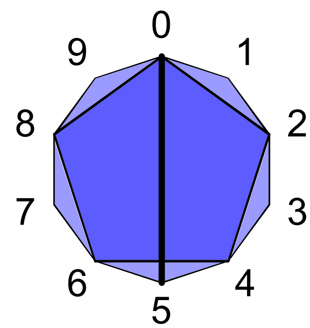
The Reciprocal Divisor Method

for Abbreviation of Multiplication Tables



This booklet is dedicated to my wife,

Laura Ann.

First Edition, 3 October 2007

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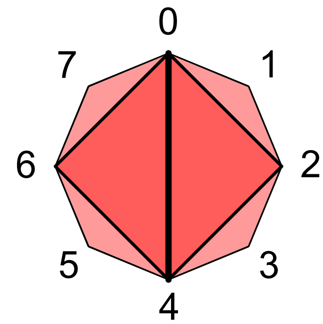

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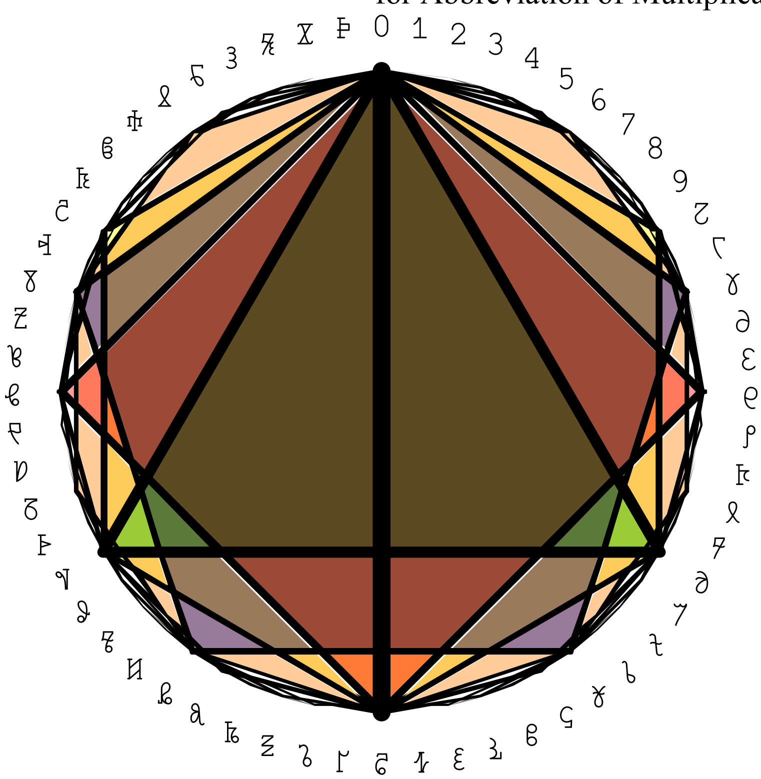

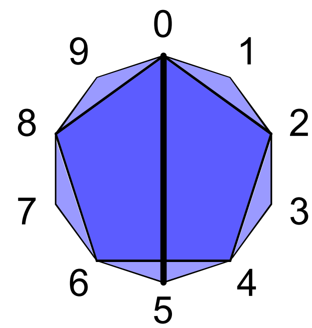

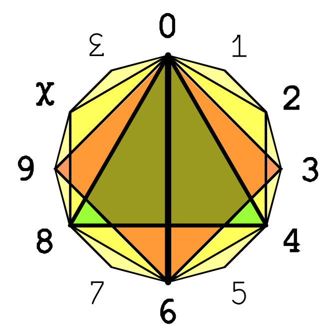

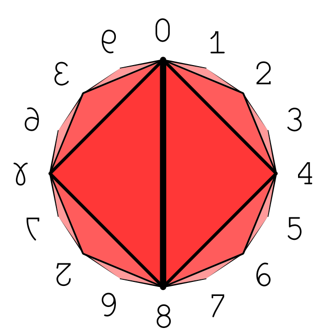

Michael De Vlieger • 3 October 2007 • Saint Louis, Missouri

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Preface

This booklet describes the Reciprocal Divisor Method for Abbreviated Multiplication Tables, which was developed in March and April 2007. The objective of this booklet is to describe a set of methods enabling the leverage of the divisors of highly composite numbers so that memorization of their multiplication tables is minimized. A secondary objective is to describe how the methods work. The Reciprocal Divisor Method is a toolset with which exploration of a range of bases between around 16 through 120 is rendered less burdensome.

