Divisibility Based Lexically Earliest Sequence with Cellular Automaton Behavior.

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Abstract.

We examine some qualities of a lexically earliest sequence (LES) based on divisibility of prime p that resemble a 1 dimensional cellular automaton that is affected by multiplication of a squarefree kernel r by kernel register $m(r)$. The sequence moves into and out of "coherence", defined in the paper, and in rare, intermittent, highly quasicoherent phases, admits primes and their perfect powers as well as composite powerful numbers. Otherwise the sequence is dominated by weak composites (as opposed to powerful numbers).

A. Introduction.

David Sycamore wrote an integer sequence A369609 defined to be as follows:

$$a(1) = 1,\ a(2) = 2;$$
$$\text{for } n > 2,\ a(n) = k = m(r) \times r,$$
$$\text{with minimal } k \neq a(j),\ j < n, \text{ where}$$
$$R = \text{RAD}(a(n-2) \times a(n-1)) \text{ and}$$
$$r = R/\text{RAD}(a(n-1)). \qquad [1.0]$$

The RAD(x) function yields the squarefree kernel A7947(x).
First terms of the sequence are shown below (see Code [C1]):

```
1, 2, 3, 4, 6, 5, 12, 10, 9, 20, 15, 8, 30, 7, 60, 14,
45, 28, 75, 42, 25, 84, 35, 18, 70, 21, 40, 63, 50, 105,
16, 210, 11, 420, 22, 315, 44, 525, 66, 140, 33, 280,
99, 350, 132, 175, 198, 245, 264, 385, 24, 770, ...
```

Lemma A1:
Minimal k implies minimal $m(r)$, since r is held constant.

The above lemma implies approaching the solution to $a(n)$ from below via incrementation on $m(r)$. An approach from below (a greedy approach) ensures that no multiple k of the squarefree number r will go missing provided input r materializes infinitely as n increases to infinity.

Let $S = \{\, p : p \mid a(n-2) \,\}$ be the set of prime factors of $a(n-2)$.
Let $T = \{\, p : p \mid a(n-1) \,\}$ be the set of prime factors of $a(n-1)$.

Theorem A2. $r = \prod(S \setminus T)$, that is, r is the product of the set difference of S and T.
Proof. The expression $\prod(S \setminus T)$ signifies removal of any prime p such that $p \mid a(n-1)$ from set S of primes p that also divide $a(n-2)$. We are left with a product r of primes p such that while $p \mid a(n-2)$, the same prime $p \nmid a(n-1)$.
Expand the expression shown below:

$$\begin{aligned} r &= R/\text{RAD}(a(n-1)) \\ &= \text{RAD}(a(n-2) \times a(n-1)) \,/\, \text{RAD}(a(n-1)) \end{aligned} \qquad [1.1]$$

The result essentially removes any prime p such that $p \mid a(n-1)$ from r, leaving us with the same product of primes p that divide $a(n-2)$ but do not divide $a(n-1)$. Logically, we may write the following equivalent expression:

$$\begin{aligned} r &= \prod \{\, p : p \mid a(n-2) \wedge p \nmid a(n-1) \,\} \\ &= \prod \{\, p : p \mid a(n-2) \vee p \mid a(n-1) \,\} \,/\, \prod \{\, p : p \mid a(n-1) \,\} \\ &= \prod(S \setminus T). \end{aligned} \qquad [1.2]$$

The expression $r = R/\text{RAD}(a(n-1))$ is necessary to remove primes p that divide $a(n-1)$ by means of simple division. ■

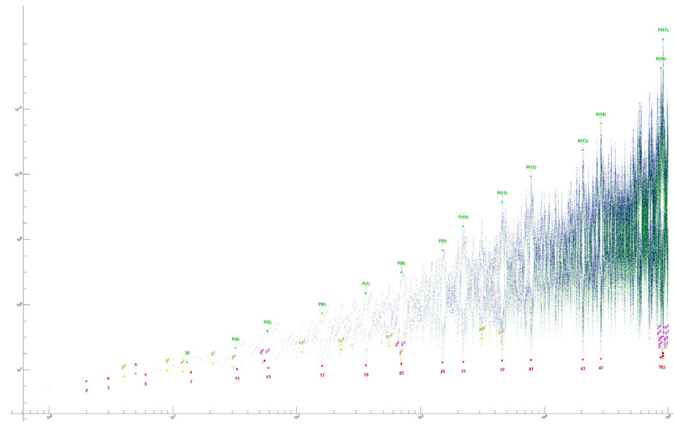

Figure 1. Log log scatterplot of 10^5 terms, showing primes in red, perfect powers of primes in yellow, squarefree composites in green, and numbers neither squarefree nor prime powers in blue or purple. We accentuate powerful numbers that are not perfect powers of primes in purple. Note clustering of powerful numbers near $n = 10^5$ and seeming association between powers of 2, primorials, and primes in the sequence for small values of n.

Given Lemma A1 and Theorem A2, we may approach generation of the sequence through the following practical means. A priori, we set $m(r) = 1$ for all r. Upon input of the kernel r, we increment $m(r)$ until $m(r) \times r \neq a(j)$, $j < n$. Hence, $m(r)$ behaves as a sort of counter or register that needs adjustment for the occasion $a(j) = m(r) \times r$, $j < n$, in other words, when the product already appears in the sequence. A natural consequence is that $a(n)$ are distinct.

We define a 2-input function $f(x, y)$ defined to be as shown below:

$$f(x, y) = m(r)^{++} \times r,\ r = \prod(\{\, p : p \mid x \,\} \setminus \{\, p : p \mid y \,\}). \qquad [1.1]$$

The result of this function is a multiple of the kernel r. Suppose that we apply the function $f(x, y)$ given x and y for the first time. Then we have the result $m(r) \times r = 1 \times r = r$. Suppose that we reiterate the function given the same input. Then we have the result $2 \times r$. A third iteration gives the output $3 \times r$, and so on. This function implies global management of the register $m(r)$. Through this function we may rewrite the sequence definition instead as follows:

$$a(1) = 1,\ a(2) = 2;$$
$$\text{for } n > 2,\ a(n) = k = f(x, y),$$
$$\text{iterating } f(x, y) \text{ until } k \neq a(j),\ j < n. \qquad [1.2]$$

Thus we describe practical means by which we may compute many of terms of the sequence, limited only by the efficacy of implementation of the RAD function, which requires factorization.

General Observations.

Examining the first 2^{26} terms, several conjectures seem evident.

Conjecture A. There is a chain $2^i \to \mathcal{P}(i) \to \text{PRIME}(i{+}1)$, where $\mathcal{P}(i)$ is the product of the smallest i primes, i.e., primorial A2110(i).

Examples include `{4, 6, 5}`, `{8, 30, 7}`, and `{16, 210, 11}`. See Appendix Tables A and B.

The conjecture is FALSE, since $a(59) = 13$ but $a(57) = 26$. Furthermore, $a(621674) = 67$ but the term preceding it is not a primorial.

Table 1: Composition of smallest 62 terms

. indicates p divides neither r nor m(r),
hence p does not divide a(n).
o indicates p | r
x indicates p | m(r).
* indicates p divides both r and m(r).

n	a(n)	prime p 2 3 5 7 11 13	r	m(r)	Cases (see Table 2) 2 3 5 7 11 13	
1	1	......	1	1	.	< empty product
2	2	x.....	1	2	F	< prime(1) = P(1)
3	3	.x....	1	3	gF	< prime(2)
4	4	*.....	2	2	Bg	< 2^2
5	6	xo....	3	2	Ha	< P(2)
6	5	..x...	1	5	CgF	< prime(3)
7	12	*o....	6	2	Bag	
8	10	x.o...	5	2	Hga	
9	9	.*....	3	3	CBg	< 3^2
10	20	*.o...	10	2	Bga	
11	15	.ox...	3	5	gaH	
12	8	*.....	2	4	BgC	< 2^3
13	30	xoo...	15	2	Haa	< P(3)
14	7	...x..	1	7	CggF	< prime(4)
15	60	*oo...	30	2	Baag	
16	14	x..o..	7	2	Hgga	
17	45	.*o...	15	3	CBag	
18	28	*..o..	14	2	Bgga	
19	75	.o*...	15	5	gaBg	
20	42	ox.o..	14	3	aHga	
21	25	..*...	5	5	gCBg	< 5^2
22	84	*o.o..	42	2	Baga	
23	35	..ox..	5	7	ggaH	
24	18	o*....	6	3	aBgC	
25	70	x.oo..	35	2	Hgaa	
26	21	.o.x..	3	7	CagH	
27	40	*.o...	10	4	BgaC	
28	63	.*.o..	21	3	gBga	
29	50	o.*...	10	5	agBg	
30	105	.oxo..	21	5	gaHa	
31	16	*.....	2	8	BgCg	< 2^4
32	210	xooo..	105	2	Haaa	< P(4)
33	11	....x.	1	11	CgggF	< prime(5)
34	420	*ooo..	210	2	Baaag	
35	22	x...o.	11	2	Hggga	
36	315	.*oo..	105	3	CBaag	
37	44	*...o.	22	2	Bggga	
38	525	.o*o..	105	5	gaBag	
39	66	ox..o.	22	3	aHgga	
40	140	x.oo..	35	4	HCaag	
41	33	.o..o.	33	1	Cagga	
42	280	*.oo..	70	4	Bgaag	
43	99	.*..o.	33	3	gBgga	
44	350	o.*o..	70	5	agBag	
45	132	xo..o.	33	4	Hagga	
46	175	..*o..	35	5	CgBag	
47	198	o*..o.	66	3	aBgga	
48	245	..o*..	35	7	ggaBg	
49	264	*o..o.	66	4	Bagga	
50	385	..oox.	35	11	ggaaH	
51	24	*o....	6	4	BaggC	
52	770	x.ooo.	385	2	Hgaaa	
53	27	.*....	3	9	CBggg	< 3^3
54	1540	*.ooo.	770	2	Bgaaa	
55	36	x*....	3	12	HBggg	< 2^2*3^2
56	1155	.xooo.	385	3	CHaaa	
57	26	o....x	2	13	aCgggF	
58	2310	xoooo.	1155	2	Haaaag	< P(5)
59	13	.....o	13	1	Cgggga	< prime(6)
60	4620	*oooo.	2310	2	Baaaag	
61	39	.x...o	13	3	gHggga	
62	3080	*.ooo.	770	4	BCaaag	
63	78	xo...o	39	2	Haggga	
64	1925	..*oo.	385	5	CgBaag	
65	156	*o...o	78	2	Baggga	
66	2695	..o*o.	385	7	ggaBag	
67	234	o*...o	78	3	aBggga	
68	3465	.xooo.	385	9	gHaaag	
69	52	*....o	26	2	BCggga	
70	5775	.o*oo.	1155	5	gaBaag	

Conjecture A.1. Primes appear in order as n increases. The conjecture is false; $a(87723) = 59$ but $a(91307) = 53$. See Appendix Table A, Note A.

Conjecture A.2. Primorials appear in order as n increases. The conjecture is false; $a(28709) = \mathcal{P}(14)$ and $a(87722) = \mathcal{P}(16)$; for $n \le 2^{26}$, $\mathcal{P}(15)$ has not appeared. See Appendix Table B, Note B.

Conjecture A.3. Powers of 2 appear in order as n increases. This conjecture seems to be true, but we see the following. For $n \le 2^{26}$, no power of 2 that exceeds 32 appears; the last power of 2 seen is $a(699) = 32$, but those powers of 2 that do appear indeed occur in order as n increases. (See Theorem G2.)

Conjecture B. Powerful numbers appear in clusters, e.g., for n roughly between 91200 and 91320. See Appendix Table D.

Conjecture C. A369609 is a permutation of natural numbers.

Therefore we can show by construction that there does not exist a chain $2^i \rightarrow \mathcal{P}(i) \rightarrow \text{PRIME}(i+1)$ except for $i < 5$. We note that 32 precedes $\mathcal{P}(8) \rightarrow \text{PRIME}(9) = 23$ (see Appendix Table F5), and that $\mathcal{P}(i) \rightarrow \text{PRIME}(i+1)$ occurs more often, yet not always.

These conjectures inspire us to undertake further examination of OEIS A369609.

B. Sequence Mechanics.

Given a dataset of terms, sensing prime factors of terms are kept small, we endeavor to examine the nature of kernel r and multiplier $m(r)$. The following notion is aided by Theorem A1 above and Theorems 5 and 8, and Corollary C4.1 below.

$$p \le \text{GPF}(a(n-1)) + 2. \qquad [2.1]$$

As a consequence it is indeed meaningful to examine divisibility patterns among primes p that satisfy [2.1].

We can employ A087207 to visualize prime divisors $p \mid a(n)$, where A087207 is defined to be as follows:

$$\text{For } x = \prod_{i=1}^{\omega} p_i^{\delta_i},$$

$$\text{A087207}(x) = \sum_{i=1}^{\omega} 2^{\pi(p_i)-1}. \qquad [2.2]$$

In the above, ω signifies the number of distinct prime factors of x. Example: $\text{A087207}(126) = 2^0 + 2^1 + 2^3 = 11$, since $126 = 2 \times 3^2 \times 7$. The function ignores multiplicity of prime power factors, retaining only the prime indices and encoding them in a binary number.

We express $\text{A087207}(r)$ as a series of bits from least to greatest, left to right. For example, we express A087207(126) as "1101" and then we replace 0's with "**.**" and 1's with "**o**" for clarity, thus "**oo.o**".

If we express primes $p \mid m$ instead by "**x**" when p does not also divide r, and by "*****" when both $p \mid r$ and $p \mid m$, we arrive at a compact means of examination of some of the sequence's mechanics.

Therefore, for example, $a(72) = 6930 = 6 \times 1155$, so we perform the following operation:

$$\text{RAD}(m(r)) = \text{RAD}(6) = 2 \times 3 \qquad \mathbf{xx......}$$

$$\text{RAD}(r) = \text{RAD}(1155) = 3 \times 5 \times 7 \times 11 \qquad \mathbf{.oooo...}$$

$$r = \prod(S \setminus T) = \mathcal{P}(5) \qquad \mathbf{x*ooo...} \qquad [2.3]$$

The downside of this protocol is that we lose multiplicity information, but such information merely pertains to the register $m(r)$. We know that $m(r)$ is a greedy function from Lemma A1; its behavior is relatively easy to understand. Therefore, the A087207 protocol focuses on relationships of $\text{PRIME}(i)$ to each of x, y, m, and k with respect to function $f(x, y)$. Table 1 exhibits notation based on A087207 for $a(n)$, $n = 1 \ldots 70$.

Define function $g(r, m(r))$ to be as follows:

$$\text{For } p = \textsc{prime}(i),\ i = 1 \ldots j,$$
$$(p \mid r \Rightarrow 1) + (p \mid m(r) \Rightarrow 2). \qquad [2.4]$$

Function output is an array such that terms are in order of prime i. Then we convert numerical output to symbols using the following replacement rules:

$$\{0 \to \texttt{.},\ 1 \to \texttt{o},\ 2 \to \texttt{x},\ 3 \to \texttt{*}\} \qquad [2.5]$$

Thereby we represent the protocol described above in a logical manner in the form of a function.

Example: for $a(72) = 6930 = 6 \times 1155$, we have $g(1155, 6)$, which yields $\{2, 3, 1, 1, 1, 0, \ldots\}$, and this converts to "**x*ooo. . .**".

C. Divisibility Truth Table.

Recognizing that the RAD function requires factorization, we consider the effects of the definition of $f(x, y)$ as regards divisibility of x, y, and m by primes p.

First, we present several corollaries that follow from the expression $k = m(r) \times r$:

Corollary C1.1. $p \mid a(n-2)$ but $p \nmid a(n-1)$ implies $p \mid a(n)$ (See cases ⒶⒷ).

Corollary C1.2. Primes p that divide both $a(n-2)$ and $a(n-1)$ do not also divide $a(n)$ unless $p \mid m$ (See cases ⒸⒹ).

Corollary C1.3. $p \mid a(n-1)$ but $p \nmid m$ implies $p \nmid a(n)$ unless $p \mid a(n-2)$ (See cases ⒸⒹⒼⒽ).

Corollary C1.4. $p \mid m$ implies $p \mid a(n)$ (see cases ⒷⒹⒻⒽ).

These inspire thought regarding a truth table whose values are consequences of sequence definition. We recognize, vis à vis the function $f(x, y) = k$, that $x = a(n-2)$, $y = a(n-1)$, and $k = a(n)$, where the latter is accepted as a solution provided $k \neq a(j)$, $j < n$.

Theorem C1. The truth table above is equivalent to the following logical formula: $(\, p \mid a(n-2) \wedge p \nmid a(n-1)\,) \vee p \mid m.$ [3.0]

This summarizes the Corollaries 1.1-1.4.

We examine some basic divisibility patterns, and conclude that R is a primorial.

Theorem C2. Case Ⓒ implies $p \nmid a(n)$, but $p \mid a(n+1)$.
Proof. With respect to $a(n+1)$, Case Ⓒ furnishes either Case Ⓐ or Case Ⓑ, both of which results in $p \mid a(n+1)$. ■

Theorem C3. Case Ⓖ implies $p \nmid a(n)$, but $p \mid a(n+1)$.
Proof. With respect to $a(n+1)$, Case Ⓖ furnishes either Case Ⓐ or Case Ⓑ, both of which results in $p \mid a(n+1)$. ■

Theorem C4. $p \mid a(j)$ implies either $p \mid a(j+1)$ or $p \mid a(j+2)$.
Proof. $p \mid a(j+1)$ results from $p \mid m$ via either Case Ⓓ or Case Ⓗ, while $p \mid a(j+2)$ results from either Case Ⓐ or Case Ⓑ. ■

Corollary C4.1. Both $a(n-2)$ and $a(n-1)$ are such that R is a primorial, i.e., $R = \mathcal{P}(i) =$ A2110 (i).

We present constraints on the constitution of k, i.e., prime power decomposition of k, given that R is a primorial.

Let $Q = \textsc{gpf}(R) =$ A6530(R).

Theorem C5. Q is nondecreasing as n increases.
Proof. Consequence of Case Ⓕ and Theorem C4.

Theorem C6. Both $R(n) = \mathcal{P}(i+1)$ and $a(n) = \mathcal{P}(i)$ imply the following:

$$a(n+1) = \textsc{prime}(i+1). \qquad [3.1]$$

Table 2.

	x	y	m	$a(n)$	$a(n+1)$	sym.
Ⓔ	·	·	·	·	Ⓔ	`. . → .`
Ⓕ	·	·	⊤	⊤	ⒼⒽ	`. . → x`
Ⓖ	·	⊤	·	·	ⒶⒷ	`.@ → .`
Ⓗ	·	⊤	⊤	⊤	ⒸⒹ	`.@ → x`
Ⓐ	⊤	·	·	⊤	ⒼⒽ	`@. → o`
Ⓑ	⊤	·	⊤	⊤	ⒼⒽ	`@. → *`
Ⓒ	⊤	⊤	·	·	ⒶⒷ	`@@ → .`
Ⓓ	⊤	⊤	⊤	⊤	ⒸⒹ	`@@ → x`

Table 2 shows "." if prime p does not divide or "⊤" if p divides the entity shown in the column heading. The a(n+1) column shows possible cases that follow the case listed in the first column. The "sym." column refers to the A087207 protocol function g defined as follows: "@" represents general divisibility, "." represents general indivisibility, "o" represents $p \nmid r \wedge p \nmid m$, "x" represents $p \nmid r \wedge p \mid m$, and "" represents $p \mid r \wedge p \mid m$. The arrow indicates output. For example, Case Ⓑ represents `@.` → x, which means that $p \mid x$ and $p \mid m$, but $p \nmid y$. Since both $p \mid r$ and $p \mid m$, we have* **x**.

