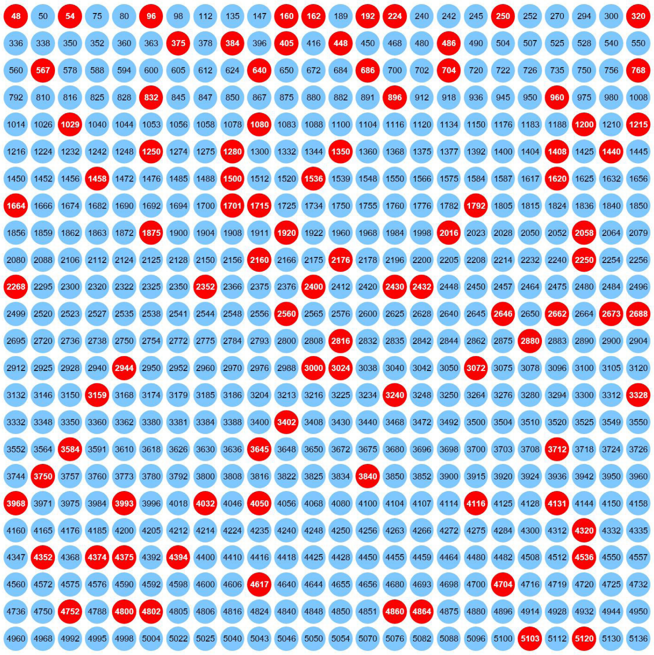

Figure A1: Red indicates terms in A366250 in the context of A364702. Chart shows the smallest 576 terms of A364702.

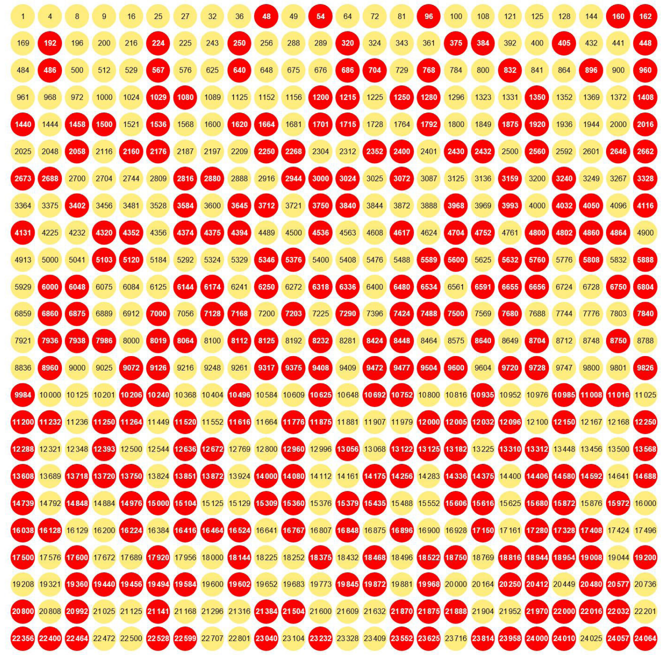

Figure B1: Red indicates terms in A366250 in the context of A341645. Chart shows the smallest 576 terms of A341645.

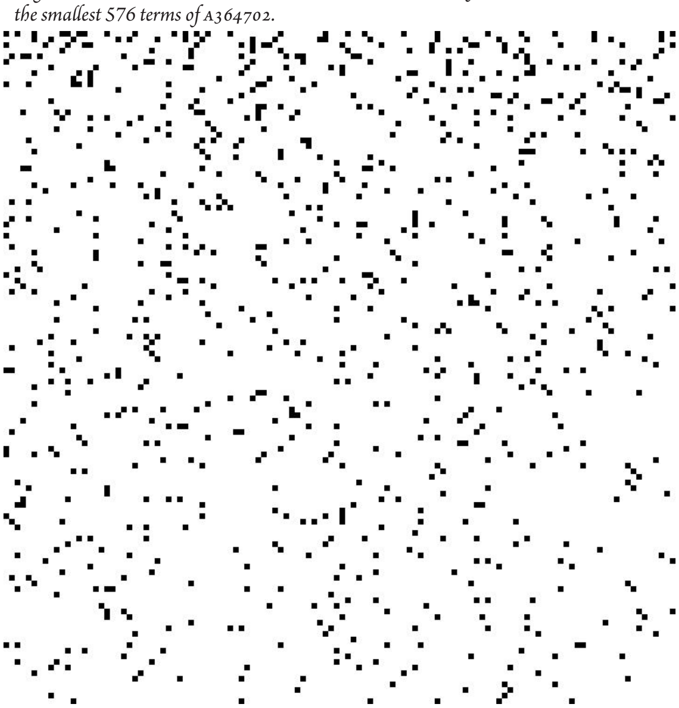

Figure A2: Black indicates terms in A366250 in the context of A364702. Chart shows the smallest 16384 terms of A364702.

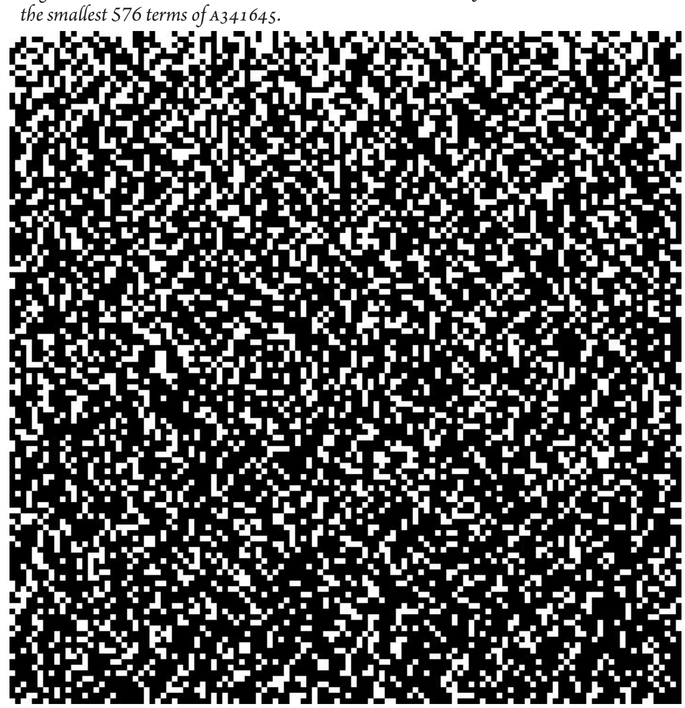

Figure B2: Black indicates terms in A366250 in the context of A341645. Chart shows the smallest 16384 terms of A341645.