I do not know whether this method is or is not my invention. It is shameful to me that I am no academic and lack the time or patience to properly research the issue. Instead of academics, my focus has been the development of a technique to facilitate the use of pure sexagesimal in daily business. Perhaps this development, at best, is akin to Baron Haussmann's in 19th century Paris; for this I apologize. I do hope you enjoy the "boulevards" these techniques might open up for your thought.

The ancient Sumerians and Babylonians, our forefathers, are known to have used sexagesimal notation and mathematics millennia before Christ. These people knew about the reciprocal divisor pairs across several sexagesimal ranks, using cuneiform digits and place notation to record their affairs. Perceivably, our fathers had some knowledge of multiplication using a method involving reciprocal divisors. They produced and used sexagesimal multiplication tables which can be viewed today on the internet.

In 1992 I expanded a set of transdecimal digits which I had developed to represent the larger bases 12, 16, and 20 to include 50 new symbols. This set was called *arqam arimaxa*, Arabic for "Reema's numbers", named after an erstwhile girlfriend. This symbol set, now simply Argam, has presently grown into the thousands, with special glyphs for special numbers [illegible] ($2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$) and [illegible] ($2187 = 3^8$), as examples. This set of symbols was instrumental to the "discovery" of the Reciprocal Divisor Method.

The dozenal system has been and remains an important tool in my work as an architect and businessman. I believe that dozenal is the optimum base for human general computation. The next greater integer in the set of superabundant numbers after 12 is 60. Sixty is even more powerful than twelve, but it cannot be wielded without the ability to multiply. The ability to effectively multiply in pure sexagesimal is limited by the human ability to memorize its multiplication table of 1830 products of unique factor combinations. So the development of a set of methods to render pure sexagesimal multiplication became a personal holy grail.

It is with a humble layman's honor that I present to you exclusively this morning the first draft of this "Reciprocal Divisor Method". May you climb mountains ever higher with the tools presented in this simple booklet. I have surveyed the heights myself during the past few months and am entirely awestruck by the patterns the Lord has laid out within His numbers, made more evident with the twin toolsets of the Argam and the RDM.

Sincerely, Michael Thomas De Vlieger, 3 October 2007, Saint Louis, Missouri

Part 1 • Properties of Integers

The Reciprocal Divisor Method for Abbreviated Multiplication tables relies on the properties of highly composite integers. The set of divisors D_r of a given integer *r*, when paired {*d*, *d'*} so that $d \cdot d' = r$, establishes a reciprocal relationship between divisors *d* and *d'*. This relationship can be leveraged so that a fraction of the full multiplication table of a large base needs to be memorized. The prime composition of the integer *r* is the source of that integer's divisors. The prime factors dictate how the base *r* will relate to quantities which will be expressed in terms of *r*. The totatives of *r* reveal "gaps in coverage" that the prime factors of the base cannot reach. These totatives are weaknesses that must be overcome by the method. Thus, the first section of this booklet deals with the elementary nature of integers.

Integral Bases

We will consider a handful of integers *r* as radices in this presentation. Integral bases are exclusively considered because the presentation focuses on practical solutions to the human perception of numbers. The numbers that will be considered are 8 (octal), 10 (decimal), 12 (dozenal), 16 (hexadecimal), and 60 (sexagesimal). Sexagesimal proficiency is the target for the Reciprocal Divisor Method.

Prime Composition

Each integer is a product of a set of prime numbers. These numbers are the *prime factors* of the integer in question. Each integer possesses a unique set of prime numbers, so that it is possible to construct a means of identifying each integer by its prime factors.

<u>Eight</u> consists of three instances of the same prime factor 2; it is the cube of 2. Octal expresses a quantity, probing deeply for content of the prime 2. Because there is no diversity among the prime factors of eight beyond repetition of the same simplest prime, eight cannot test for any other content. All the odd digits of an octal number are relatively prime to eight, and may harbor a prime number besides 2.

<u>Ten</u> is the product of the first and third primes, namely 2 and 5. Because it is the product of two primes, some mathematicians call ten a "diprime". Ten does not represent the second prime number, 3, which occurs more often than 5. Users of decimal, the system based on ten, do benefit from the fact that ten is one more than nine, the square of three. This fact means that decimal users have an easy way to "detect" the divisibility by three of an integer represented in decimal.