Proof. Consequence of Case Ⓕ and Theorems 5 and 6.

Theorem C7. For any kernel r, there are $2^{\omega(k)}$ combinations of factors of r.
Proof. The kernel r is squarefree by definition, the result of taking squarefree kernels. The divisor counting function $\tau(r)$ is defined to be as follows:

$$\tau(r) = \prod_{p^\delta \mid r} \delta + 1,$$
$$\text{where } \delta \text{ is maximal such that } p^\delta \mid r. \qquad [3.2]$$

Since r is squarefree, $\delta = 1$ in all cases, therefore, we have $2^{\omega(k)}$ divisors of r. ■

Corollary C7.1. Powerful number $k \in$ A1694 implies $m(r) \geq r$.
Proof. A powerful number k is such that $\textsc{rad}(k)^2 \mid k$, hence, since both $m(r) \mid k$ and $r \mid k$, with r squarefree such that $r \leq \textsc{rad}(k)$, we minimize $m(r)$ by maximizing r, which occurs when $r = \textsc{rad}(k)$. Therefore the multiplier $m(r)$ for squarefree r must be at least as large as r. ■

Corollary C7.2. Perfect prime power $k = p^\delta$, i.e., $k \in$ A246547, implies $m(r) \geq r$, where $r = p$, hence, $m(p) \geq p$.

Theorem C8. Regarding an arbitrary index $n > 1$, let $Q = \textsc{gpf}(R)$ and let $\mathcal{M}$ be the maximum value of $m(r)$. Then we have the following:

$$\mathcal{M} < \textsc{prime}(\pi(Q)+1). \qquad [3.3]$$

Proof. Consequence of Case Ⓕ and Theorem C5.
See Table F15A for values of $\mathcal{M}$ for $n \leq 2^{27}$.

Corollary C8.1. $\mathcal{M}$ implies no powerful number k can appear as $a(j)$, $j \leq n$, such that $\textsc{rad}(k) > \mathcal{M}$.
Example: if $\textsc{gpf}(R) = Q = 19$, then $\mathcal{M} < 23$, hence there can be no powerful number $k = a(1 \ldots n)$ such that $k \geq 23^2 = 529$, which is equivalent to saying $\textsc{rad}(k) \geq 23$.

Corollary C8.2. The largest powerful number K in the sequence is governed by Q such that $K < \textsc{prime}(\pi(Q)+1)^2$.

Theorem C9. Let s be a squarefree number. All s may appear in the sequence. Consequence of Corollary 1.4, i.e., Cases ⒶⒷⒹⒻⒽ.

Table 2 summarizes logic in the above theorems and corollaries.

D. Extended Divisibility Patterns.

Through the logical formula [3.0] in Theorem C1 and the truth table (Table 2), we explore extended patterns of divisibility of x, y, m, and k by a given prime p by assuming both $p \mid m$ and $p \nmid m$, then following the resultant term by setting $x = a(n-1)$ and $y = k$. In this manner we can examine flow structures based on the 8 cases laid out in the truth table.

Theorem D1. Some extended divisibility patterns that are consequences of the truth table, replacing ⊤ with 1 for divisibility by p:

Table 3.

`AGA 1.1.1`	`AGB 1.1.1`	`AHC 1.11.`	`AHD 1.111`
`BGA 1.1.1`	`BGB 1.1.1`	`BHC 1.11.`	`BHD 1.111`
`CAG 11.1.`	`CAH 11.11`	`CBG 11.1.`	`CBH 11.11`
`DCA 111.1`	`DCB 111.1`	`DDC 1111.`	`DDD 11111`
`EEE .....`			
`FGA ..1.1`	`FGB ..1.1`	`FHA ..1.1`	`FHB ..1.1`
`GAG .1.1.`	`GAH .1.1.`	`GBG .1.1.`	`GBH .1.1.`
`HCA .11.1`	`HCB .11.1`	`HDC .111.`	`HDD .1111`

Table 3 is a consequence of the 8 cases described in the truth table, i.e., Table 2. In Table 3, we write a string of successive cases followed by the divisibility patterns.

Example: The entry `AGA 1.1.1` represents the following:

Case Ⓐ followed by Case Ⓖ, then in turn followed by Case Ⓐ. Holding n constant, this results in the following:

$p \mid a(n-2)$, $p \nmid a(n-1)$, $p \mid a(n)$, $p \nmid a(n+1)$, $p \mid a(n+2)$.

From this we can see the following divisibility patterns:

Case Ⓐ has $p \mid a(n-2)$, $p \nmid a(n-1)$, and assume $p \mid m$, therefore we have k such that $p \mid k$.

We assume that $a(n) = k$.

Now for $a(n+1)$, we set $x = a(n-1)$ and $y = a(n)$. Assuming $p \mid m$, we obtain k such that $p \nmid k$. We assume that $a(n+1) = k$.

Finally, we we set $x = a(n)$ and $y = a(n+1)$ to project $a(n+2)$. Assuming $p \mid m$, we obtain k such that $p \mid k$.

We may summarize these patterns in the example using the respective cases showin in the truth table:

	x	y	m	k	k'
Ⓐ	⊤	·	·	⊤	ⒼⒽ
Ⓖ	·	⊤	·	·	ⒶⒷ
Ⓐ	⊤	·	·	⊤	ⒼⒽ

For concision, we might abbreviate all of the above example as the entry `AGA 1.1.1`.

Some dependencies based on Table 3:

Corollary D1.1. Cases Ⓐ or Ⓑ lead to Cases Ⓖ or Ⓗ, which in turn lead to Ⓐ, Ⓑ, Ⓒ, or Ⓓ.

Corollary D1.2. Cases Ⓒ or Ⓖ lead to Cases Ⓐ or Ⓑ.

Corollary D1.3. Case Ⓗ leads to Cases Ⓒ or Ⓓ.

Corollary D1.4. A run of repeated Cases Ⓓ implies $p \mid a(n)$ for as long as the run is unbroken by Case Ⓒ.

For $n \le 2^{20}$, Case Ⓓ appears at most only twice in a row. Duplex Ⓓ first appears at $a(1662)$, see Appendix Table F8 for analysis.

Corollary D1.5. Case Ⓓ leads to Case Ⓒ or itself; through Case Ⓒ, to either Ⓐ or Ⓑ.

Theorem D1.6. Cases Ⓐ, Ⓑ, Ⓒ, Ⓓ, Ⓖ, and Ⓗ comprise a closed system. (See Theorem D1.8.)

Corollary D1.7. Case Ⓔ is idempotent, i.e., Case Ⓔ gives rise to itself. This is to say, if a prime p divides none of x, y, or m, then it also does not divide k. Prime p will not divide y so long as it does not divide m as we iterate function f and accept output.

Theorem D1.8. Case Ⓕ introduces prime $p \mid a(n)$ solely through $p \mid m$. Then via Case Ⓖ or Case Ⓗ, p does not divide $a(n+1)$, and thereafter, through either Case Ⓐ or Case Ⓑ, $p \mid a(n+2)$.

Corollary D1.9. Patterns that alternate either Case Ⓐ or Case Ⓑ, followed by Case Ⓖ, imply alternating divisibility by prime p. This is to say, if prime p divides either $a(n-2)$ or $a(n-1)$ but not both, regardless of whether p also divides m, repeats divisibility or nondivisibility of $a(n)$ by p.

Theorem D2. For $p \le Q$, $p \nmid a(n)$ implies $p \mid a(n+1)$. This is to say that, after Case Ⓕ introduces $p \mid a(n)$, such divisibility is interrupted at most by singleton terms as n increases through either Case Ⓓ or Case Ⓖ. Consequence of Theorem C4.

Theorem D3. Change in alternating ⒶⒼ derives from m. Consequence of definitions of Cases in the truth table (Table 2).

Figure 2 summarizes extended divisibility patterns presented through the logic of Tables 2 and 3.

Figure 2.

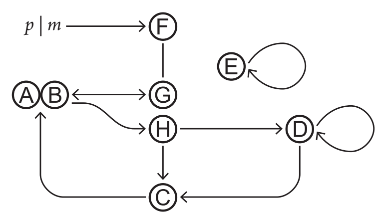

Figure 2 demonstrates the following:

- Repeated Nondivisibility Case Ⓔ
- Introduction of Divisibility Case Ⓕ
- Alternating Divisibility Cases ⒶⒷⒼ
- Repeated Divisibility Cases ⒹⒽ
- Transition from repeated to alternating cases, Case Ⓒ

For prime p, Case Ⓔ implies Case Ⓔ until $m(r)$ increments to p for some r, hence $a(j) = k$ such that $p \mid k$, and we have Case Ⓕ. Therefore, repeated Case Ⓔ represents repeated nondivisibility with respect to $p \mid a(n)$. This repeated nondivisibility is finally broken through Case Ⓕ, introducing divisibility of $a(j)$ by p, thereafter, for $n > j$, through Theorem C4, p divides either $a(n-1)$ or $a(n)$ or both.

In the course of sequence generation for $n > j$, so long as we have duplexes where Cases ⒶⒷ are followed by Case Ⓖ, we have an alternating divisibility pattern. However, if Case Ⓗ comes in place of Case Ⓖ (i.e, in addition to $p \mid a(n-1)$, p also divides m), we exit the alternating divisibility pattern.

Case Ⓗ induces repeated divisibility, that is, p divides both $a(n)$ and $a(n+1)$. We have Case Ⓓ if p divides all of $a(n-2)$, $a(n-1)$, and m; so long as this situation lasts, p divides $a(n)$. When p fails to divide m as n increases, we have Case Ⓒ and we exit repeated divisibility.

Among all the divisibility cases, alternating ⒶⒼ is the commonest. Primes p enter divisibility via sequence …ⒺⒻⒼⒶ… until they are perturbed by $p \mid m$, transmuting Ⓖ to Ⓗ. When Ⓗ is followed by Ⓒ, we have changed index parity of divisibility by p. See Appendix Table E for a study of case frequencies.

This, in a nutshell, completely describes divisibility patterns in this sequence with regard to an arbitrary prime p.

E. Alternating Divisibility Patterns.

Consider the bisection of A369609 by index parity, thus, we create 2 interleaved sequences $a(1, 3, 5, \ldots)$ and $a(2, 4, 6, \ldots)$. Bisection by index parity creates partially dependent sequences.

Define "alternating divisibility (patterns)" to be a sustained relationship $p \mid a(n)$ and $p \mid a(n+2)$ as n increments by 2, which ends when such is no longer true for some n. The qualifier "alternating" is necessary given sequence definition.

Theorem E1. Cases Ⓐ, Ⓑ, and Ⓖ do not affect the other bisection, consequence of Corollary D1.9.

Theorem E2. Cases Ⓗ→Ⓐ, Ⓗ→Ⓑ, and Ⓓ→Ⓓ move divisibility by p to the opposite bisection.

Theorem E3. Case Ⓓ→Ⓒ ends runs of divisibility by p. This is shown by Figure 2 and follows from sequence definition and Table 2.

Given $a(n) = p = \text{PRIME}(i)$ and Theorem C4, we find interest in the duration ℓ of alternating divisibility. We define $\ell(i)$ to be as follows:

$$\text{With } a(n) = p = \text{PRIME}(i) \text{ and } k = 1 \ldots \ell(i)/2, \quad \ell(i) \text{ such that } p \mid a(n+2k). \qquad [5.1]$$

We present data associated with for $n \le 2^{24}$ in Table 4.

Table 4: Alternating divisibility duration ℓ(i).
Key: i = table index, j = prime index, p = prime(j).
a(n) = p = prime(i).

i	j	p	n	ℓ
1	1	2	2	2
2	2	3	3	16
3	3	5	6	4
4	4	7	14	8
5	5	11	33	16
6	6	13	59	52
7	7	17	161	74
8	8	19	363	78
9	9	23	701	164
10	10	29	1509	212
11	11	31	2222	924
12	12	37	4581	1708
13	13	41	7827	7278
14	14	43	20543	4702
15	15	47	28710	23612
16	17	59	87723	3244
17	16	53	91307	97778
18	20	71	384195	338418
19	19	67	621674	126438
20	18	61	810244	86074
21	21	73	1080885	205632
22	22	79	2814146	99986
23	24	89	16009512	612522

The table shows that primes $a(n) = p = \text{PRIME}(i)$ can enjoy protracted alternating divisibility duration. Such protracted duration increases roughly along with the increase in n.

Indeed, certain alternating divisibility duration for prime(i) intercalates with same for prime(i+1), noting index i versus $j = \pi(\text{PRIME}(i))$. For example, 2 divides $a(2)$ and $a(4)$ while 3 divides $a(3)$, $a(5)$, etc. but 5 and 7 do not have intercalated alternating divisibility durations.

Durations of 71 and 67 overlap for 100939 terms, meaning that for $n = 621693 \ldots 722613$, $71 \mid a(n)$ for n odd, and $67 \mid a(n)$ for n even. In these intercalating cases, kernels r that are products of smaller primes are locked out. This comprises some of the reason for paucity of powerful numbers in the sequence, as well as some of the reason for delayed emergence of primes other than the intercalated pair.

Let us examine those numbers that lie within the alternating divisibility duration. More precisely, let us examine an irregular triangle Λ defined to be as follows:

$$\text{For } a(n) = \text{PRIME}(i) = p, \quad \Lambda(i, j) = a(n + 2j)/p, \quad j = 0 \ldots k{-}1, \text{ where } p \nmid a(n + 2k). \qquad [5.1]$$

For example, for $i = 5$, $a(33) = \text{PRIME}(5) = 11$. Thereafter, we have the following:

$a(35) = 2 \times 11$, $a(37) = 4 \times 11$, $a(39) = 6 \times 11$, $a(41) = 3 \times 11$, $a(43) = 9 \times 11$, $a(45) = 12 \times 11$, $a(47) = 18 \times 11$, $a(49) = 24 \times 11$, but $a(51) = 24$, indivisible by 11. Hence we have $\Lambda(5, j)$, $j = 0 \ldots 8$.

Table 5 below shows $\Lambda(i, j)$ for $i = 1 \ldots 5$.

Table 5: Λ(i,j) for i = 1..5.

i:	n	j: 0	1	2	3	4	5	6	7	8
1:	2	1	2							
2:	3	1	2	4	3	5	10	20	15	25
3:	6	1	2	4						
4:	14	1	2	4	6	12				
5:	33	1	2	4	6	3	9	12	18	24
		...								

An extended Table 5 (too large to print) demonstrates the following points:

1.) $\Lambda(i, j)$ is not always smaller than $\Lambda(i, j{+}1)$; the terms in the rows are not nondecreasing.

2.) $\Lambda(i, 2)$ is not always 2. For $i \in \{6, 7, 12, \ldots\}$, $\Lambda(i, 2) = 3$, while for $i \in \{11, 14, 15, \ldots\}$, $\Lambda(i, 2) = 4$.

3.) Not all multiples of $\text{PRIME}(i)$ in A369609 occur in row i of Λ. For instance, $a(21) = 5^2$, $a(91273) = 5^3$, despite row 3 of Λ missing 5 and 25, respectively.

4.) Though not apparent in Λ, Appendix Table B shows that for some i, $\text{PRIME}(i) \mid a(N)$ and $a(n) = \text{PRIME}(i)$ for $N < n$. An example is $a(87723) = \mathcal{P}(16)$, $a(91307) = \text{PRIME}(16)$.

5.) Irregular table Λ does not demonstrate mp in the sequence for m less than some limit.

6.) Despite point 5 above, powers $\text{PRIME}(i)^\delta$ in row i of Λ do demonstrate mp in the sequence for $m \le \text{PRIME}(i)^\delta$, since $a(n) = mp = \text{PRIME}(i)^\delta \times p$ is coprime to any $r \ne p$. Therefore it cannot arise by means of another kernel.

Despite the points shown above, Λ demonstrates "many small multiples" of $\text{PRIME}(i)$ appear in an alternating run following the emergence of $a(n) = \text{PRIME}(i)$. We need some other method of examining the entry of a certain $m(r) \times r$ in the sequence.

Particularly, we find interest in the entry of $m(r) \times r$ such that $\text{RAD}(m(r)) \mid r$, since, as consequence of sequence definition, it can be shown that $k = m(r) \times r$ as described, in a sequence K_r of numbers that have the same squarefree kernel as r enter the sequence in order. We explore this in section G below.

Sequence definition, the truth table (Table 2), and Figure 2 indicate that alternating divisibility patterns bear significant insight into the gross operation of the sequence. Corollary D1.9 and Theorem E1 are the drivers of the exhibited alternating divisibility patterns in the sequence. Theorems E2 and E3 show that not everything about the sequence can be explained by alternating divisibility patterns. Sequence mechanics, particularly multiplier $m(r)$, explains the points listed above regarging irregular table Λ.

We have focused on terms that follow primes, but there is cause to find interest instead in terms that precede them in A369609.

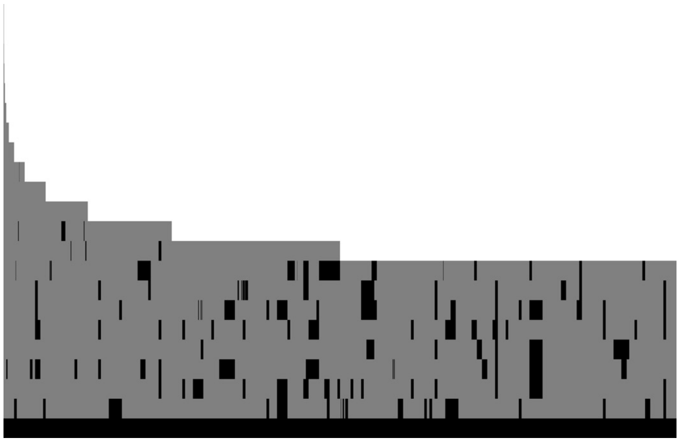

Figure 3. Plot of $s(i)$ for $i = 2^s + j$, $j \le 2^{12}$, i.e., arranged according to [6.1], showing rows $k = 0 \ldots 20$, given 1048576 terms in A369609. Black indicates undefined terms while gray indicates defined terms.

F. Kernel Coverage.

We turn to the question of whether all squarefree r occur in A369609. Given the truth table and its extension, Corollary C4.1 and Theorem C5, we can approach the question in a particularly organized manner.

Consider that $R = \mathcal{P}(k)$ is nondecreasing as n increases. In fact, k increments when R increases. Therefore, the question of whether or not r covers all divisors d_k of $\mathcal{P}(k)$, where $\text{PRIME}(k) = \text{GPF}(d_k)$. Perhaps the ordering of coverage resembles irregular table A019565, which begins as follows, where $\mathcal{P}(k)$ is the last d_k in row k.

```
  1;                                                  [6.1]
  2;
  3,  6;
  5, 10, 15, 30;
  7, 14, 21, 35, 42,  70, 105, 210;
 11, 22, 33, 66, 55, 110, 165, 330, 77, 154, ..., 2310;
 ...
```

Lemma F1. For $k > 1$, $\text{A019565}(k, 1) = \text{PRIME}(k)$ is the smallest term.

Lemma F2. For $k > 1$, $\text{A019565}(k, 2^{(k-1)}) = \mathcal{P}(k)$ is the largest term.

This sequence maps to the natural numbers through $\pi(p) \to 2^{(k-1)}$ for $p \mid d_k$, then taking the sum of the powers 2^k. Define functions $g(x)$ and $h(x)$ to be as follows:

$$g(x) = \sum 2^{(k-1)} \text{ for } \text{PRIME}(k) \mid x.$$

$$h(x) = \prod \text{PRIME}(k{+}1) \text{ for } x \text{ expressed in binary as a sum of } 2^k. \quad [6.2]$$

Then we take mappings $g(x)$ across A019565. This transform yields the index of A019565 as shown below:

```
  0;                                                  [6.3]
  1;
  2,  3;
  4,  5,  6,  7;
  8,  9, 10, 11, 12, 13, 14, 15;
 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, ..., 31;
 ...
```

The mappings $h(x)$ across natural numbers yields A019565. It is thus plain to see that $\text{PRIME}(k) \to 2^{(k-1)}$ while $\mathcal{P}(k) \to (2^k - 1)$. This transform becomes handy in tracking coverage of d_k.

A consequence of failure of r to cover an arbitrary d_k, a squarefree number, is that d_k along with any d_k-coregular nonsquarefree number is missing from the sequence.

If we miss d_k, and if the smallest missing prime or powerful number exceeds d_k, then squarefree d_k is the smallest missing number u.

For $n = 2^{20}$, $671 = 11 \times 61 = \text{A019565}(131088)$ is the smallest missing d_k, but the term in A019565 with the smallest index missing from A369609 is $746130 = \mathcal{P}(8)/13 = \text{A019565}(223)$.

Define sequence S20240329 = s with offset 0 to be as follows:

$$s(i) = n \text{ such that } a(n) = h(i). \quad [6.4]$$

The first terms S20240329 of appear below:

```
1, 2, 3, 5, 6, 8, 11, 13, 14, 16, 26, 20, 23, 25, 30,
32, 33, 35, 41, 39, 100, 102, 96, 92, 80, 82, 76, 74,
50, 52, 56, 58, 59, 57, 61, 63, 73, 75, 83, 79, 93, 95,
103, 99, 91, 188, 107, 109, 112, 114, 118, 120, ...
```

Seen as an irregular triangle as [6.1] above, s begins as follows:

```
  1;                                                  [6.5]
  2;
  3,  5;
  6,  8, 11, 13;
 14, 16, 26, 20,  23,  25, 30, 32;
 33, 35, 41, 39, 100, 102, 96, 92, 80, ..., 52, 56, 58;
 ...
```

This is the sequence of indices n in A369609 such that $a(n)$ is the squarefree number $h(i)$.

For example, suppose we are interested in the index n such that $a(n) = 6$. Since $\text{A019565}(3) = 6$ and $s(3) = 5$, $a(5) = 6$.

Given $\text{A369609}(1 \ldots 2^{24})$, sequence s is defined for $i < 223$, however the sequence features some singleton missing terms, but more often, runs of undefined terms (see Figure 3). Despite this, the sequence seems mostly defined for $n \le \mathcal{P}(k)$ as k increases.

In order to demonstrate that the sequence is a permutation of natural numbers, a necessary but insufficient condition is the coverage of the set of squarefree numbers A5117, represented by a completely defined s, i.e., a fully populated Figure 3.

Sequence s harbors implications for the nature of the smallest missing number u. Naively, we expect u to either be prime or powerful. Is it possible that the smallest missing number u is squarefree for some n? Is it possible that u is in A332785 for some n?

Table 6 shows the smallest squarefree $r = d_k$ missing from row k of s presented in the form of [6.3] for $\text{A369609}(1 \ldots 2^{24})$.

Table 6

k	i	2^(k-1)-i	r	Prime decomposition
				1111223344455667778 (tens)
				23571379391713739171393 (units)
1..6	----- COMPLETELY COVERED -----			
7	223	95	746130	xxxxx.xx
8..9	----- COMPLETELY COVERED -----			
10	1112	88	40579	...xx.x...x
11	2456	408	1245013	...xx..xx..x
12	4384	288	12259	.....x..x...x
13	8384	192	13889	......xx.....x
14	16576	192	15181	......xx......x
15	32960	192	17119	......xx.......x
16	69760	4224	45961	.......x....x...x
17	131088	16	671	....x............x
18	327680	65536	3953	................x.x
19	524352	64	1207	......x............x
20	1050624	2048	2701	...........x........x
21	2621440	524288	5609	...................x.x
22	4194304	0	83	x
...				1111223344455667778
				23571379391713739171393

In row $k = 22$, the second smallest missing $r = d_k = 3649 = 41 \times 89 = \text{A019565}(8392704)$.

Investigation related to Table 6 suggests that u "narrowly escapes" being weak (i.e., $u \in \text{A052485}$) as n increases, but does not prove that the smallest number missing from $\text{A369609}(1 \ldots n)$ is certainly either prime or a powerful number. The question remains open.

G. Coregular Sequences in A369609.

In section E, we examined $\Lambda(i, j)$, an array of terms divisible by PRIME(i) that follow $a(n)$ = PRIME$(i) = p$ with same index parity. We called this pattern "alternating divisibility" and is a consequence of Theorems E1 through E3. Having examined alternating divisibility duration, we saw in Table 4 that such can be quite protracted.

In this section, we go beyond the question of whether r appears, and attempt to determine to what extent it appears. Rather, how many k such that RAD$(k) = r$ appear in A369609, knowing that for squarefree r > 1, there are an infinite number of such k.

Theorem G1. Though $p \mid \Lambda(i, j)$, RAD$(\Lambda(i, j)) \neq p$ implies $r \neq p$. The proposition is tautological since r = RAD$(\Lambda(i, j))$.

Interest in the entry of $m(r) \times r$ such that RAD$(m(r)) \mid r$ arises so as to examine the depth of the occurrence of the "template" r in the sequence. We generalize interest from prime p to squarefree r. We are lead to the following:

Question: How many numbers that share the same set of prime factors as r appear in the sequence? This question probes several other questions about the sequence:

1.) Section F explored coverage of A5117 by r and remains inconclusive. For $r > 1$, there are an infinite number of k such that RAD$(k) = r$. This question attempts to find out how many k appear in the sequence such that RAD$(k) = r$.

2.) Numbers k that are perfect powers of primes and powerful numbers have squarefree kernels RAD$(k) = r$. Therefore the question addresses the paucity of such numbers in the sequence.

Let $K_r(i)$ be the sorted set of numbers k that share the same set of prime factors as squarefree r. This is to say that k is such that the squarefree kernel RAD$(k) = r$. We may say that all the terms $k \in K_r$ are r-coregular, since k such that rad$(k) \mid r$ are said to be r-regular. For example, is shown below.

$$K_6 = \{6, 12, 18, 24, 36, 48, 54, 72, 96, 108, \dots\}$$
$$= \text{A033845} = 6 \times \text{A3586}.$$

The following basic lemmas are self evident:

Lemma G2.0. The set K_r is countably infinite for $r > 1$. For $r = 1$, the cardinality of K_1 is 1, since there exists only 1 empty product.

Lemma G2.1. Then $K_r(1) = r$ is the minimum, and for $i > 1$, $K_r(i) = k = mr$, where RAD$(m) \mid r$.

Lemma G2.2. Prime r implies prime $K_r(1)$, and $K_r(i)$, $i > 1$ is a perfect power of prime r.

Lemma G2.3. Composite r implies composite squarefree $K_r(1) = r$, and $K_r(i) = k$ is a tantus number, meaning that at least 1 prime $p \mid k$ is such that p^2 also divides k, i.e., $k \in$ A126706.

Theorem G2. Terms k in K_r enter in order. Consequence of Lemma 16.1 and the greedy nature of m. Proves Conjecture A.3 provided no prohibition.

Corollary G2.4. The first term in K_r to appear in A369609 is the number $K_r(1) = r$ itself.

Corollary G2.5. Powers p^δ enter A369609 in order of δ, where p itself is the first power of p that appears in A369609.

Therefore we have interest in the "penetration" $D(r)$ of kernel r in A369609 defined to be the following:

$$D(r) = j \text{ such that } a(n) = K_r(i),\ i = 1 \dots j \text{ for some } n. \qquad [7.1]$$

Showing that $a(n) = K_r(i)$ for all i, and additionally, all r appear in A369609 enables a conclusion that A369609 is a permutation of $\mathbb{N}$.

This amounts more precisely to the following question:
Is $D(r) = \infty$ for all $r \in$ A5117?

Given the nature of the sequence, we could settle for showing $D(r)$ can reach ∞ as n increases, for all divisors d_k of $\mathcal{P}(k)$, where primorial $\mathcal{P}(k) = R \times$ PRIME$(k) = \mathcal{P}(k-1) \times$ PRIME(k) through Corollary C4.1 and Theorem C5.

Given Appendix Table D, we see powerful numbers rarely enter the sequence, and from this we conclude that $D(r)$ is relatively shallow. For example, after 2^{27} terms, we have not seen $64 = 2^6$, hence we conclude that $D(2) = 5$.

Using the functions g and h, we can create a sequence Δ that registers penetration at a given threshold N. Define sequence Δ to be the following:

$$\Delta(i) = j \text{ such that } a(n) = K_{h(i)}(j),$$
$$\text{where } j \text{ is maximal and } n \leq N. \qquad [7.2]$$

Setting $N = 2^{24}$ and using offset 0, sequence Δ begins as follows:

```
1, 5, 4, 10, 3, 9, 5, 8, 2, 5, 4, 6, 3, 7, 4, 4, 2, 7,
6, 10, 4, 8, 5, 7, 4, 11, 6, 9, 5, 9, 6, 7, 2, 6, 4, 6,
3, 5, 6, 8, 3, 6, 5, 9, 4, 8, 5, 6, 3, 7, 5, 9, 4, 8,
3, 4, 4, 5, 5, 6, 5, 7, 5, 4, 2, 6, 4, 6, 1, 3, 2, ...
```

Seen as an irregular triangle as [6.1] above, Δ begins as follows:

```
1;                                                  [7.3]
5;
4, 10;
3,  9, 5,  8;
2,  5, 4,  6, 3, 7, 4, 4;
2,  7, 6, 10, 4, 8, 5, 7, 4, 11, 6, 9, 5, 9, 6, 7;
...
```

This sequence is useful merely because it is relatively stable for small values. Many of the first columns above advanced 1-3 terms after the spate of powerful numbers entered for $n = 91217 \dots 91305$.

From this data, we see that the largest power of 2 in the sequence is $t(1) = 5$. The largest 3-smooth number in the sequence is $K_6(t(3))$ = 108, the 10th term in K_6. For $N = 2^{24}$, the kernel with the deepest penetration $j = 22$ is $r = 2 \times 3 \times 5 \times 7 \times 11 \times 19 \times 29 \times 83$ = 105643230, diagrammed below:

```
ooooo..o.o...........o
```

The following table shows the indices of first terms in K_r for $r \in \{6, 10, 15, 30\}$:

```
     n  K_6        n  K_10       n  K_15       n  K_30
----------  ----------  ----------  -----------
     5    6        8    10      11    15      13    30
     7   12       10    20      17    45      15    60
    24   18       27    40      19    75     681    90
    51   24       29    50     691   135     683   120
    55   36      685    80   91267   225     689   150
   695   48      687   100       ?   375     693   180
   697   54    91271   160                 91265   240
 91299   72    91275   200                 91269   270
 91303   96    91277   250                     ?   300
 91305  108        ?   320
     ?  144      (? means n > 2^27, if it exists)
```

Though some headway is made to support Conjecture A.3 via Theorem G2, we are not able to show $D(r) = \infty$ for all $r \in$ A5117. Our sense remains that indeed, Theorem G2 is only repressed by circumstance in A369609, and thus, we desire to explore the repression. Section J explores what might be repressing Theorem G2 and causing what is thwarting Conjecture A.3 and inducing Conjecture B.

H. Coherent Alternating Divisibility Patterns.

In Section C we developed 8 cases of divisibility of $a(n-2)$, $a(n-2)$, and m regarding an arbitrary prime p in Tables 1, 2, and Figure 2.

We introduced alternating divisibility patterns in Section E regarding prime p. We attempted to determine whether kernel r covers all squarefree numbers A5117 through $\Lambda(i, j)$.

Perhaps more relevant to this section, in Section G, we attempted to determine how deep this sequence penetrates the set K_r through the function $D(r)$.

Now we move beyond examining individual arbitrary primes p to examine these patterns across all primes $p \le Q$, the latter as defined before Theorem C5.

We note a "coherent" resonance across a range of primes that can be seen in Table 1 in the vicinity of $a(59) = 13$. A more prominent example regards Table 7 below. In this way we might explore the emergence of primorials and primes in the sequence, but also the appearance of powerful terms.

Table 7: Coherent zone that yields {100, 32, P(8), 23}:

n	a(n)	Primes 111122 2357137939	r	m(r)	notes
680	323323	...ooxoo..	24871	13	
681	90	o*o.......	30	3	
682	646646	x..ooooo..	323323	2	
683	120	xoo.......	15	8	
684	969969	.x.ooooo..	323323	3	
685	80	*.o.......	10	8	
686	1939938	xo.ooooo..	969969	2	
687	100	x.*.......	5	20	< 2^2*5^2
688	2909907	.*.ooooo..	969969	3	
689	150	ox*.......	10	15	
690	1293292	x..ooooo..	323323	4	
691	135	.*o.......	15	9	
692	2586584	*..ooooo..	646646	4	
693	180	x*o.......	15	12	
694	1616615	..xooooo..	323323	5	
695	48	*o........	6	8	
696	3233230	x.oooooo..	1616615	2	
697	54	x*........	3	18	
698	4849845	.xoooooo..	1616615	3	
699	32	*.........	2	16	< 2^5
700	9699690	xooooooo..	4849845	2	< P(8)
701	23	........x.	1	23	< prime(10)
702	19399380	*ooooooo..	9699690	2	
703	46	x.......o.	23	2	
704	14549535	.*oooooo..	4849845	3	
705	92	*.......o.	46	2	
706	24249225	.o*ooooo..	4849845	5	
707	138	ox......o.	46	3	
708	6466460	x.oooooo..	1616615	4	

Remark H1. Some remarks on Table 7 and coherence in general:

1.) For $n < 701$ in Table 7, a "resonant" or coherent state exists among many primes $p \le Q$ where all primes p divide one term, but generally do not divide the next, etc.
2.) The coherent state is characterized by alternating divisibility in phase across many primes, shown by alternating **o.o.**, etc. (vertically) for a given prime. This equates to alternating Cases Ⓐ and Ⓖ, which is stable unless perturbed by substitution of Case Ⓖ with Case Ⓗ. In the graph above, this is an occurrence of an **x** where there should be a "**.**".
3.) Most change induced by Case Ⓗ and follow-on Cases Ⓒ or Ⓓ occurs for small p.
4.) The appearance of prime $a(701) = 23$ introduces unraveling and erosion of coherence as n increases. This is because divisibility by 23, the largest $p \le Q$, occurs out of phase with divisibility of $a(n)$ by smaller primes.
5.) Table 4 shows that for 23 = prime(9), $\ell(9) = 164$. This goes to show that the alternating divisibility pattern associated with the largest $p \le Q$ prove rather stubborn. This seems to suggest that once coherence is lost, it may take a long time for it to materialize again.

Already by $n = 680$, we can see r is a product 24871 of 7, 11, 13, 17, and 19. As n increases, m that are products of small primes confer divisibility of r by this or that small prime. Since even n harbor a product of 24871, $a(n)$ with odd n are generally shielded from divisibility by large primes, and we see a couple powerful numbers enter.

Association with powerful numbers. Appendix Table H shows a protracted cluster of powerful numbers that enter the sequence at $91207 \le n \le 91305$, which demands explanation. Why do so many powerful numbers enter the sequence within this narrow range, when the sequence proves generally one of weak numbers?

Through examination of the various coherent intervals in A369609 documented in Appendix Tables F and H, we see that such intervals are shallow aside from the intervals $n = 682 \ldots 700$ in Table 6 and $n = 90970 \ldots 91306$ in Appendix Table H. In the interval $n = 754467 \ldots 754786$, gpf$(r) \approx$ prime(9) tends to be too high to supply powerful numbers.

Intervals $n = 682 \ldots 700$ and $n = 90970 \ldots 91306$ are characterized both by small gpf(r) and $\omega(r)$.

Conjecture H2. Protracted coherent alternating divisibility patterns across primes $p \le Q$ may yield a rash of powerful numbers.

1.) Coherent divisibility patterns such that $a(2n)$ approaches R and $a(2n+1)$ has minimized gpf(r) and $\omega(r)$ (or parity reversed), along with $m(r) \in K_r$, make for powerful numbers in the sequence.
2.) If $r = 1$, then the smallest missing number u enters the sequence via $m(r) \times r = m(1) \times 1 = u$. See Appendix Table F5 and Sections J and K.
3.) If prime p is already in the sequence and m is a power of p, then we have a perfect power of p in the sequence. (Section K, specifically Theorem K1, addresses the appearance of primes.)
4.) If $r = \mathcal{P}(k)$ new to the sequence, $\mathcal{P}(k)$ enters the sequence via $m(r) \times r = m(\mathcal{P}(k)) \times 1 = \mathcal{P}(k)$.

In order to prove this conjecture, we would need to address the emergence of coherence and show how the actions described in Remark H1 arise. This would seem to present significant complexity.

Recovery of lost coherence. Appendix Table F15 or F19 serve as examples of incoherent intervals that predominate A369609. Appendix Table G illustrates gradual partial recovery of coherent alternating divisibility pattern from a more disorganized state, for $n = 3940 \ldots 3980$. Recovery of coherence is a complex process that might be described as "random".

Suppose we want to create a $\pi(Q)$-bit binary number that is predominantly comprised of zeros, except for 1s in small places. It follows that such becomes increasingly less likely as $\pi(Q)$ increases. Therefore we expect proper coherence to arise increasingly rarely as n increases.

We have attempted to find a new protracted coherent phase but such has not materialized for $n \le 2^{27}$.

J. On the Smallest Missing Number u.

Lexically earliest sequences (LES) normally involve a greedy approach to solutions such that we can identify the smallest number u that is not in the sequence $a(1 \ldots n)$. We present the general theorem for lexically earliest sequences (LES):

Let $R(n)$ be the largest number in $a(1 \ldots n)$.

Let $u(n)$ be the minimum of the sequence U of numbers that are not in the sequence. If the reference range $\mathcal{Y} = \mathbb{N}$ as it is for A369609, then we have the following:

$$U = \mathbb{N} \setminus a(1 \ldots n),$$
$$u(n) = \text{MIN}(U). \qquad [9.1]$$
$$R(n) = \text{MAX}(a(1 \ldots n)). \qquad [9.2]$$

Theorem. We can break the reference range $\mathcal{Y}$ of a lexically earliest sequence (LES) into at least 2 or at most 3 intervals.

① The saturated interval $[\text{MIN}(U) \ldots u(n))$,

② The mixed interval $[u(n) \ldots R(n)]$

③ The clear interval of $k > R(n)$.

Proof. A priori, before definition, we begin with ③. A sequence that begins with its first term $\text{MIN}(\mathcal{Y})$ either by definition or natural operation of $f(x)$ given an initial value for x has $R(n) = \text{MIN}(\mathcal{Y})$. Since a number k is either in the sequence or not, and given the greedy nature of the sequence function, given $R(n) = \text{MIN}(\mathcal{Y})$, we have at least ① and ③, otherwise we have ② and ③. A sequence that proceeds from $R(n) = \text{MIN}(\mathcal{Y})$ to incorporate all terms of $\mathcal{Y}$ in order becomes $\mathcal{Y}$ itself, and only ever has the intervals ① and ③. Sequences that through operation of $f(x)$ incorporates certain terms in $\mathcal{Y}$ before others has either ② and ③, but if it began with $R(n) = \text{MIN}(\mathcal{Y})$, it has all three intervals. ■

Corollary. For $f(x) = k$, $k < u(n)$ implies reiteration of $f(x)$ until its output $k \ge u(n)$.

Corollary. For $f(x) = k$, $u(n) \le k < R(n)$ requires testing to see if k is in the sequence; if so, then we reiterate $f(x)$ until either output $k > R(n)$ or we can show that k is not already a term.