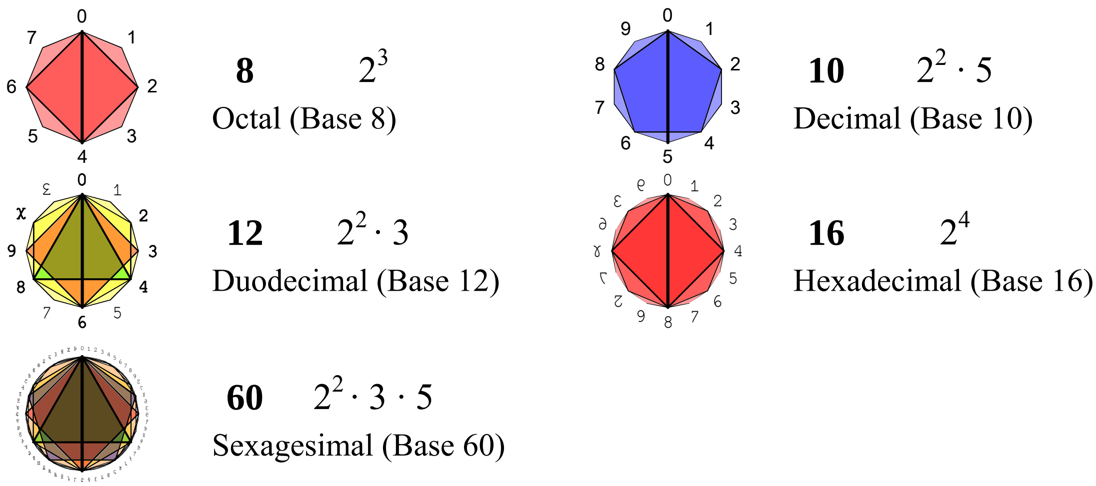


Figure 1A. *Prime factors of certain integers.*

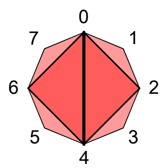
Twelve is the product of the second prime and the square of the simplest prime, namely $2^2 \cdot 3$. It is thus the simplest expression of the prime factor "shape" $\Pi_0^2 \cdot \Pi_1$. Because the prime factors involved are the simplest and occur so frequently, these interact often. The dozenal multiplication table is highly rhythmic and intuitive compared to that of other small integers. The dozen is naturally pervasive in life, simply because its factors are the simplest and commonest factors; thus multiples of the dozen are very common multiples. Dozenal does not support five the way decimal does three; five is relatively prime to twelve; it is a totative, so detecting divisibility by five in base twelve is complicated.

Sixteen is the fourth power of two; it is sheerly a product of the first prime. Hexadecimal bundles quantities terms of two, four times over; thus it is a deep study of a given number for that number's content of two. Where ten includes two as a factor, offering the basic ability to test for evenness, and twelve includes the second power of two for additional power to detect content of two, sixteen focuses its power on two to the exclusion of any other prime. Each additional hexadecimal digit yields only four additional divisors because the prime factors of 16 are indeed the same prime number. This slower compounding occurs despite the fact that this base has four prime factors,

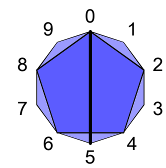
Sixty is the product of the simplest three primes, representing the first prime twice. Thus sixty's prime factors are $2^2 \cdot 3 \cdot 5$. Sixty offers the user a diverse set of primes through which to see the world while putting a little power behind its ability to test for content of two. Where dozenal yields a "compact" set of prime factors, sixty continues the trend established by dozenal. In fact, 12 and 60 are the third and fourth integers in the series of superabundant numbers, integers that represent the peak divisibility among integers up to twice their size. Sixty is also the reconciliation between those doubled composites of three, and those doubled composites of five. Ten and twelve meet and intertwine at 60. Sixty possesses four prime factors, just as sixteen. However, with each additional digit, sixty compounds divisors at an enormous velocity.

Divisors

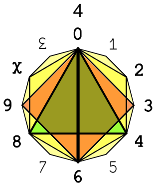
The multiplicative permutations of the prime factors of each integer *r* yields a set of integer divisors of base *r*. A number is a divisor of the radix if the number divides the radix, yielding an integer. The number of divisors (σ_0) is a measure of the versatility of the base. The consideration of the sums of the divisors (σ_1) perhaps is a better indicator of versatility. We can also consider greater powers of the integers we are examining as bases to see how their "hundreds" and "thousands" might function as decimal percents do in today's society. Figure 1B shows all the divisors for each of the integers considered. The list of divisors runs from left to right until it reaches $r^{1/2}$, then proceeds right to left so that the divisors which align yield *r* as a product. In the case of 16, there is a divisor which is precisely $r^{1/2}$; this divisor appears once in the list, and is understood to be multiplied by another instance of itself to yield *r*. This method of listing divisors illustrates symmetry among the divisors of a base. This symmetry, and the paired nature of divisors, is a crucial concept in the method described in this booklet.