Corollary. For $f(x) = k$, $k > R(n)$ is immediately acceptable; furthermore, $R(n+1) = k$.

Remark. The mixed interval ② tends to be dense with terms already in the sequence for k not much larger than $u(n)$, and progressively rarifed in such terms as k approaches $R(n)$.

Conjecture J1. Given the nature of the sequence presented thus far, especially the summaries in Appendix Tables A and D, we might expect $u(n)$ to feature terms that are either prime or powerful as n increases.

Define sequence W to be sorted A8578 ∪ A1694, a sequence that begins as follows:

```
1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27,
29, 31, 32, 36, 37, 41, 43, 47, 49, 53, 59, 61, 64, 67,
71, 72, 73, 79, 81, 83, 89, 97, 100, 101, 103, 107, 108,
109, 113, 121, 125, 127, 128, 131, 137, 139, 144, ...
```

Therefore, Conjecture J1 expects $u \in W$.

Challenges to this conjecture include faults in coverage described in Section F, and sufficiently slow incorporation of r-coregular terms described in Section G. For example, suppose that for $R \ge \mathcal{P}(k)$, some small composite kernel $r = d_k$ in row k of A019565 does not materialize. Then $u = d_k$ if all nonsquarefree numbers smaller than d_k enter the sequence ahead of it. If we can show that some reason prevents d_k from entering, then A369609 is not a permutation of $\mathbb{N}$.

Table 8 below summarizes distinct smallest missing $u(i)$ that first emerges at n for $n \le 2^{27}$. Asterisks denote composite u. Parenthetic $u(i)$ appear by definition, while bracketed $u(i)$ appear via Theorem J6. The abbreviated divisibility pattern which brings about $a(n(i+1)) = u(i)$ appears in the "patt." column. The suppressed primes appear in column Ⓒ. The circumstance of $a(n(i+1)) = u(i)$ appears in the listed table.

Table 8: Smallest missing number u(i)

i	n	u		patt.	(C)	Table
1	0	(1)		.	.	1
2	1	(2)		.	.	1
3	2	[3]		_gF	.	1
4	3	4	*	_gB	.	1
5	4	[5]		CgF	2	1
6	6	[7]		CgF	2	1
7	14	[11]		CgF	2	1
8	33	13		Cga	2	1
9	59	17		Cga	2	F2
10	161	19		Cga	3	F4
11	363	[23]		CgF	2	F5,7
12	701	29		Cga	3	F7
13	1509	31		Cga	2	F10
14	2222	37		Cga	2	F12
15	4581	41		Cga	3	F14
16	7827	43		Cga	2	F16
17	20543	47		Cga	2	F18
18	28710	49	*	CgB	2	H
19	91283	[53]		CgH	2×3	F22, H
20	91307	61		Cga	2	F28
21	810244	64	*			

For example, $u(11) = 23$, which emerges when $a(363) = 19$, therefore, for n = 363. When $a(701) = 23$, it appears through the divisibility pattern **CgggggggF**, where Case Ⓒ pertains to $p = 2$, suppressing divisibility by 2 while Case Ⓕ furnishes divisibility by $p = 23$, and Case Ⓖ suppresses divisibility by all other primes $p \le Q$, hence **CgF**. Appendix Table F5 and Table 7 detail the entry of 23 into the sequence A369609.

<u>Circumstances for entry of missing numbers</u>. The smallest missing number is merely a special case of a number that is not in the sequence, meaning a potential term. There are several modes of admitting missing numbers into the sequence. These can be stated in terms of the cases described in Section C.

Returning to divisibility cases in Table 2, we note the following consequences of the truth table:

Theorem J2. Cases that suppress divisibility of primes p such that $p \mid a(n-1)$ and $a(n)$ include Case Ⓖ, i.e., p only divides $a(n-1)$, and Case Ⓒ, where p divides both $a(n-2)$ and $a(n-1)$ but not $m(r)$.

Theorem J3. All other cases deliver divisibility by p to $a(n)$ via TheoremC4. The most common mode in the sequence is alternating Cases Ⓐ and Ⓖ.

Theorem J4. Case Ⓐ implies p only divides $a(n-2)$ and thus $p \mid r$. It is distinguished from Cases ⒷⒹⒻⒽ since it does not require prime $p \mid m(r)$.

Corollary J4.1. Case Ⓐ alone (i.e., aside from Ⓖ and Ⓒ) cannot generate nonsquarefree $a(n)$ since the case produces squarefree r.

Theorem J5. Case Ⓕ implies prime $m(r) = p$, with $r = R = \mathcal{P}(k)$.

Corollary J5.1. Case Ⓕ is the only case that can yield terms alone. $a(2) = 2$ can be construed as the result of singleton Case Ⓕ. Consequence of Theorem J5, the definition of Case Ⓕ, and $a(1) = 1$.

Corollary J5.2. For $a(n) = p = \text{PRIME}(k+1)$ brought in by Case Ⓕ, $a(n-1) = m(\mathcal{P}(k)) \times \mathcal{P}(k)$.

Theorem J6. Kernel $r = 1$ implies $a(n) = u$.

Proof. Consequence of sequence definition, specifically, given the greedy approach to $m(r)$ and the following:

$$\begin{aligned} a(n) &= k = m(r) \times r, \text{ with minimal } k \neq a(j),\, j < n \\ &= m(1) \times 1 = u. \end{aligned} \qquad [9.3]$$

This, since we increment $m(r)$ until we encounter the smallest number not in the sequence, which is u by definition. ■

Expected smallest missing numbers. Whereupon we see $a(n) = 2^6$, $W(22) = 83$, but if 83 enters the sequence before 64, then we expect $W(22) = 101$ instead. The smallest powerful numbers not in the sequence after 64 are 128 and 144. We might expect these to enter in a flurry of powerful numbers that attend a new deeply coherent phase.

Coherent Divisibility Modes of Entry. We examine various modes for numbers to enter the sequence, with attention to the smallest missing number u. Entry modes are governed by Theorem C7 and corollaries. For composite u, there is more than 1 way for $u = m(r) \times r$, with squarefree r to enter.

Mode `CgF`.

This mode is restricted to bringing in primes p = NEXTPRIME(Q).

Early in the sequence a few smallest missing numbers $u(i)$ enter with $m(r) = p$ = NEXTPRIME(Q), which is Case Ⓕ introducing prime p. Primes $q < p$ are suppressed by Case Ⓖ and for $i > 4$, Case Ⓒ for at least 1 small prime q that divided $m(r)$ for $n-1$. See Table 1 for examples. Corollary J5.2 pertains to Mode `CgF` when it results in $a(n) = p$.

Mode `Cga`.

Mode `Cga` is the means of delivery associated with a prime $p \mid r$, but is not restricted to prime $a(n) = p$.

This is the most common mode of furnishing divisibility by primes p such that p divides u seems to be Case Ⓐ, where $p \mid a(n-2)$ but does not divide $a(n-1)$, hence $p \mid r$, and divisibility by all other primes $q \neq p$, $q \leq Q$ are suppressed by Case Ⓖ and for $i > 4$, Case Ⓒ for at least 1 small prime q that divided $m(r)$ for $n-1$. Aside from Case Ⓒ applying to 3 rather than 2, the entry of $u(10) = 19$ is exemplary:

```
                    prime p   Cases
                       1111      1111
  n        a(n) 23571379  23571379
--------------------------------
361        171  .*.....o  gBggggga
362     510510  oxooooo.  aHaaaaag  P(7)
363         19  .......o  gCggggga  prime(8)
```

Modes `CgB-D-H`.

Smallest missing u that are prime squares p^2 are often brought about through Case Ⓑ instead of Case Ⓐ, where p divides both $m(r)$ and $a(n-2)$ but p does not divide $a(n-1)$.

Case Ⓗ may substitute, where p divides both $m(r)$ and $a(n-1)$ but p does not divide $a(n-2)$.

Case Ⓓ may also appear instead of Ⓑ or Ⓗ, where p divides all of $a(n-2)$, $a(n-1)$, and $m(r)$. This mode has not yet been observed.

Divisibility by all other primes $q \neq p$, $q \leq Q$ are suppressed by Cases Ⓖ or Ⓒ. Theorem C7 also admits entry of $u = p^2$ via Theorem J2, that is, via $m(r) = p^2$.

Prime $u(19) = 53$ enters via `CgH`.

Mode `CgB-D-H` is the means of delivery associated with primes p that divide $m(r)$.

Combination Modes.

Theorem J7. Any combination of Cases ⒶⒷⒹⒻⒽ may usher a number k such that $\omega(k) > 1$ into the sequence. This is a consequence of Theorem J3, specifically, all these cases serve to confer divisibility by primes that produce $a(n)$.

Corollary J7.1. Any combination of Cases ⒷⒹⒽ may usher a nonsquarefree number into A369609. Hence, numbers $a(n)$ that are neither prime powers nor squarefree, including powerful numbers are the fruit of any combination of Cases ⒷⒹⒽ, excluding Ⓐ via Corollary J4.1 and Ⓕ through Corollary J5.2.

For $n \leq 2^{27}$, The most common combination mode that produces powerful numbers is `CgB`. Mode `_gB` pertains to {4, 108, 250, 1089}, `_gB-D` to {54, 100}, `_gB-H` to {36, 200}, and `_gH` to 96. Therefore, missing number 144 is expected to come via Corollary J7.1.

Singleton Modes for Perfect Powers of Primes.

Theorem J8. Perfect powers of primes p^δ, $\delta > 1$, may enter through one of Cases Ⓑ, Ⓓ, or Ⓗ, excluding Ⓐ via Corollary J4.1 and Ⓕ through Corollary J5.2.

Corollary J8.1. For $a(n) = p^\delta$, $\delta > 1$, Case Ⓑ implies the squarefree $r = p$ and $m(r) = p^{(\delta-1)}$, since Case Ⓑ implies $p \mid a(n-2)$ but p does not divide $a(n-1)$ by definition.

Corollary J8.2. For $a(n) = p^\delta$, $\delta > 1$, both Cases Ⓓ and Ⓗ imply squarefree $r = 1$ and $m(r) = p^\delta$, since these cases imply $p \mid a(n-1)$ but p does not divide $a(n-2)$ by definition.

We anticipate $u = 64$ to enter through Case Ⓑ since $m(r)$ is minimized, but it is possible that it comes in through either Ⓓ or Ⓗ.

K. Occasion of Primes in A369609.

We focus attention on the appearance of primes p in A369609. Appendix Table A lists primes in the sequence for $n \leq 2^{27}$, while Appendix Table B shows primorials.

Since we are dealing with a single prime factor, we can trace emergence of a given prime to a certain case in the truth table (Table 2). Theorem J2 shows that Cases Ⓖ and Ⓒ suppress divisibility of $a(n)$ by primes q such that both $q \mid R$ and $q \neq p$. Theorem J3 shows that Cases Ⓐ, Ⓑ, Ⓓ, Ⓕ, and Ⓗ furnish $p \mid a(n)$. Therefore we have winnowed the provenance of primes in the sequence to those 5 cases.

Theorem K1. Cases Ⓑ and Ⓓ imply composite $a(n)$.

Proof. Case Ⓑ implies prime p divides both r and $m(r)$. Because $a(n) = m(r) \times r$, $p^2 \mid a(n)$. ■

Corollary K1.1. Cases Ⓐ, Ⓕ, and Ⓗ may produce prime $a(n)$, as consequence of both Theorem J3 and K1. Case Ⓕ is a consequence of Theorem J5.

Lemma K2.1. Case Ⓐ implies prime $a(n) = p$, when Case Ⓐ applies to a sole prime p such that $p \mid R$, while all other prime factors $q \mid R$ are suppressed by Theorem J2. Consequence of squarefree R.

Lemma K2.2. Case Ⓗ implies prime $m(r) = a(n) = p$, when Case Ⓗ applies to a sole prime p such that $p \mid R$, while all other prime factors $q \mid R$ are suppressed by Theorem J2. Consequence of squarefree R.

Therefore, Case Ⓐ has prime $a(n) = p$ derive from $p \mid a(n-2)$ while Case Ⓗ has prime $m(r) = a(n) = p$.

Theorem K2. Primes $p \le Q$ such that $p \nmid a(n{-}1)$ enter the sequence as consequence of Theorem C7 and sequence definition. Primes arise through 1 of the following 4 modes:

⓪. By definition. Applies to $p = 2$.
①. $r = p$, $m(r) = 1$ through Case Ⓐ, p only divides $a(n{-}2)$.
②. $r = 1$, $m(r) = p$ through Case Ⓗ, p only divides $m(r)$.
③. $r = 1$, $m(r) = p$ through Case Ⓕ, $p = Q$.

Mode ①, a consequence of Theorems J4 and Lemma K2.1, applies to most primes, first instance is $a(59) = 13$.

Mode ② is a consequence of Theorems J4 and Lemma K2.2. Induced by $a(n{-}1) = R = \mathcal{P}(k)$, the mode is only observed for $a(91307) = 53$.

Mode ③, a consequence of Theorems J5 and J6. Induced by $a(n{-}1) = R = \mathcal{P}(k)$, this mode yields the primes $\{(2), 3, 5, 7, 11, 23\}$.

Theorem K2 summarizes the entry modes of primes p in A369609.

Primes "coming over the top" of R. Theorem J5 describes introduction of prime $p = Q$ to the sequence through Mode ③, i.e., Case Ⓕ, for examples see Table 1 or Appendix Table F5.

Skipping primes. Conjecture A.1, proved wrong, anticipated that primes appear in order in A369609 as n increases. This observed contradiction raises a couple key questions.

1.) How does a skipped prime enter the sequence?
2.) How do 59, 71, 89, and 103 enter ahead of schedule?

Turning to question 1 above, in essence, we see that primes enter through primes $q \le Q$ such that $q \nmid a(n{-}1)$. The following corollaries address the issue of skipped primes.

Corollary K2.3. Suppose $a(n{-}1) = R/p$, where $R = \mathcal{P}(k)$, a primorial, and $p = \text{PRIME}(i)$, $i \le k$. Then if $p \nmid m$, and if $a(h) \ne p$, $h < n$, $a(n) = p$. Consequence of Case Ⓐ, Theorems J4, and Lemma K2.1. For 2 examples, see Appendix Tables F27 for $a(621674) = 67$, and F28 for $a(810244) = 61$.

```
810242  122  o.................o..  agCgggggggggggggagg
810243    *  xoooooooooooooooo.oo  Haaaaaaaaaaaaaaagaa
810244   61  ..................o..  Cgggggggggggggggagg
           * = P(20)/61
```

Corollary K2.4. Special case of Corollary K2.3: With $R = \mathcal{P}(k)$ and $a(n{-}1) = \mathcal{P}(k{-}1)$, if both $m \ne \text{PRIME}(k)$ and $a(h) \ne \text{PRIME}(k)$, $a(n) = \text{PRIME}(k)$. This is the most common mode of entry for prime p through Case Ⓐ. Appendix Tables F21 for an example.

```
87721   531  .*..............o  gBggggggggggggga
87722 P(16)  oxoooooooooooooo.  aHaaaaaaaaaaaaag
87723    59  ................o  gCggggggggggggga
```

Corollary K2.5. Suppose $a(n{-}1) = R = \mathcal{P}(k)$, and $p = \text{PRIME}(i)$, $i \le k$. Then if $p \mid m$, and if $a(h) \ne p$, $h < n$, $a(n) = p$. Consequence of Case Ⓗ, Theorems J4 and Lemma K2.2. The only observed example is $a(91307) = 53$, see Appendix Table F22.

```
91305   108   **...............  BBgggggggggggggg
91306  P(17)  xxooooooooooooooo  HHaaaaaaaaaaaaaa
91307    53   ...............x.  CCggggggggggggHg
```

Hence A369609 is able to "cure" the issue of skipped primes through a single prime gap in $a(n{-}1)$ via Mode ①, Case Ⓐ described in Corollaries K2.3 and K2.4, or for $a(n{-}1) = R$ and $p \mid m$ via Mode ②, Case Ⓗ and Corollary K2.5.

Corollary K2.6. Corollaries K2.4 and K2.5 have primes succeed primorials in the sequence, while Corollary K2.3 furnishes primes that do not succeed primorials.

We address question 2 above. How do primes "jump the queue" and enter "early"?

Dilation. Theorem C4 and the original definition of A369609 define R to be a primorial with greatest factor $Q = \text{PRIME}(k{+}j)$, $j > 0$. Examination of the primorials $\mathcal{P}(k)$ in A369609$(1 \dots 2^{27})$ shows that primorials do not enter the sequence in order. Hence we turn attention to the difference j which we call "dilation".

Appendix Table C shows the advancement of $R = \mathcal{P}(k)$ as $n \le 2^{27}$ increases. Table J tracks change in dilation as n increases to 2^{27}.

We note the following regarding Table J:

1.) "Hitting the ceiling": $j = 0$.
Increase in R conflated with emergence of a prime.
$a(n) = \mathcal{P}(k) = R \rightarrow a(n{+}1) = \text{prime}(k{+}1) \rightarrow R = \mathcal{P}(k{+}1)$.
Prime Mode ③ (Case Ⓕ) through Theorem J6.
Pertains to cases $k \in \{(1), 2, 3, 4, 5, 9, \dots\}$.
2.) "Topping off": $j = 1$.
Increase in R ahead of emergence of corresponding prime.
$a(n) = \mathcal{P}(k{-}1) \rightarrow a(n{+}1) = \text{prime}(k)$.
Prime Mode ① (Case Ⓐ) through Corollary K2.4.
Pertains to most observed cases.
3.) "Catching up": $j > 1$.
$R = \mathcal{P}(k{+}j)$, $a(n) = \mathcal{P}(k{+}j{-}1)$.
Followed by $a(n{+}1) = \text{prime}(k{+}j)$ in observed cases.
Prime Mode ① (Case Ⓐ) through Corollary K2.4.
Pertains to $\mathcal{P}(k)$ with $k \in \{16, 19, 23, 26, \dots\}$.
4.) "Coming up short": $j > 1$. (Not observed).
$R = \mathcal{P}(k{+}j)$, $a(n) = \mathcal{P}(k{+}i)$, $i < j{-}1$.
Not followed by a prime, but instead a number that is not a prime power, via Cases Ⓐ and Ⓑ.

We see $a(28709) = \mathcal{P}(14)$. For $n = 82279$, $R = \mathcal{P}(17)$, then $a(87722) = \mathcal{P}(16)$. $\mathcal{P}(15)$ is not seen after 67 million terms. Hence $n = 82279$ is a landmark in the sequence that represents dilation $j = 3$.

Skipped primorials. A hypothetical way that the skipped primorial $\mathcal{P}(15)$ could enter the sequence for $n > 2^{27}$ and $Q = \text{PRIME}(27)$ appears below.

```
x..............ooooooooooooo  2 × P(28)/P(15)
xoooooooooooooo.............  2 × P(15)/2 = P(15)
.x.............ooooooooooooo  3 × P(28)/P(15)
```

Note that we do not have a prime follow this skipped primorial, but some composite number that is not a prime power, since it is the product of a run of largest primes $p \le Q$.

The trouble with this arrangement is that a number composed of complementary runs of divisibility and nondivisibilty, i.e., contiguous repeated Case Ⓖ and repeated Case Ⓐ except for limited small primes is not observed for $n > 80000$. Much more common are numbers like the arrangement that are products of largest primes $p < Q$, with Q stubbornly out of phase.

What is more likely, very much later in the sequence, is that $\mathcal{P}(15)$ appears for an immense primorial R, when m has sufficient incrementation to be in the vicinity of $\mathcal{P}(15)$. We have not proved that the pattern shown is impossible but should be quite rare.

It appears that item 3 "Catching up" is more common because of the observed prevalence of prime Q out of phase with smaller primes.

Primes do not always follow primorials: this we see regarding 61 and 67. Remark 4 above shows that primorials do not always precede primes.

L. On the Potential of a Reverse Permutation.

Conjecture C asserts that A369609 is a permutation of natural numbers. We aren't able to prove such through the methods used in this sequence.

Theorem L1. The sequence is infinite.
Proof. Since $a(n) = k = m(r) \times r$, minimal $m(r)$ such that $a(h) \neq k$, and given squarefree r resulting from Theorem A2. Theorem J6 covers the occasion $R = a(n-1)$, hence $r = 1$, and $a(n) = u$, the smallest missing number. ■

The following questions are not answered. If the answer to at least 1 of these questions is negatory, then we show the sequence is not a permutation of natural numbers.

1.) Section F: are all squarefree numbers r in the sequence?
2.) Section G: r-coregular numbers enter the sequence in order per Theorem G2. We demonstrated some penetration. Are all r-coregular numbers in the sequence?
3.) Section H: do all powerful numbers appear in the sequence? This question is a special case of Question 2.
4.) Section J: do all smallest missing numbers eventually enter the sequence? Critically, do all primes appear in the sequence? This is a special case of Question 1.
5.) Section K: do skipped primorials eventually appear in the sequence? This question is a special case of Question 1.

It is our hunch that the sequence is a permutation of natural numbers. The range of frequent values of m increases but remains small for large values of $\mathcal{M}$ and Q. Furthermore, it seems plausible that all r appear and do so infinitely. Countervailing this assertion is the fact that r is a product of a rather uniform jumble of primes $p \le Q$, and that a small r requires protracted coherence that seems to arise only in rare, relatively short runs.

An analogy is inviting kindergartners to flip light switches and expecting the room to go dark. Suppose we ask a single 5 year old to toggle $k = 1$ switches. It is easy for the room to go dark. As we increase the number k of kids, one per switch (i.e., k such that $R = \mathcal{P}(k)$) it is easy to see that as k increases, it becomes less likely to observe any coordination, much less the room to go totally dark.

Laying these questions aside, we contemplate the reverse permutation. Essentially, we create a new sequence $b(k) = n$, where $a(n) = k$. The reverse permutation begins with the following terms. Asterisks denote terms $n > 2^{27}$ if they exist.

```
1, 2, 3, 4, 6, 5, 14, 12, 9, 8, 33, 7, 59, 16, 11, 31,
161, 24, 363, 10, 26, 35, 701, 51, 21, 57, 53, 18, 1509,
13, 2222, 699, 41, 159, 23, 55, 4581, 365, 61, 27, 7827,
20, 20543, 37, 17, 703, 28710, 695, 91283, 29, 163, 69,
91307, 697, 100, 91279, 359, 1511, 87723, 15, 810244,
2220, 28, *, 73, 39, 621674, 177, 709, 25, 384195,
91299, 1080885, 4579, 19, 367, 80, 63, 2814146, 685,
91301, 7829, *, 22, 151, 20541, 1505, 108, 16009512,
681, 93, 705, 2214, 28708, 375, 91303, 29524905, 91281,
43, 687, *, 165, *, 71, 30, 91309, *, 91305, *, 102,
4583, 91285, *, 357, 725, 1513, 224, 87725, 131, ...
```

M. Conclusion.

This investigation concerns a lexically earliest sequence A369609 based on prime decomposition that behaves like a cellular automaton, alternating states unless perturbed by multiplier m. We are able to create a truth table (Table 2) and study extended patterns of the states in the truth table so as to arrive at dependencies that appear in Figure 3.

Certain naive questions arose before study regarding the relationship of primorials with primes, conjecturing that these numbers appeared in order. These questions furnished impetus to study the sequence further.

With data in Appendix Tables A and B we found counterexamples and know that these numbers do not appear together all the time, and do not appear in order. Conjecture A.3 asserted that powers of 2 appear in order; Theorem G2 confirms that those powers in the sequence indeed do appear in order, but it is unknown whether r = 2 occurs infinitely.

Conjecture C asserts that A369609 is a permutation of natural numbers. Section L lays down unanswered questions associated with the matter, and gives the first 119 terms of the reverse permutation if indeed A369609 is such.

The following list is a summary of findings in this paper.

1.) Conjecture A. There is a chain $2^i \to \mathcal{P}(i) \to \text{PRIME}(i+1)$, where $\mathcal{P}(i)$ is the product of the smallest i primes, i.e., primorial A2110(i), shown to be FALSE; $a(59) = 13$ but $a(57) = 26$.
2.) Conjecture A.1. Primes appear in order as n increases. Shown to be FALSE; $a(87723) = 59$ but $a(91307) = 53$.
3.) Conjecture A.2. Primorials appear in order as n increases. Shown to be FALSE; $a(28709) = \mathcal{P}(14)$ and $a(87722) = \mathcal{P}(16)$.
4.) Conjecture A.3. Powers of 2 appear in order as n increases. True via Theorem G2, however, it is uncertain whether A79 is a subset of A369609.
5.) Conjecture B. Powerful numbers appear in clusters, e.g., for n roughly between 91200 and 91320. Explored in Section H, Tables 1 and 7, and in Appendix Tables F and H.
6.) Conjecture C. A369609 is a permutation of natural numbers.
7.) Theorem C1 summarizes logic associated with divisibility relations $(\, p \mid a(n-2) \wedge p \nmid a(n-1)\,) \vee p \mid m$. See truth Table 2.
8.) Figure 2 summarizes extended divisibility patterns and dependencies of cases presented in Table 2.
   a.) Repeated Nondivisibility Case Ⓔ
   b.) Introduction of Divisibility Case Ⓕ
   c.) Alternating Divisibility Cases ⒶⒷⒼ
   d.) Repeated Divisibility Cases ⒹⒽ
   e.) Transition from repeated to alternating cases, Case Ⓒ
9.) Disruption of alternating Cases ⒶⒼ and introduction of divisibility of $a(n)$ by p through Case Ⓕ arises from $m(r) = p = \mathcal{M}$.
10.) Record $m(r) = \mathcal{M}$ implies no powerful number $a(j) = k$, $j \le n$, such that $\text{RAD}(k) > \mathcal{M}$. (Corollary C8.1.)
11.) Given $a(n) = p$, $p \mid a(n + 2j)$, $j \ge 0$ for a significantly large j.
12.) All squarefree numbers may appear in the sequence (Theorem C9). Do all squarefree r occur in A369609? See Section F, which examines coverage of r across A5117, specifically Table 6.
13.) Do all numbers k such that $\text{RAD}(k) = r$ appear in A369609? See Section G.
14.) Numbers k such that $\text{RAD}(k) = r$ appear in order, Theorem G2.

15.) Conjecture J1. The smallest missing number u is either prime or a powerful number.
16.) Suppression of $p \mid a(n)$ via Cases Ⓖ and Ⓒ. Theorem J2.
17.) Deliverance of $p \mid a(n)$ via Cases ⒶⒷⒹⒻⒽ. Theorem J3.
18.) Among the above, Case Ⓐ alone cannot generate nonsquarefree $a(n)$.
19.) Case Ⓕ implies prime $m(r) = p$, $r = R = \mathcal{P}(k)$. Theorem J5.
20.) Kernel $r = 1$ implies $a(n) = u$, smallest missing number. Theorem J6. For $n \le 2^{27}$, smallest missing number $u = 64$.
21.) Any combination of Cases ⒶⒷⒹⒻⒽ may usher a number k such that $\omega(k) > 1$ into the sequence. Theorem J7.
22.) Any combination of Cases ⒷⒹⒽ may usher a nonsquarefree number into A369609 (including powerful k). Corollary J7.1.
23.) Case Ⓑ is the most likely source of $a(n) = p^{\delta}$, $\delta > 1$.
24.) Cases Ⓑ and Ⓓ imply composite $a(n)$. Theorem K1.
25.) Lone Case Ⓐ has prime $a(n) = p$ derive from $p \mid a(n-2)$ while Lone Case Ⓗ has prime $m(r) = a(n) = p$.
26.) Primes $p \le Q$ such that $p \nmid a(n-1)$ enter the sequence as consequence of Theorem C7 and sequence definition. Primes arise through 1 of the following 4 modes:
⓪. By definition. Applies to $p = 2$.
①. $r = p$, $m(r) = 1$ through Case Ⓐ, p only divides $a(n-2)$.
②. $r = 1$, $m(r) = p$ through Case Ⓗ, p only divides $m(r)$.
③. $r = 1$, $m(r) = p$ through Case Ⓕ, $p = Q$.

22.) Section K addresses skipped primes and primorials using the concept of dilation j, where $\mathcal{P}(k)$ is the largest primorial in the sequence and $Q = \text{PRIME}(k+j)$, $j > 0$. See Appendix Tables C and J. Nicknames for 4 consequences of dilation:
  a.) "Hitting the ceiling", $j = 0$ for $k \in \{(1), 2, 3, 4, 5, 9, \dots\}$.
  b.) "Topping off": $j = 1$, for most k.
  c.) "Catching up": $j > 1$ for $k \in \{16, 19, 23, 26, \dots\}$.
  d.) "Coming up short": $j > 1$. (Not observed).
  Cases a-c involve a primorial followed by prime, but case d would have a primorial not followed by a prime.
23.) Some loss of confidence in Conjecture C:
  a.) Case 22d involves a special case of coherence with 2 or 3 runs of the same divisibility cases, which seems hard to get.
  b.) Skipped primorials $\mathcal{P}(k)$ could also show when average m ranges in the scale of $\mathcal{P}(k)$, therefore, for very large n.
24.) We can project a reverse permutation $b(k) = n$, where $a(n) = k$, shown in Section L.

Sycamore's sequence A369609 presents interesting questions related to prime decomposition and a behavior akin to a cellular automaton through alternating divisibility Cases ⒶⒼ. Some of these questions are not answered, including whether the sequence is a permutation of natural numbers, whether all powerful numbers and primorials appear. Can the smallest missing number be anything but either prime or powerful?

Concerns Sequences:

A1694, A2110, A3586, A5117, A6530, A7947, A8578, A019565, A033845, A052485, A067255, A087207, A126706, A246547, A332785, A369609.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved April 2024.

Code:

[c1] Generate a million terms of the sequence:

```
nn = 2^20;
c[_] := False; m[_] := 1;
f[x_] := f[x] = Times @@ FactorInteger[x][[All, 1]];
Array[Set[{a[#], c[#], m[#]}, {#, True, 2}] &, 2];
i = 1; j = r = 2;
Monitor[Do[(While[c[Set[k, # m[#]]], m[#]++]) &[r/f[j]];
  Set[{a[n], c[k], i, j, r},
    {k, True, j, k, f[j*k]}], {n, 3, nn}], n];
a369609 = Array[a, n];
```

[c2] Generate the sequence of multipliers m and a sequence of binary-compactified squarefree kernels r:

```
nn = 2^20;
c[_] := False; m[_] := 1;
f[x_] := f[x] = Times @@ FactorInteger[x][[All, 1]];
Array[Set[{a[#], c[#], m[#]}, {#, True, 2}] &, 2];
i = 1; j = r = 2;
A067255[n_] :=
 If[n == 1, {0},
  Function[f,
    ReplacePart[Table[0, {PrimePi[f[[-1, 1]]]}], #] &@
     Map[PrimePi@ First@ # -> Last@ # &, f]]@
    FactorInteger@ n]
Set[{a369609pb, a369609m},
 Transpose@
  Reap[Monitor[
    Do[(While[c[Set[k, # m[#]]], m[#]++];
     Sow[{FromDigits[Reverse@ A067255[#], 2], m[#]}]) &
       [r/f[j]];
     Set[{a[n], c[k], i, j, r},
       {k, True, j, k, f[j*k]}],
    {n, 3, nn}], n]][[-1, 1]] ]
```

[c3] Generate data associated with Appendix Tables A, B, and C:

```
nn = 2^20;
Q = FoldList[Times, Prime@ Range[64]];
c[_] := False; m[_] := 1;
f[x_] := Times @@ FactorInteger[x][[All, 1]];
Array[Set[{a[#], c[#], m[#]}, {#, True, 2}] &, 2];
i = ii = 1; j = jj = r = 2; u = 3; mm = 1; sa = sb = 2;
ra[1] = rb[1] = {2, 0, 1, 2}; rc[1] = {2, 2};
Reap[Monitor[
  Do[(While[c[Set[k, # m[#]]], m[#]++]) &[r/f[j]];
      If[PrimeQ[k], Set[ra[sa], {n, ii, jj, k}]; sa++];
      If[MemberQ[Q, k],
       Set[rb[sb], {n, ii, jj,
         StringJoin["P(",
           ToString[FirstPosition[Q, k][[1]]], ")"]}];
       sb++];
      Set[{c[k], h, i, j, hh, ii, jj, q},
        {True, i, f[j], k, ii, jj,
        k, f[j*k]}];
      If[q != r, mm++; Set[rc[mm], {n, k}]]; r = q,
  {n, 3, nn}], n] ][[-1, 1]];
Set[{a369609pp, a369609qq, a369609mm},
  {Array[ra, sa - 1],
   Array[rb, sb - 1],
   Array[rc, mm]}];
```

[C4] Generate data associated with Appendix Table D:

```
nn = 2^20;
c[_] := False; m[_] := 1;
f[x_] := f[x] = Times @@ FactorInteger[x][[All, 1]];
Array[Set[{a[#], c[#], m[#]}, {#, True, 2}] &, 2];
i = 1; j = r = 2;
Reap[Monitor[
  Do[(While[c[Set[k, # m[#]]], m[#]++]) &[r/f[j]];
    If[Divisible[k, f[k]^2], Sow[{n, i, j, k}]];
    Set[{c[k], i, j, r},
      {True, f[j], k, f[j*k]}], {n, 3, nn}],
  n]][[-1, 1]]
```

[C5] Generate a textual plot of divisibility patterns between $a(n-k)$ and $a(n+k)$ as seen in Tables F. Set j to show patterns associated with PRIME$(1\ldots j)$. Key to the plot appears below code:

```
n = 1509; j = 24;
w = ConstantArray[0, j]; k = 12;
rule1 = {0 -> ".", 1 -> "x", 2 -> "o", 3 -> "*"};
t = StringJoin @@ # & /@
  Array[#2 + #1 /. Dispatch[rule1] & @@
    {If[a369609m[[#]] == 1, w,
       ReplacePart[w, Map[# -> 1 &, PrimePi /@
         FactorInteger[a369609m[[#]]][[All, 1]] ] ] ],
       ReplacePart[w, Map[# -> 2 &,
         Position[Reverse@
           IntegerDigits[a369609pb[[#]], 2], 1]
             [[All, 1]] ] ]} &,
         2 k, n - k - 2];
Array[{n - k + # - 1, a369609[[n - k + # - 1]], t[[#]],
  Times @@ Prime@ Position[Reverse@
    IntegerDigits[a369609pb[[n - k + # - 3]], 2], 1]
      [[All, 1]], a369609m[[n - k + # - 3]]} &,
    Length[t]] ] // TableForm
(*
  Key:
     . indicates p divides neither r nor m(r),
       hence p does not divide a(n).
     o indicates p | r
     x indicates p | m(r).
     * indicates p divides both r and m(r).  *)
```

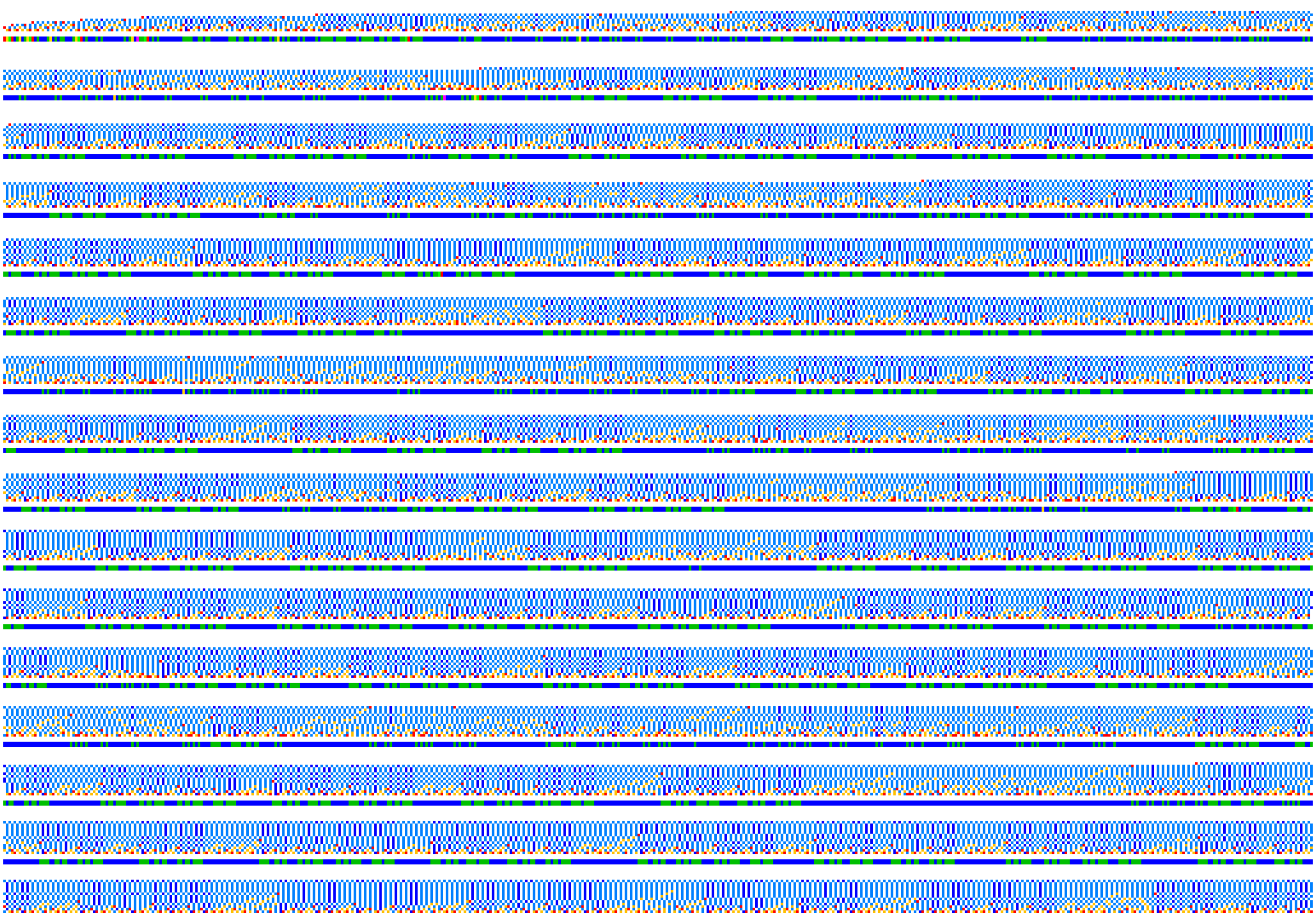

Figure 4. Aggregate divisibility pattern exhibited in A369609. Plot PRIME(i) | A369609(n) at (x, y) = (n, i) for n = 2…8194 in strips of 512 terms. If PRIME(i) | r but not m, we show such in blue. We show PRIME(i) | m in red, but if PRIME(i) also divides r, we use gold. If m = 1, we show dark blue. The strip under the plot shows primes in red, perfect powers of primes in gold, squarefree composites in green, primorials in bright green, and numbers neither prime powers nor squarefree in blue or magenta, the latter color representing numbers that are also powerful.