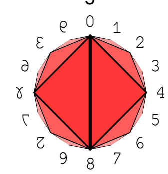
8 1 2 / 8 4
Octal (Base 8)



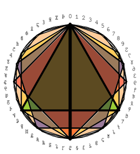
10 1 2 / 10 5
Decimal (Base 10)



12 1 2 3 / 12 6 4
Duodecimal (Base 12)



16 1 2 4 / 16 8
Hexadecimal (Base 16)



60 1 2 3 4 5 6 / 60 30 20 15 12 10
Sexagesimal (Base 60)

Figure 1B. *Divisors of certain bases. The "reciprocal divisors" of each integer are paired.*

Eight has four divisors: {1, 2, 4, 8}; of the integers possessing four divisors, only six is simpler. Half of the octal digits are divisors, if we regard the divisor *r* as zero. The sum of these divisors is 15; when divided by eight, this yields a ratio of 1.875.

Ten has four divisors: {1, 2, 5, 10}. Thus 40% of decimal digits are divisors. The sum of these divisors is 18; when divided by ten, this leaves us with a ratio of 1.8 or 9/5.

Twelve has six divisors: {1, 2, 3, 4, 6, 12}. Thus, 50% of dozenal digits are divisors. The sum of these divisors is 28; when divided by twelve this yields a ratio of 2.33… or 7/3. In both cases, twelve offers a greater divisibility, thus versatility, than decimal.

Sixteen has five divisors: {1, 2, 4, 8, 16}. Only 31.25% of hexadecimal digits are divisors. Summing these, we arrive at 31; when divided by sixteen, we get 1.9375. When society wants to improve the "resolution" of their base, they turn to "percents" or "per-mils"; when we consider the divisors of the second and third powers of 16, we see that its ability to yield integral divisors winces even in the face of decimal powers.

Decimal thousandths are superior to hexadecimal's; this is because the interaction of dissimilar primes is not confined to the powers of one prime.

Sixty has ten divisors: {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}. One fifth of sexagesimal digits are divisors. Summing these, we obtain the number 168; dividing this figure by 60 yields 2.8 or 14/5.

It's easy to see that sixty yields far more divisors than any of the smaller bases. We can see that the sum of the divisors, (σ_1), indicate that dozenal is pretty attractive at a ratio of 2.33, but sexagesimal offers 2.8. This wonderful versatility is not necessarily the result of a magnitude represented by sixty; base 61 features fewer divisors than any we are considering. The product of sixty's prime factors is indeed large, and this is the paramount barrier to the application of pure sexagesimal in human society.

Compounding Divisibility and Rank

A recent poll determined that the average person perceives fractions written as percents as being "more accurate" than "vulgar" fractions. This is, of course, incorrect: 1/3 is precise, whereas 33% is deficient. But this poll underscores the effectiveness of the technique of using the powers of a base to obtain greater resolution. If we examine the powers of integers to be considered as bases, we will see that the number of divisors compound at different rates for different classes of integers. Prime integers will add one divisor per power. Table 1A shows that integers *r* which are powers of one prime number add divisibility at a slower rate than integers with a diverse set of prime factors. Decimal thousandths, or per mils, offer a greater set of resolved fractions than hexadecimal fractions carried out to three places. Examination of the trends established in the table for each *r* reveals a sequence which is easy to extrapolate to additional ranks.

Table 1A • Extension of Divisibility via Additional Places

r	Prime Factors	1 Place $\sigma_0(r^1)$	2 Places $\sigma_0(r^2)$	3 Places $\sigma_0(r^3)$
5	5	2	3	4
6	$2 \cdot 3$	4	9	16
8	2^3	4	7	10
10	$2 \cdot 5$	4	9	16
12	$2^2 \cdot 3$	6	15	28
16	2^4	5	9	13
18	$2 \cdot 3^2$	6	15	28
20	$2^2 \cdot 5$	6	15	28
24	$2^3 \cdot 3$	8	21	40
27	3^3	4	7	10
30	$2 \cdot 3 \cdot 5$	8	27	64
36	$2^2 \cdot 3^2$	9	25	49
60	$2^2 \cdot 3 \cdot 5$	12	45	112
360	$2^3 \cdot 3^2 \cdot 5$	24	105	280