Some data based on a dataset of 2^27 = 134217728 terms:

Table A. Primes in the sequence:

n	a(n-2)	a(n-1)	a(n)	SMN	Case Mode	Diagram Table *	
2	-	1	(2)	(u)	(F)	1	
3	1	2	[3]	u	F	1	
6	4	6	[5]	u	F	1	
14	8	30	[7]	u	F	1	
33	16	210	[11]	u	F	1	
59	26	2310	13	u	A	1	
161	34	P(6)	17	u	A	F2	
363	171	P(7)	19	u	A	F4	
701	32	P(8)	[23]	u	F	F5	
1509	261	P(9)	29	u	A	F7	
2222	62	P(10)	31	u	A	F10	
4581	74	P(11)	37	u	A	F12	
7827	369	P(12)	41	u	A	F14	
20543	86	P(13)	43	u	A	F16	
28710	94	P(14)	47	u	A	F18	
87723	531	P(16)	59		A	F21	<-A
91307	108	P(17)	[53]	u	H	F22	
384195	639	P(19)	71		A	F26	
621674	134	P(20)/67	67		A	F27	
810244	122	P(20)/61	61	u	A	F28	
1080885	657	P(20)	73		A	F30	
2814146	711	P(21)	79		A	F32	
16009512	178	P(23)	89		A	F35	
29524905	873	P(24)	97		A		
94188167	927	P(26)	103		A		

Parentheses indicate given terms.
Brackets indicate primes that come in via Theorem J6.
SMN = smallest missing number.
Note A: For n >= 87723, primes are not in order.

Table B: Primorials in the sequence:

n	a(n-2)	a(n-1)	a(n)	Diagram Table *	
2	-	1	P(1)	1	
5	P(2)/2	4	P(2)	1	
13	P(3)/2	8	P(3)	1	
32	P(4)/2	16	P(4)	1	
58	P(5)/2	26	P(5)	1	
160	P(6)/2	34	P(6)	F2	
362	2×P(7)/3	171	P(7)	F4	
700	P(8)/2	32	P(8)	F5	
1508	2×P(9)/3	261	P(9)	F7	
2221	P(10)/2	62	P(10)	F10	
4580	P(11)/2	74	P(11)	F12	
7826	2×P(12)/3	369	P(12)	F14	
20542	P(13)/2	86	P(13)	F16	
28709	P(14)/2	94	P(14)	F18	
87722	2×P(16)/3	531	P(16)	F21	<-B
91306	5×P(17)/6	108	P(17)	F22	
384194	2×P(19)/3	639	P(19)	F26	
1080884	2×P(20)/3	657	P(20)	F30	
2814145	2×P(21)/3	711	P(21)	F32	
16009511	P(23)/2	178	P(23)	F35	
29524904	2×P(24)/3	873	P(24)		
94188166	2×P(26)/3	927	P(26)		

Note B: P(15) is missing. For n > 87722, primorials are not in order.

* For a diagram of terms around the landmark (primes, primorials, etc.) see the noted Table, either Table 1 or one of the Appendix Tables F. For instance, to see how the sequence behaves around primorial P(8) = 9699690, prime(9) = 23, and R(9) = P(9) = 223092870, see Appendix Table F9.

Table C: First occasions of R(k) = rad(a(n-2)*a(n-1)) = P(k):

k	n	a(n)	Diagram Table *	
1	2	2	1	
2	3	3	1	
3	6	5	1	
4	14	7	1	
5	33	11	1	
6	57	26	1	
7	125	595	F1	
8	287	209	F3	
9	701	23	F5	<-C
10	1029	21489	F6	
11	1898	84227	F9	
12	4557	4255	F11	
13	7125	4879	F13	
14	15595	582521	F15	
15	26138	595631	F17	
16	52449	4036109	F19	
17	82279	42067	F20	<-D
18	135396	2257	F23	
19	328641	91321	F24	
20	373179	2627	F25	
21	1037245	4199179	F29	
22	2067803	8943961661600459	F31	
23	5238559	818560837103471656403	F33	
24	6177592	18769372247	F34	
25	22983553	231478957		
26	41827189	3999317898971997931		
27	56618797	656499995352641		

Note C: n = 701 represents R(9) and a(701) = prime(9) = 23.
Note D: rad(a(n-2)*a(n-1)) increases to P(17) before P(16) enters the sequence.

Table D: Powerful numbers in the sequence:

k	n	a(n-2)	a(n-1)	a(n)		
1	4	2	3	4	2^2	
2	9	6	10	9	3^2	
3	12	10	15	8	2^3	
4	21	15	42	25	5^2	
5	31	10	105	16	2^4	
6	53	6	770	27	3^3	
7	55	3	1540	36	2^2×3^2	
8	110	22	2730	121	11^2	
9	228	39	39270	169	13^2	
10	558	34	1141140	289	17^2	
11	687	10	1939938	100	2^2×5^2	
12	699	6	4849845	32	2^5	
13	3145	58	13831757940	841	29^2	
14	4505	46	17440042620	529	23^2	
15	91217	154	87398197734282392685	484	2^2×11^2	<-E
16	91247	33	291327325780941308950	1089	3^2×11^2	
17	91267	30	128184023343614175938	225	3^2×5^2	
18	91273	10	384552070030842527814	125	5^3	
19	91275	5	769104140061685055628	200	2^3×5^2	
20	91283	14	274680050022030377010	49	7^2	
21	91299	42	320460058359035439845	72	2^3×3^2	
22	91301	6	640920116718070879690	81	3^4	
23	91305	6	1602300291795177199225	108	2^2×3^3	
24	135394	74	103932991900227710220	1369	37^2	

Note E: A cluster of powerful numbers appear in the sequence in the interval n = [91217..91305].

Table E: Number of instances of cases for n <= 2^20:

Case	Count
Ⓕ	21
Ⓖ	9450758
Ⓗ	323929
Ⓐ	9200604
Ⓑ	613456
Ⓒ	323907
Ⓓ	1322

Appendix Table F, a group of tables showing composition and cases for sequence landmarks shown in Appendix Tables A through D.

Table F1: R(7) for n = 125.

n	a(n)	prime p 111 2357137	Cases 111 2357137	
120	858	xo..oo.	Haggaa	
121	560	x.oo...	Dgaagg	
122	1287	.*..oo.	CBggaa	
123	700	*.*o...	BgBagg	
124	1716	xo..oo.	Haggaa	
125	595	..oo..x	CgaaggF	<- R(7)
126	2574	o*..oo.	aBggaag	
127	1190	x.oo..o	Hgaagga	
128	2145	.ox.oo.	CaHgaag	
129	238	o..o..o	agCagga	

Table F2: P(6) and prime(7).

n	a(n)	prime p 111 2357137	Cases 111 2357137	
155	340	*.o...o	Bgaggga	
156	9009	.*.ooo.	gBgaaag	
157	680	*.o...o	Bgaggga	
158	15015	.oxooo.	gaHaaag	
159	34	o.....o	agCggga	
160	30030	xooooo.	Haaaaag	P(6)
161	17	......o	Cgggggа	prime(7)
162	60060	*ooooo.	Baaaaag	
163	51	.x....o	gHgggga	
164	10010	o.oooo.	aCaaaag	

Table F3: R(8) for n = 287.

n	a(n)	prime p 1111 23571379	Cases 1111 23571379	
282	23205	.ooo.oo.	Caaagaa	
283	242	o...*...	agggBgg	
284	69615	.*oo.oo.	gBaagaa	
285	352	*...o...	Bgggagg	
286	92820	xooo.oo.	Haaagaa	
287	209	....o..x	CgggaggF	<- R(8)
288	139230	o*oo.oo.	aBaagaag	
289	418	x...o..o	Hgggagga	
290	116025	.o*o.oo.	CaBagaag	
291	836	*...o..o	Bgggagga	

Table F4: P(7) and prime(8) = 19.

n	a(n)	prime p 1111 23571379	Cases 1111 23571379	
357	114	oo.....o	aagCggga	
358	170170	x.ooooo.	Hgaaaaag	
359	57	.o.....o	Caggggga	
360	340340	*.ooooo.	Bgaaaaag	
361	171	.*.....o	gBggggga	
362	510510	oxooooo.	aHaaaaag	P(7)
363	19	.......o	gCggggga	prime(8)
364	1021020	*oooooo.	Baaaaaag	
365	38	x......o	Hgggggga	
366	255255	.oooooo.	Caaaaaag	

Table F5: P(8), prime(9) = 23, R(9).

n	a(n)	prime p 11112 235713793	Cases 11112 235713793	
695	48	*o......	BaCggggg	
696	3233230	x.oooooo	Hgaaaaaa	
697	54	x*......	DBgggggg	
698	4849845	.xoooooo	CHaaaaaa	
699	32	*.......	BCgggggg	2^5
700	9699690	xooooooo	Haaaaaaa	P(8)
701	23	x	CgggggggF	< E
702	19399380	*ooooooo.	Baaaaaaag	
703	46	x.......o	Hggggggga	
704	14549535	.*oooooo.	CBaaaaaag	

Note E: prime(9) = u, R(9) at n = 701

Table F6: R(10) for n = 1029.

n	a(n)	prime p 111122 2357137939	Cases 111122 2357137939	
1024	1204280	x.ooo.o.o.	Hgaaagaga	
1025	14079	.o...o.*..	CagggagBg	
1026	1505350	o.*oo.o.o.	agBaagaga	
1027	20007	.*...o.o..	gBgggagag	
1028	2107490	o.o*o.o.o.	agaBagaga	
1029	21489	.o...o.o.x	gagggagagF	<- R(10)
1030	2408560	*.ooo.o.o.	Bgaaagagag	
1031	42978	xo...o.o.o	Hagggagaga	
1032	1655885	..oo*.o.o.	CgaaBgagag	
1033	85956	*o...o.o.o	Bagggagaga	

Table F7: P(9) and prime(10) = 29.

n	a(n)	prime p 111122 2357137939	Cases 111122 2357137939	
1503	174	oo.......o	aagCgggga	
1504	74364290	x.ooooooo.	Hgaaaaaaag	
1505	87	.o.......o	Cagggggga	
1506	148728580	*.ooooooo.	Bgaaaaaaag	
1507	261	.*.......o	gBgggggga	
1508	223092870	oxoooooooo.	aHaaaaaaag	P(9)
1509	29	.........o	gCgggggga	prime(10)
1510	446185740	*oooooooo.	Baaaaaaaag	
1511	58	x........o	Hgggggggga	
1512	111546435	.oooooooo.	Caaaaaaaag	

Table F8: First case of duplex case D.

n	a(n)	prime p 111122 2357137939	Cases 111122 2357137939	
1657	135575	..*.o.o..o	CgBgagagga	
1658	954408	*o.o.o.oo.	Bagagagaag	
1659	189805	..oxo.o..o	ggaHagagga	
1660	102258	o*...o.oo.	aBgCgagaag	
1661	379610	x.ooo.o..o	Hgaaagagga	
1662	136344	xo...o.oo.	Dagggagaag	<- Case D for p = 2
1663	759220	x.ooo.o..o	Dgaaagagga	<- Case D for p = 2
1664	153387	.*...o.oo.	CBgggagaag	
1665	1518440	*.ooo.o..o	Bgaaagagga	
1666	204516	x*...o.oo.	HBgggagaag	

Table F9: R(11) for n = 1898.

n	a(n)	prime p 1111223 23571379391	Cases 1111223 23571379391	
1893	1428714	o*.o..o.oo.	aBgaggagaa	
1894	176605	..o.o*.o...	ggagaBgagg	
1895	1904952	*o.o..o.oo.	Bagaggagaa	
1896	258115	..o.oo.*...	ggagaagBgg	
1897	2381190	ooxo..o.oo.	aaHaggagaa	
1898	84227	....oo.o..x	ggCgaagaggF	<- R(11)
1899	4762380	*ooo..o.oo.	Baaaggagaag	
1900	168454	x...oo.o..o	Hgggaagagga	
1901	1190595	.ooo..o.oo.	Caaaggagaag	
1902	336908	*...oo.o..o	Bgggaagagga	

Table F10: P(10) and prime(11) = 31.

n	a(n)	prime p 1111223 23571379391	Cases 1111223 23571379391	
2216	186	xo........o	Haggggggga	
2217	1078282205	..oooooooo.	Cgaaaaaaaag	
2218	372	*o........o	Bagggggggga	
2219	3234846615	.xoooooooo.	gHaaaaaaaag	
2220	62	o.........o	aCgggggggga	
2221	6469693230	xooooooooo.	Haaaaaaaaag	P(10)
2222	31	..........o	Cggggggggga	prime(11)
2223	12939386460	*ooooooooo.	Baaaaaaaaag	
2224	124	x.........o	Hggggggggga	
2225	9704539845	.*oooooooo.	CBaaaaaaaag	

```
Key: . indicates p divides neither r nor m(r),
       hence p does not divide a(n).
     o indicates p | r       x indicates p | m(r).
     * indicates p divides both r and m(r).
```

Table 2 (Key to Case Letters)

	x	y	m	$a(n)$	$a(n+1)$	sym.
Ⓔ	·	·	·	·	Ⓔ	.. → .
Ⓕ	·	·	⊤	⊤	ⒼⒽ	.. → **x**
Ⓖ	·	⊤	·	·	ⒶⒷ	.@ → .
Ⓗ	·	⊤	⊤	⊤	ⒸⒹ	.@ → **x**
Ⓐ	⊤	·	·	⊤	ⒼⒽ	@. → **o**
Ⓑ	⊤	·	⊤	⊤	ⒼⒽ	@. → *****
Ⓒ	⊤	⊤	·	·	ⒶⒷ	@@ → .
Ⓓ	⊤	⊤	⊤	⊤	ⒸⒹ	@@ → **x**

Table 2 shows "." if prime p does not divide or "⊤" if p divides the entity shown in the column heading. The a(n+1) column shows possible cases that follow the case listed in the first column. The "sym." column refers to the A087207 *protocol function g defined as follows: "*`@`*" represents general divisibility, "* `.` *" represents general indivisibility, "***o***" represents* $p \nmid r \wedge p \nmid m$, *"***x***" represents* $p \nmid r \wedge p \mid m$, *and "********" represents* $p \mid r \wedge p \mid m$. *The arrow indicates output. For example, Case Ⓑ represents* `@.` → **x**, *which means that* $p \mid x$ *and* $p \mid m$, *but* $p \nmid y$. *Since both* $p \mid r$ *and* $p \mid m$, *we have* **x**.

```
Table F11: R(12) for n = 4557.
                  prime p   Cases
                      11112233      11112233
   n         a(n) 235713793917  235713793917
---------------------------------------------
4552   9592023441  .o.o*ooo.oo.  gagaBaaagaa
4553         5290  o.o.....*...  agaggggBgg
4554  10464025572  x*.ooooo.oo.  HBgaaaaagaa
4555         2875  ..*.....o...  CgBgggggagg
4556  12208029834  oo.*oooo.oo.  aagBaaaagaa
4557         4255  ..o.....o..x  ggagggggaggF  <- R(12)
4558  13952034096  *o.ooooo.oo.  Bagaaaaagaag
4559         8510  x.o.....o..o  Hgaggggagga
4560  11336027703  .o.oo*oo.oo.  CagaaBaagaag
4561        17020  *.o.....o..o  Bgaggggagga
```

```
Table F12: P(11) and prime(12) = 37.
                   prime p      Cases
                       11112233      11112233
   n          a(n) 235713793917  235713793917
----------------------------------------------
4575           740  *.o........o  Bgaggggggga
4576   60168147039  .*.oooooooo.  gBgaaaaaaaag
4577          1480  *.o........o  Bgaggggggga
4578  100280245065  .oxoooooooo.  gaHaaaaaaaag
4579            74  o..........o  agCgggggggga
4580  200560490130  xoooooooooo.  Haaaaaaaaaag  P(11)
4581            37  ...........o  Cgggggggggga  prime(12)
4582  401120980260  *oooooooooo.  Baaaaaaaaaag
4583           111  .x.........o  gHgggggggga
4584   66853496710  o.ooooooooo.  aCaaaaaaaaag
```

```
Table F13: R(13) for n = 7125.
                   prime p        Cases
                       111122334      111122334
   n          a(n)  2357137939171  2357137939171
-------------------------------------------------
7120   62359143990  xoo.oo.ooooo.  Haagaagaaaaa
7121          3808  x..o..o......  Dggaggaggggg
7122   93538715985  .*o.oo.ooooo.  CBagaagaaaaa
7123          4046  o..o..*......  aggaggBggggg
7124  124718287980  xoo.oo.ooooo.  Haagaagaaaaa
7125          4879  ...o..o.....x  CggaggaggggF  <- R(13)
7126  187077431970  o*o.oo.ooooo.  aBagaagaaaaag
7127          9758  x..o..o.....o  Hggaggaggggga
7128  155897859975  .o*.oo.ooooo.  CaBgaagaaaaag
7129         19516  *..o..o.....o  Bggaggaggggga
```

```
Table F14: P(12) and prime(13) = 41.
                      prime p        Cases
                          111122334      111122334
   n              a(n) 2357137939171  2357137939171
----------------------------------------------------
7821               246  oo..........o  aagCgggggggga
7822     2473579378270  x.ooooooooooo.  Hgaaaaaaaaaag
7823               123  .o...........o  Caggggggggggga
7824     4947158756540  *.ooooooooooo.  Bgaaaaaaaaaag
7825               369  .*...........o  gBggggggggggga
7826     7420738134810  oxooooooooooo.  aHaaaaaaaaaag  P(12)
7827                41  .............o  gCggggggggggga  prime(13)
7828    14841476269620  *oooooooooooo.  Baaaaaaaaaaag
7829                82  x............o  Hggggggggggga
7830     3710369067405  .oooooooooooo.  Caaaaaaaaaaag
```

```
Table F15: R(14) for n = 15595.
                    prime p        Cases
                       1111223344      1111223344
    n          a(n)  23571379391713  23571379391713
----------------------------------------------------
15590  101064898935  .*ooooo..o.oo.  gBaaaaaggagaa
15591        839914  o......oo.*...  aggggggaagBgg
15592  123523765365  .ooo*oo..o.oo.  gaaaBaaggagaa
15593        867008  *......oo.o...  Bggggggaagagg
15594  134753198580  x*ooooo..o.oo.  HBaaaaaggagaa
15595        582521  .......oo.o..x  CggggggaagaggF  <- R(14)
15596  157212065010  ooo*ooo..o.oo.  aaaBaaaggagaag
15597       1165042  x......oo.o..o  Hggggggaagagga
15598  145982631795  .oooo*o..o.oo.  CaaaaBaggagaag
15599       2330084  *......oo.o..o  Bggggggaagagga
```

```
Table F15A: Record setting multiplier M(i).
                                111122334445566777
 i          n       r  M(i) 2357137939171373917139 Table
--------------------------------------------------------
  3         3       1     3 .                      1
  4         6       1     5 .   .   .   .   .   .  1
  5        14       1     7 .                      1
  6        31       2     8 x   .   .   .   .   .  1
  7        33       1    11 .                      1
  8        55       3    12 .x  .   .   .   .   .  1
  9        57       2    13 x                      1
 10       121      35    16 ..xx    .   .   .   .  F1
 11       125      35    17 ..xx                   F1
 12       287      11    19 ....x   .   .   .   .  F3
 13       577    5005    20 ..xxxx                 -
 14       701       1    23 .       .   .   .   .  F5
 15       869     429    24 .x..xx                 -
 16      1027     741    27 .x...x.x    .   .   .  F6
 17      1029     741    29 .x...x.x               F6
 18      1898    2717    31 ....xx.x    .   .   .  F9
 19      4435    2185    35 ..x....xx              -
 20      4557     115    37 ..x.....x   .   .   .  F11
 21      7125     119    41 ...x..x                F13
 22     15595   13547    43 .......xx.x .   .   .  F15
 23     26138   12673    47 .......xxx             F17
 24     52447   76153    49 ...xx...x....x  .   .  F19
 25     52449   76153    53 ...xx...x....x         F19
 26     82279     713    59 ........x.x .   .   .  F20
 27    135396      37    61 ...........x           F23
 28    328597    6815    64 ..x......x....x .   .  -
 29    328641    1363    67 .........x....x        F24
 30    373179      37    71 ...........x    .   .  F25
 31   1037245   57523    73 ........x...x....x     F29
 32   2029833       .    74 .....xx.x..x.xxxx   .  -
 33   2067803       .    79 .....x.x.x..xxxxxx     F31
 34   2949082       .    81 .....xx.x.x.xxxxxxx.x  -
 35   5238559       .    83 ....x..x.xxxxxx.xxxx   F33
 36   6177592       .    89 ...........xxx.x.x     F34
 37  22336678       .    94 .....x....xxx.x.xxxx.x -
 38  22983495       .    95 .......x.x.......x.x   -
 39  22983553       .    97 .......x.x.......x.x   -
 40  41827189       .   101 ...x....x.....xxxxxxxx -
 41  56618797       .   103 ....x....x..xxx.xxx    -
...
                            ----------------------
                                111122334445566777
                            2357137939171373917139
```

Table 2 (Key to Case Letters)

	x	y	m	a(n)	a(n+1)	sym.
Ⓔ	·	·	·	·	Ⓔ	.. → .
Ⓕ	·	·	⊤	⊤	ⒼⒽ	.. → **x**
Ⓖ	·	⊤	·	·	ⒶⒷ	.@ → .
Ⓗ	·	⊤	⊤	⊤	ⒸⒹ	.@ → **x**
Ⓐ	⊤	·	·	⊤	ⒼⒽ	@. → **o**
Ⓑ	⊤	·	⊤	⊤	ⒼⒽ	@. → *****
Ⓒ	⊤	⊤	·	·	ⒶⒷ	@@ → .
Ⓓ	⊤	⊤	⊤	⊤	ⒸⒹ	@@ → **x**

Table F16: P(13) and prime(14) = 43.

n	a(n)	prime p 1111223344 23571379391713	Cases 1111223344 23571379391713	
20537	258	xo...........o	Haggggggggggga	
20538	50708377254535	..ooooooooooo.	Cgaaaaaaaaaaag	
20539	516	*o...........o	Baggggggggggga	
20540	152125131763605	.xooooooooooo.	gHaaaaaaaaaaag	
20541	86	o............o	aCggggggggggga	
20542	304250263527210	xoooooooooooo.	Haaaaaaaaaaaag	P(13)
20543	43	.............o	Cgggggggggggga	prime(14)
20544	608500527054420	*oooooooooooo.	Baaaaaaaaaaaag	
20545	172	x............o	Hgggggggggggga	
20546	456375395290815	.*ooooooooooo.	CBaaaaaaaaaaag	

Table F17: R(15) for n = 26138.

n	a(n)	prime p 11112233444 235713793917137	Cases 11112233444 235713793917137	
26133	3613166942385	.oo*ooo...oooo.	gaaBaaagggaaaa	
26134	582958	o......o*o.....	aggggggaBagggg	
26135	4129333648440	xoooooo...oooo.	Haaaaaagggaaaa	
26136	367517	.......oo*.....	CggggggaaBgggg	
26137	5161667060550	oo*oooo...oooo.	aaBaaaagggaaaa	
26138	595631	.......ooo....x	gggggggaaaggggF	<- R(15)
26139	6194000472660	**ooooo...oooo.	BBaaaaagggaaaag	
26140	1191262	x......ooo....o	Hggggggaaagggga	
26141	4645500354495	.*ooooo...oooo.	CBaaaaagggaaaag	
26142	2382524	*......ooo....o	Bggggggaaagggga	

Table F18: P(14) and prime(15) = 47.

n	a(n)	prime p 11112233444 235713793917137	Cases 11112233444 235713793917137	
28704	282	xo............o	Hagggggggggggga	
28705	2180460221945005	..oooooooooooo.	Cgaaaaaaaaaaaag	
28706	564	*o............o	Bagggggggggggga	
28707	6541380665835015	.xoooooooooooo.	gHaaaaaaaaaaaag	
28708	94	o.............o	aCgggggggggggga	
28709	13082761331670030	xooooooooooooo.	Haaaaaaaaaaaaag	P(14)
28710	47	..............o	Cggggggggggggga	prime(15)
28711	26165522663340060	*ooooooooooooo.	Baaaaaaaaaaaaag	
28712	188	x.............o	Hggggggggggggga	
28713	19624141997505045	.*oooooooooooo.	CBaaaaaaaaaaaag	

Table F19: R(16) for n = 52449.

n	a(n)	prime p 111122334445 2357137939171373	Cases 111122334445 2357137939171373	
52444	64595199935760	xoo..ooo.oooo.o.	Haaggaaagaaaaga	
52445	3274579	...oo...o....*..	CggaagggaggggBg	
52446	72669599927730	o*o..ooo.oooo.o.	aBaggaaagaaaaga	
52447	3731497	...*o...o....o..	gggBagggaggggag	
52448	80743999919700	*o*..ooo.oooo.o.	BaBggaaagaaaaga	
52449	4036109	...oo...o....o.x	gggaagggaggggagF	<- R(16)
52450	96892799903640	**o..ooo.oooo.o.	BBaggaaagaaaagag	
52451	8072218	x..oo...o....o.o	Hggaagggaggggaga	
52452	68632399931745	.oo..o*o.oooo.o.	CaaggaBagaaaagag	
52453	16144436	*..oo...o....o.o	Bggaagggaggggaga	

Table F20: R(17) for n = 82279.

n	a(n)	prime p 1111223344455 23571379391713739	Cases 1111223344455 23571379391713739	
82274	159974831234453235	.oo*oooo.o.ooooo.	gaaBaaaagagaaaaa	
82275	44206	o.......o.*......	agggggggagBggggg	
82276	182828378553660840	xooooooo.o.ooooo.	Haaaaaaagagaaaaa	
82277	22103	........o.*......	CgggggggagBggggg	
82278	228535473192076050	oo*ooooo.o.ooooo.	aaBaaaaagagaaaaa	
82279	42067	........o.o.....x	ggggggggagagggggF	<- R(17)
82280	274242567830491260	**oooooo.o.ooooo.	BBaaaaaagagaaaaag	
82281	84134	x.......o.o.....o	Hgggggggagaggggga	
82282	205681925872868445	.*oooooo.o.ooooo.	CBaaaaaagagaaaaag	
82283	168268	*.......o.o.....o	Bgggggggagaggggga	

Table F19A: m such that a(n) = m × r for small r.
Asterisks denote powerful m × r.

r = 2 n	m	r = 3 n	m	r = 5 n	m	r = 6 n	m
4	2*	5	2	8	2	7	2
12	4*	9	3*	21	5*	24	3
31	8*	11	5	23	7	51	4
57	13	26	7	687	20	695	8
699	16*	53	9*	91273	25*	91299	12*
91279	28	55	12*	91275	40	91305	18*
		697	18				
		91301	27*				
		91303	32				

r = 7 n	m	r = 10 n	m	r = 11 n	m	r = 13 n	m
16	2	10	2	35	2	59	1
91283	7*	27	4	110	11*	61	3
91285	16	29	5	112	13	73	5
		685	8	281	16	228	13*
		689	15	287	19	230	16
		91271	16	91219	25		
		91277	25	91249	55		

r = 14 n	m	r = 15 n	m	r = 17 n	m	r = 22 n	m
18	2	13	2	161	1	37	2
20	3	17	3	163	3	39	3
91281	7	19	5	181	11	108	4
91287	9	683	8	552	16	283	11
		691	9	558	17*	285	16
		693	12	560	19	91217	22*
		91267	15*				
		91269	18				

r = 23 n	m	r = 29 n	m	r = 30 n	m	r = 33 n	m
703	2	1509	1	15	2	41	1
4501	16	1511	2	681	3	43	3
4505	23*	3133	24	91265	8	45	4
4507	25	3145	29*			277	9
		3147	31			279	11
						91237	18
						91239	22
						91245	27
						91247	33*

r = 37 n	m
4581	1
4583	3
4601	7
135338	25
135394	37*
135396	61

Next expected powerful numbers, in no particular order: 64, 144, 196, 216, 400

```
Key: . indicates p divides neither r nor m(r),
       hence p does not divide a(n).
     o indicates p | r
     x indicates p | m(r).
     * indicates p divides both r and m(r).
```

Table 2 (Key to Case Letters)

	x	y	m	$a(n)$	$a(n+1)$	sym.
Ⓔ	·	·	·	·	Ⓔ	`.. → .`
Ⓕ	·	·	⊤	⊤	Ⓖ Ⓗ	`.. → x`
Ⓖ	·	⊤	·	·	Ⓐ Ⓑ	`.@ → .`
Ⓗ	·	⊤	⊤	⊤	Ⓒ Ⓓ	`.@ → x`
Ⓐ	⊤	·	·	⊤	Ⓖ Ⓗ	`@. → o`
Ⓑ	⊤	·	⊤	⊤	Ⓖ Ⓗ	`@. → *`
Ⓒ	⊤	⊤	·	·	Ⓐ Ⓑ	`@@ → .`
Ⓓ	⊤	⊤	⊤	⊤	Ⓒ Ⓓ	`@@ → x`

```
Table F21: P(16) and prime(17) = 59.
                               prime p            Cases
                                   1111223344455      1111223344455
    n                     a(n) 23571379391713739  23571379391713739
-------------------------------------------------------------------
87717                      354  oo..............o  aagCgggggggggggga
87718  10863052825730014910  x.oooooooooooooo.  Hgaaaaaaaaaaaaaag
87719                      177  .o..............o  Cagggggggggggggga
87720  21726105651460029820  *.oooooooooooooo.  Bgaaaaaaaaaaaaaag
87721                      531  .*..............o  gBgggggggggggggga
87722  32589158477190044730  oxoooooooooooooo.  aHaaaaaaaaaaaaaag  P(16)
87723                       59  ................o  gCgggggggggggggga  prime(17)
87724  65178316954380089460  *ooooooooooooooo.  Baaaaaaaaaaaaaaag
87725                      118  x...............o  Hggggggggggggggga
87726  16294579238595022365  .ooooooooooooooo.  Caaaaaaaaaaaaaaag
```

```
Table F22: P(17) and prime(16) = 53, entering via r = 1.
                                 prime p            Cases
                                     1111223344455      1111223344455
    n                       a(n) 23571379391713739  23571379391713739
---------------------------------------------------------------------
91301                         81  .*...............  CBggggggggggggggg
91302  128184023343614175938O  *.ooooooooooooooo  Bgaaaaaaaaaaaaaaa
91303                         96  xo...............  Hagggggggggggggggg
91304  160230029179517719922 5  ..*oooooooooooooo  CgBaaaaaaaaaaaaaa
91305                        108  **...............  BBggggggggggggggg
91306  192276035015421263907O  xxooooooooooooooo  HHaaaaaaaaaaaaaaa  P(17)
91307                         53  ...............x.  CCgggggggggggggHg  prime(16) = u
91308    3627849717272099319O  ooooooooooooooo.o  aaaaaaaaaaaaaaaCa
91309                        106  x..............o.  Hggggggggggggggag
91310    181392485863604965 95  .oooooooooooooo.o  Caaaaaaaaaaaaaaga
```

```
Table F23: R(18) for n = 52449.
                                 prime p             Cases
                                     11112233444556      11112233444556
     n                      a(n) 235713793917137391  235713793917137391
-----------------------------------------------------------------------
135391   7794974392517078266 5  .xooooooooo.ooooo.  CHaaaaaaaaagaaaaa
135392                      1184  *..........o......  BCgggggggggaggggg
135393  1039329919002277102 20  xoooooooooo.ooooo.  Haaaaaaaaaagaaaaa
135394                      1369  ...........*......  CggggggggggBggggg
135395  1558994878503415653 30  o*ooooooooo.ooooo.  aBaaaaaaaaagaaaaa
135396                      2257  ...........o.....x  gggggggggggagggggF  <- R(18)
135397  2078659838004554204 40  *oooooooooo.ooooo.  Baaaaaaaaaagaaaaag
135398                      4514  x..........o.....o  Hggggggggggaggggga
135399  1299162398752846377 75  .o*oooooooo.ooooo.  CaBaaaaaaaagaaaaag
135400                      9028  *..........o.....o  Bggggggggggaggggga
```

```
Table F24: R(19) for n = 328641.
                                 prime p              Cases
                                     111122334445566      111122334445566
     n                      a(n) 2357137939171373917  2357137939171373917
-------------------------------------------------------------------------
328636  12907745564131361443 5  .*ooooooo.oooo.ooo.  CBaaaaaaagaaaagaaa
328637                    128122  o........o....*....  aggggggggaggggBggg
328638  21512909273552269072 5  .o*oooooo.oooo.ooo.  gaBaaaaaagaaaagaaa
328639                    158108  *........*....o....  BggggggggBggggaggg
328640  25815491128262722887 0  x*ooooooo.oooo.ooo.  HBaaaaaaagaaaagaaa
328641                     91321  .........o....o...x  CggggggggaggggagggF  <- R(19)
328642  34420654837683630516 0  *oooooooo.oooo.ooo.  Baaaaaaaagaaaagaaag
328643                    182642  x........o....o...o  Hggggggggaggggaggga
328644  30118072982973176701 5  .oo*ooooo.oooo.ooo.  CaaBaaaaagaaaagaaag
328645                    365284  *........o....o...o  Bggggggggaggggaggga
```

```
Table F25: R(20) for n = 373179.  prime p               Cases
                                      1111223344455667      1111223344455667
     n                       a(n) 23571379391713739171  23571379391713739171
-----------------------------------------------------------------------------
373174   6371612068443459775037 1  .*.oooooooo.ooooooo.  CBgaaaaaaaagaaaaaaa
373175                       11840  *.o........o........  Bgaggggggggaggggggg
373176  10619353447405766291728 5  .oxoooooooo.ooooooo.  gaHaaaaaaaagaaaaaaa
373177                        2368  *..........o........  BgCggggggggaggggggg
373178  21238706894811532583457 0  xoooooooooo.ooooooo.  Haaaaaaaaaagaaaaaaa
373179                        2627  ...........o.......x  CggggggggggagggggggF  <- R(20)
373180  42477413789623065166914 0  *oooooooooo.ooooooo.  Baaaaaaaaaagaaaaaaag
373181                        5254  x..........o.......o  Hggggggggggaggggggga
373182  31858060342217298875185 5  .*ooooooooo.ooooooo.  CBaaaaaaaaagaaaaaaag
373183                       10508  *..........o.......o  Bggggggggggaggggggga
```

Key: . indicates p divides neither r nor m(r),
hence p does not divide a(n).
o indicates p | r
x indicates p | m(r).
* indicates p divides both r and m(r).

Table F26: P(19) and prime(20) = 71.

n	a(n)	prime p	Cases	
		1111223344455667	1111223344455667	
		23571379391713739171	23571379391713739171	
384189	426	oo.................o	aagCggggggggggggggga	
384190	2619440517026755685293030	x.ooooooooooooooooo.	Hgaaaaaaaaaaaaaaaaag	
384191	213	.o.................o	Caggggggggggggggggga	
384192	5238881034053511370586060	*.ooooooooooooooooo.	Bgaaaaaaaaaaaaaaaaag	
384193	639	.*.................o	gBggggggggggggggggga	
384194	7858321551080267055879090	oxooooooooooooooooo.	aHaaaaaaaaaaaaaaaaag	P(19)
384195	71	...................o	gCggggggggggggggggga	prime(20)
384196	15716643102160534111758180	*oooooooooooooooooo.	Baaaaaaaaaaaaaaaaaag	
384197	142	x..................o	Hgggggggggggggggggga	
384198	3929160775540133527939545	.oooooooooooooooooo.	Caaaaaaaaaaaaaaaaaag	

: Prime(19) = 67.

n	a(n)	prime p	Cases	
		1111223344455667	1111223344455667	
		23571379391713739171	23571379391713739171	
621669	2498242522955368481943651	.*.ooooooooooooooo.o	CBgaaaaaaaaaaaaaaaga	
621670	2680	*.o...............o.	Bgaggggggggggggggaag	
621671	4163737538258947469906085	.oxooooooooooooooo.o	gaHaaaaaaaaaaaaaaaga	
621672	134	o.................o.	agCgggggggggggggggag	
621673	8327475076517894939812170	xooooooooooooooooo.o	Haaaaaaaaaaaaaaaaaga	
621674	67	..................o.	Cgggggggggggggggggag	prime(19)
621675	16654950153035789879624340	*ooooooooooooooooo.o	Baaaaaaaaaaaaaaaaaga	
621676	201	.x................o.	gHggggggggggggggggag	
621677	2775825025505964979937390	o.oooooooooooooooo.o	aCaaaaaaaaaaaaaaaaga	
621678	402	xo................o.	Haggggggggggggggggag	

Table F28: Prime(18) = 61.

n	a(n)	prime p	Cases	
		1111223344455667	1111223344455667	
		23571379391713739171	23571379391713739171	
810239	2743971295705076857216797	.*.oooooooooooooo.oo	gBgaaaaaaaaaaaaaagaa	
810240	2440	*.o..............o..	Bgaggggggggggggggagg	
810241	4573285492841794762027995	.oxoooooooooooooo.oo	gaHaaaaaaaaaaaaaagaa	
810242	122	o................o..	agCggggggggggggggagg	
810243	9146570985683589524055990	xoooooooooooooooo.oo	Haaaaaaaaaaaaaaaagaa	
810244	61	.................o..	Cggggggggggggggggagg	prime(18)
810245	18293141971367179048111980	*oooooooooooooooo.oo	Baaaaaaaaaaaaaaaagaa	
810246	183	.x...............o..	gHgggggggggggggggagg	
810247	3048856995227863174685330	o.ooooooooooooooo.oo	aCaaaaaaaaaaaaaaagaa	
810248	366	xo...............o..	Hagggggggggggggggagg	

Table F29: R(21) for n = 1037245.

n	a(n)	prime p	Cases	
		11112233444556677	11112233444556677	
		235713793917137391713	235713793917137391713	
1037240	38797756036833889815720	xooooooo.ooo.oooo.oo.	Haaaaaaagaaagaaaagaa	
1037241	2358443	........o...*....o...	CgggggggagggBggggagg	
1037242	48497195046042362269650	oo*ooooo.ooo.oooo.oo.	aaBaaaaagaaagaaaagaa	
1037243	3508903	........o...o....*...	ggggggggagggaggggBgg	
1037244	58196634055250834723580	**oooooo.ooo.oooo.oo.	BBaaaaaagaaagaaaagaa	
1037245	4199179	........o...o....o..x	ggggggggagggaggggaggF	<- R(21)
1037246	67896073064459307177510	ooo*oooo.ooo.oooo.oo.	aaaBaaaagaaagaaaagaag	
1037247	8398358	x.......o...o....o..o	Hgggggggagggaggggagga	
1037248	43647475541438126042685	.*oooooo.ooo.oooo.oo.	CBaaaaaagaaagaaaagaag	
1037249	16796716	*.......o...o....o..o	Bgggggggagggaggggagga	

Table F30: P(20) and prime(21) = 73.

n	a(n)	prime p	Cases	
		11112233444556677	11112233444556677	
		235713793917137391713	235713793917137391713	
1080879	438	oo..................o	aagCgggggggggggggggga	
1080880	185980276708899653655805130	x.oooooooooooooooooo.	Hgaaaaaaaaaaaaaaaaaag	
1080881	219	.o..................o	Cagggggggggggggggggga	
1080882	371960553417799307311610260	*.oooooooooooooooooo.	Bgaaaaaaaaaaaaaaaaaag	
1080883	657	.*..................o	gBgggggggggggggggggga	
1080884	557940830126698960967415390	oxoooooooooooooooooo.	aHaaaaaaaaaaaaaaaaaag	P(20)
1080885	73	....................o	gCgggggggggggggggggga	prime(21)
1080886	1115881660253397921934830780	*ooooooooooooooooooo.	Baaaaaaaaaaaaaaaaaaag	
1080887	146	x...................o	Hggggggggggggggggggga	
1080888	278970415063349480483707695	.ooooooooooooooooooo.	Caaaaaaaaaaaaaaaaaaag	

Table 2 (Key to Case Letters)

	x	y	m	$a(n)$	$a(n+1)$	sym.
Ⓔ	·	·	·	·	Ⓔ	`..` → `.`
Ⓕ	·	·	⊤	⊤	Ⓖ Ⓗ	`..` → `x`
Ⓖ	·	⊤	·	·	Ⓐ Ⓑ	`.@` → `.`
Ⓗ	·	⊤	⊤	⊤	Ⓒ Ⓓ	`.@` → `x`
Ⓐ	⊤	·	·	⊤	Ⓖ Ⓗ	`@.` → `o`
Ⓑ	⊤	·	⊤	⊤	Ⓖ Ⓗ	`@.` → `*`
Ⓒ	⊤	⊤	·	·	Ⓐ Ⓑ	`@@` → `.`
Ⓓ	⊤	⊤	⊤	⊤	Ⓒ Ⓓ	`@@` → `x`

Table F31: R(22) for n = 1037245.

n	a(n)	prime p	Cases	
		111122334445566777	111122334445566777	
		2357137939171373917139	2357137939171373917139	
2067798	5036585402212980	xoo*o.o.o.oo......ooo.	HaaBagagagaaggggggaaa	
2067799	6679667570056039	.....o.o.o..oooo*o....	CggggagagaggaaaaBaggg	
2067800	5396341502371050	o**oo.o.o.oo......ooo.	aBBaagagagaaggggggaaa	
2067801	6906096979210481	.....o.o.o..ooooo*....	gggggagagaggaaaaaBggg	
2067802	5756097602529120	*oooo.o.o.oo......ooo.	Baaaagagagaaggggggaaa	
2067803	8943961661600459	.....o.o.o..oooooo...x	gggggagagaggaaaaaagggF	<- R(22)
2067804	6115853702687190	ooooo.*.o.oo......ooo.	aaaaagBgagaagggggaaag	
2067805	17887923323200918	x....o.o.o..oooooo...o	Hggggagagaggaaaaaaggga	
2067806	5216463452292015	.oooo.o.oxoo......ooo.	CaaaagagaHaaggggggaaag	
2067807	616824942179342	o....o.o....oooooo...o	aggggagagCggaaaaaaggga	

Table F32: P(21) and prime(22) = 79.

n	a(n)	prime p	Cases	
		111122334445566777	111122334445566777	
		2357137939171373917139	2357137939171373917139	
2814140	474	oo....................o	aagCggggggggggggggggga	
2814141	13576560199749674716873774490	x.oooooooooooooooooooo.	Hgaaaaaaaaaaaaaaaaaaag	
2814142	237	.o....................o	Caggggggggggggggggggga	
2814143	27153120399499349433747548980	*.oooooooooooooooooooo.	Bgaaaaaaaaaaaaaaaaaaag	
2814144	711	.*....................o	gBggggggggggggggggggga	
2814145	40729680599249024150621323470	oxoooooooooooooooooooo.	aHaaaaaaaaaaaaaaaaaaag	P(21)
2814146	79	......................o	gCggggggggggggggggggga	prime(22)
2814147	81459361198498048301242646940	*ooooooooooooooooooooo.	Baaaaaaaaaaaaaaaaaaaag	
2814148	158	x.....................o	Hgggggggggggggggggggga	
2814149	20364840299624512075310661735	.ooooooooooooooooooooo.	Caaaaaaaaaaaaaaaaaaaag	

Table F33: R(23) for n = 1037245.

n	a(n)	prime p	Cases	
		1111223344455667778	1111223344455667778	
		23571379391713739171393	23571379391713739171393	
5238554	4893915703950	o**o.oo.o......o....oo.	aBBagaagagggggggaggggaa	
5238555	660765976938946999747	....o..o.oooooo.oo*o...	ggggaggagaaaaaagaaBagg	
5238556	5220176750880	*ooo.oo.o......o....oo.	Baaagaagagggggggaggggaa	
5238557	700214691980078163911	....o..o.oooooo.ooo*...	ggggaggagaaaaaagaaaBgg	
5238558	5546437797810	oooo.o*.o......o....oo.	aaaagaBgaggggggaggggaa	
5238559	818560837103471656403	....o..o.oooooo.oooo..x	ggggaggagaaaaaagaaaaggF	<- R(23)
5238560	5872698844740	**oo.oo.o......o....oo.	BBaagaagaggggggaggggaag	
5238561	1637121674206943312806	x...o..o.oooooo.oooo..o	Hgggaggagaaaaaagaaaagga	
5238562	4404524133555	.*oo.oo.o......o....oo.	CBaagaagaggggggaggggaag	
5238563	3274243348413886625612	*...o..o.oooooo.oooo..o	Bgggaggagaaaaaagaaaagga	

Table F34: R(24) for n = 6177592.

n	a(n)	prime p	Cases	
		11112233444556677788	11112233444556677788	
		235713793917137391713939	235713793917137391713939	
6177587	10764041730337001907005205	.ooooo*oooo...o.o.ooooo.	gaaaaaBaaaagggagagaaaaa	
6177588	25728802406	o..........ooo.o.*......	aggggggggggaaagagBggggg	
6177589	12030399580964884484299935	.oooooo*ooo...o.o.ooooo.	gaaaaaaBaaagggagagaaaaa	
6177590	26994153344	*..........ooo.o.o......	Bggggggggggaaagagaggggg	
6177591	12663578506278825772947300	xo*oooooooo...o.o.ooooo.	HaBaaaaaaaagggagagaaaaa	
6177592	18769372247	...........ooo.o.o.....x	CggggggggggaaagagagggggF	<- R(24)
6177593	13929936356906708350242030	oooo*oooooo...o.o.ooooo.	aaaaBaaaaaagggagagaaaaag	
6177594	37538744494	x..........ooo.o.o.....o	Hggggggggggaaagagaggggga	
6177595	13296757431592767061594665	.*o*ooooooo...o.o.ooooo.	CBaBaaaaaaagggagagaaaaag	
6177596	75077488988	*..........ooo.o.o.....o	Bggggggggggaaagagaggggga	

Table F35: P(23) and prime(24) = 89.

n	a(n)	prime p	Cases	
		11112233444556677788	11112233444556677788	
		235713793917137391713939	235713793917137391713939	
16009506	534	xo......................o	Haggggggggggggggggggggga	
16009507	44510752614879308559270669665465	..oooooooooooooooooooooo.	Cgaaaaaaaaaaaaaaaaaaaaag	
16009508	1068	*o......................o	Baggggggggggggggggggggga	
16009509	133532257844637925677812008996395	.xoooooooooooooooooooooo.	gHaaaaaaaaaaaaaaaaaaaaag	
16009510	178	o.......................o	aCgggggggggggggggggggga	
16009511	267064515689275851355624017992790	xooooooooooooooooooooooo.	Haaaaaaaaaaaaaaaaaaaaaag	P(23)
16009512	89	........................o	Cgggggggggggggggggggggga	prime(24)
16009513	534129031378551702711248035985580	*ooooooooooooooooooooooo.	Baaaaaaaaaaaaaaaaaaaaaag	
16009514	356	x.......................o	Hgggggggggggggggggggggga	
16009515	400596773533913777033436026989185	.*oooooooooooooooooooooo.	CBaaaaaaaaaaaaaaaaaaaaag	

Table G: Example of a transition from low to high alternating divisibility coherence.

n	a(n)	1111223344 23571379391713	r	m(r)
3940	6533219	...*o.o.o.o...	933317	7
3941	1289340	**o..o.o.o....	214890	6
3942	7466536	x..oo.o.o.o...	933317	8
3943	967005	.*o..o.o.o....	107445	9
3944	13066438	o..*o.o.o.o...	1866634	7
3945	1396785	.oo..*.o.o....	107445	13
3946	14933072	*..oo.o.o.o...	1866634	8
3947	1611675	.**..o.o.o....	107445	15
3948	20532974	o..o*.o.o.o...	1866634	11
3949	1719120	xoo..o.o.o....	107445	16
3950	10266487	...o*.o.o.o...	933317	11
3951	1934010	o*o..o.o.o....	214890	9
3952	15866389	...oo.*.o.o...	933317	17
3953	2148900	*o*..o.o.o....	214890	10
3954	17733023	...oo.oxo.o...	933317	19
3955	33930	o*o..o...o....	11310	3
3956	35466046	x..oo.ooo.o...	17733023	2
3957	16965	.*o..o...o....	5655	3
3958	70932092	*..oo.ooo.o...	35466046	2
3959	28275	.o*..o...o....	5655	5
3960	106398138	ox.oo.ooo.o...	35466046	3
3961	9425	..*..o...o....	1885	5
3962	212796276	*o.oo.ooo.o...	106398138	2
3963	18850	x.*..o...o....	1885	10
3964	53199069	.o.oo.ooo.o...	53199069	1
3965	30160	*.o..o...o....	3770	8
3966	159597207	.*.oo.ooo.o...	53199069	3
3967	37700	*.*..o...o....	3770	10
3968	265995345	.oxoo.ooo.o...	53199069	5
3969	6032	*....o...o....	754	8
3970	531990690	xoooo.ooo.o...	265995345	2
3971	4901	.....*...o....	377	13
3972	1063981380	*oooo.ooo.o...	531990690	2
3973	9802	x....*...o....	377	26
3974	797986035	.*ooo.ooo.o...	265995345	3
3975	12064	*....o...o....	754	16

Table J: Dilation j.

n	Q k+j	P k	j	p
2	1	1	0	>2F
3	2	.	1	(2F)
5	.	2	0	>3F
6	3	.	1	(3F)
13	.	3	0	>4F
14	4	.	1	(4F)
32	.	4	0	>5F
33	5	.	1	(5F)
57	6	.	2	
58	.	5	1	>6A
125	7	.	2	
160	.	6	1	>7A
287	8	.	2	
362	.	7	1	>8A
700	.	8	0	>9F
701	9	.	1	(9F)
1029	10	.	2	
1508	.	9	1	>10A
1898	11	.	2	
2221	.	10	1	>11A
4557	12	.	2	
4580	.	11	1	>12A
7125	13	.	2	
7826	.	12	1	>13A
15595	14	.	2	
20542	.	13	1	>14A
26138	15	.	2	
28709	.	14	1	>15A
52449	16	.	2	
82279	17	.	3	
87722	.	16	1	>17A
91306	.	17	0	>16H
135396	18	.	1	
328641	19	.	2	
373179	20	.	3	
384194	.	19	1	>20A
621674	.	.	.	(19A)
810244	.	.	.	(18A)
1037245	21	.	2	
1080884	.	20	1	>21A
2067803	22	.	2	
2814145	.	21	1	>22A
5238559	23	.	2	
6177592	24	.	3	
16009511	.	23	1	>24A
22983553	25	.	2	
29524904	25	24	1	>25A
41827189	26	.	2	
56618797	27	.	3	
94188166	.	26	1	>27A

Table J Key:

$Q(k+j)$ is the largest prime factor seen in $a(1\ldots n)$.

$\mathcal{P}(k)$ is the largest primorial seen in $a(1\ldots n)$.

j represents dilation.

"." represents no change from the figure above.

p: ">" represents the prime that follows $\mathcal{P}(k)$.

Parentheses represent $a(n)$ = PRIME(i), where i is the number in parentheses. The letter is the mode of entry of PRIME(i).

For example, for $n = 362$, $a(32) = \mathcal{P}(4)$ and $a(33)$ = PRIME(5), coming in via Case Ⓕ. $a(33)$ is the point where Q = PRIME(5).

$a(810244)$ = PRIME(18), coming in through Case Ⓐ, while $\mathcal{P}(19)$ is the largest primorial in the sequence and Q = PRIME(20).

Table H: Coherent interval n = 91217..91305

n	a(n)	primes 1111223344455 23571379391713739	s	m(s)	notes
91200	8323637879455465970	x.o..oooooooooooo	4161818939727732985	2	
91201	4851	.*.*o............	231	21	
91202	16647275758910931940	*.o..oooooooooooo	8323637879455465970	2	
91203	5082	xo.o*............	231	22	
91204	12485456819183198955	.xo..oooooooooooo	4161818939727732985	3	
91205	2156	*..*o............	154	14	
91206	24970913638366397910	xoo..oooooooooooo	12485456819183198955	2	
91207	2464	x..oo............	77	32	
91208	37456370457549596865	.*o..oooooooooooo	12485456819183198955	3	
91209	3388	*..o*............	154	22	
91210	49941827276732795820	xoo..oooooooooooo	12485456819183198955	4	
91211	3773	...*o............	77	49	
91212	74912740915099193730	o*o..oooooooooooo	24970913638366397910	3	
91213	4312	x..*o............	77	56	
91214	62427284095915994775	.o*..oooooooooooo	12485456819183198955	5	
91215	4928	*..oo............	154	32	
91216	87398197734282392685	.oox.oooooooooooo	12485456819183198955	7	
91217	484	*...*............	22	22	2^2*11^2
91218	174796395468564785370	xooo.oooooooooooo	87398197734282392685	2	
91219	275	..x.o............	11	25	
91220	34959279093712957074	oo.o.oooooooooooo	34959279093712957074	1	
91221	550	x.*.o............	55	10	
91222	17479639546856478537	.o.o.oooooooooooo	17479639546856478537	1	
91223	880	*.o.o............	110	8	
91224	52438918640569435611	.*.o.oooooooooooo	17479639546856478537	3	
91225	990	oxo.o............	110	9	
91226	5826546515618826179	...o.oooooooooooo	5826546515618826179	1	
91227	1320	*oo.o............	330	4	
91228	11653093031237652358	x..o.oooooooooooo	5826546515618826179	2	
91229	825	.o*.o............	165	5	
91230	23306186062475304716	*..o.oooooooooooo	11653093031237652358	2	
91231	1485	.*o.o............	165	9	
91232	46612372124950609432	*..o.oooooooooooo	11653093031237652358	4	
91233	1650	xo*.o............	165	10	
91234	29132732578094130895	..xo.oooooooooooo	5826546515618826179	5	
91235	528	*o..o............	66	8	
91236	58265465156188261790	x.oo.oooooooooooo	29132732578094130895	2	
91237	594	x*..o............	33	18	
91238	116530930312376523580	x.oo.oooooooooooo	29132732578094130895	4	
91239	726	xo..*............	33	22	
91240	145663662890470654475	..*o.oooooooooooo	29132732578094130895	5	
91241	792	**..o............	66	12	
91242	203929128046658916265	..o*.oooooooooooo	29132732578094130895	7	
91243	1056	*o..o............	66	16	
91244	233061860624753047160	x.oo.oooooooooooo	29132732578094130895	8	
91245	891	.*..o............	33	27	
91246	291327325780941308950	o.*o.oooooooooooo	58265465156188261790	5	
91247	1089	.*..*............	33	33	3^2*11^2
91248	349592790937129570740	*xoo.oooooooooooo	58265465156188261790	6	
91249	605	..x.*............	11	55	
91250	69918558187425914148	*o.o.oooooooooooo	34959279093712957074	2	
91251	1100	x.*.o............	55	20	
91252	104877837281138871222	x*.o.oooooooooooo	17479639546856478537	6	
91253	1210	x.o.*............	55	22	
91254	122357476827995349759	.o.*.oooooooooooo	17479639546856478537	7	
91255	1760	*.o.o............	110	16	
91256	139837116374851828296	xo.o.oooooooooooo	17479639546856478537	8	
91257	1375	..*.o............	55	25	
91258	209755674562277742444	**.o.oooooooooooo	34959279093712957074	6	
91259	1815	.xo.*............	55	33	
91260	81571651218663566506	o..*.oooooooooooo	11653093031237652358	7	
91261	1980	x*o.o............	165	12	
91262	40785825609331783253	...*.oooooooooooo	5826546515618826179	7	
91263	2640	*oo.o............	330	8	
91264	64092011671807087969	...oxoooooooooooo	5826546515618826179	11	
91265	240	*oo..............	30	8	
91266	128184023343614175938	x..oooooooooooooo	64092011671807087969	2	
91267	225	.**..............	15	15	3^2*5^2
91268	256368046687228351876	*..oooooooooooooo	128184023343614175938	2	
91269	270	x*o..............	15	18	
91270	192276035015421263907	.x.oooooooooooooo	64092011671807087969	3	
91271	160	*.o..............	10	16	
91272	384552070030842527814	xo.oooooooooooooo	192276035015421263907	2	
91273	125	..*..............	5	25	5^3
91274	769104140061685055628	*o.oooooooooooooo	384552070030842527814	2	
91275	200	x.*..............	5	40	2^3*5^2
91276	576828105046263791721	.*.oooooooooooooo	192276035015421263907	3	
91277	250	o.*..............	10	25	
91278	961380175077106319535	.oxoooooooooooooo	192276035015421263907	5	
91279	56	*..x.............	2	28	
91280	137340025011015188505	.oo.ooooooooooooo	137340025011015188505	1	
91281	98	o..*.............	14	7	
91282	274680050022030377010	xoo.ooooooooooooo	137340025011015188505	2	
91283	49	...*.............	7	7	7^2
91284	549360100044060754020	*oo.ooooooooooooo	274680050022030377010	2	
91285	112	x..o.............	7	16	
91286	412020075033045565515	.*o.ooooooooooooo	137340025011015188505	3	
91287	126	ox.o.............	14	9	
91288	45780008337005062835	..o.ooooooooooooo	45780008337005062835	1	
91289	168	*o.o.............	42	4	
91290	91560016674010125670	x.o.ooooooooooooo	45780008337005062835	2	
91291	147	.o.*.............	21	7	
91292	183120033348020251340	*.o.ooooooooooooo	91560016674010125670	2	
91293	189	.*.o.............	21	9	
91294	366240066696040502680	*.o.ooooooooooooo	91560016674010125670	4	
91295	252	x*.o.............	21	12	
91296	228900041685025314175	..*.ooooooooooooo	45780008337005062835	5	
91297	294	oo.*.............	42	7	
91298	320460058359035439845	..oxooooooooooooo	45780008337005062835	7	
91299	72	**...............	6	12	2^3*3^2
91300	640920116718070879690	x.ooooooooooooooo	320460058359035439845	2	
91301	81	.*...............	3	27	3^4
91302	1281840233436141759380	*.ooooooooooooooo	640920116718070879690	2	
91303	96	xo...............	3	32	
91304	1602300291795177199225	..*oooooooooooooo	320460058359035439845	5	
91305	108	**...............	6	18	2^2*3^3
91306	1922760350154212639070	xxooooooooooooooo	320460058359035439845	6	P(17)
91307	53	...............x.	1	53	prime(16)
91308	36278497172720993190	ooooooooooooooo.o	36278497172720993190	1	
91309	106	x..............o.	53	2	
91310	18139248586360496595	.oooooooooooooo.o	18139248586360496595	1	
91311	212	*..............o.	106	2	
91312	54417745759081489785	.*ooooooooooooo.o	18139248586360496595	3	
91313	318	ox.............o.	106	3	
91314	6046416195453498865	..ooooooooooooo.o	6046416195453498865	1	
91315	636	*o.............o.	318	2	
91316	12092832390906997730	x.ooooooooooooo.o	6046416195453498865	2	
91317	159	.o.............o.	159	1	
91318	24185664781813995460	*.ooooooooooooo.o	12092832390906997730	2	
91319	477	.*.............o.	159	3	