	8			10			12			16	
1		10	1		10	1		10	1		10
2		4	2		5	2		6	2		8
						3		4		4	
1		100	1		100	1		100	1		100
2		40	2		50	2		60	2		80
4		20	4		25	3		40	4		40
	10		5		20	4		30	8		20
				10		6		20		10	
						8		16			
						9		14			
							10				
1		1000	1		1000	1		1000	1		1000
2		400	2		500	2		600	2		800
4		200	4		250	3		400	4		400
10		100	5		200	4		300	8		200
20		40	8		125	6		200	10		100
			10		100	8		160	20		80
			20		50	9		140		40	
			25		40	10		100			
						14		90			
						16		80			
						20		60			
						23		54			
						28		46			
						30		40			

Figure 1C. *The number of divisors of r^x increases as x increases at a faster rate for integers which have a diverse set of prime factors. Even though 12 has fewer prime factors than 16, the dozenal prime factors are more diverse. Even decimal wins out over hexadecimal by the time three digits are in play. The divisors are arranged in reciprocal pairs such that* d · d' = r*, and are expressed in base* r.

Table 1A shows that divisibility compounds at rates which are the same for integers which have the same prime factorization "template". Thus, integers 12, 18, and 20, each being an instance of the "template" ($\Pi_0^2 \cdot \Pi_1$), exhibit an addition of 9 divisors when the number of digits in a given figure rises to 2 from 1, and an addition of 13 divisors when the number of digits in a given figure rises to 3 from 2.

Primes add 1 new divisor for each additional place, starting with the 2 divisors for r^1. The "diprimes", composite integers of the "template" ($\Pi_0 \cdot \Pi_1$), have $(r + 1)^2$ divisors for their x-th power.

The powers of primes which follow the "template" of "templates" (Π_0^x) add x divisors for each new digit to the 1 divisor for $r^0 = 1$.

60

1	10
2	ƃ
3	∂
4	ɘ
5	ɣ
6	Ƨ

1	100
2	ƃ0
3	∂0
4	ɘ0
5	ɣ0
6	Ƨ0
8	7ƃ
9	6ʠ
Ƨ	60
ɣ	50
ɘ	40
ʃ	3ɞ
ℓ	3∂
∂	30
ɤ	2ƃ
ς	2ɤ
ƃ	20
ɮ	1ʠ
ʠ	1ƃ
ɞ	1∂
ȣ	1ɘ
Ɔ	1ɣ

10

1	1000
2	ƃ00
3	∂00
4	ɘ00
5	ɣ00
6	Ƨ00
8	7ƃ0
9	6ʠ0
Ƨ	600
ɣ	500
ɘ	400
ʃ	3ɞ0
ℓ	3∂0
∂	300
ɤ	2ƃ0
ς	2ɤ0
ʓ	2∂∂
ƃ	200
ɀ	1ꞵƃ
ɮ	1ʠ0
ʠ	1ƃ0
ɞ	1∂0
ȣ	1ɘ0
Ɔ	1ɣ0
ȴ	16ʠ
10	100
14	ʒɘ
1ɣ	Ɔ0
1ɘ	ȣ0
1∂	ɞ0
1ƃ	ʠ0
1ʠ	ɮ0
1ȣ	Ʒ∂
20	ƃ0
25	ɛȣ
2ɘ	ϐʠ
2ɤ	ς0

Figure 1D. The sexagesimal first, second, and third rank divisors, expressed in a sexagesimal single digit notation. This simply illustrates the further compounding of a more diverse set of prime factors.

Table 1A shows the divisors for each integer per the number of places or "rank" used in a figure. The rank is another name for the power of an integer, thus r^2 represents the second rank. The square of 12 or 144 is the second rank of twelve. Divisors pertaining to the square of an integer are called "second rank divisors". Dozenal clearly possesses more divisors when two dozenal digits are used than any base of comparable size. Thus, the dozenal "percent" or per gross resolves 15 fractional denominations in two digits. Sexagesimal "percents" resolve 45 fractional denominations in two digits.

Magnitude and Human Scale

One thing that becomes obvious is that the magnificent versatility of sixty comes at the price of its large size. Studies have illustrated that the human mind can deal effectively with group sizes of around 7 to 12 objects, and that a group size of 60 is simply beyond the ability of the great majority of people to manipulate. The usage of a base as large as sixty suggests the application of a mixed radix, usually 6 on 10, to yield a less pure form of sexagesimal. We'll return to this consideration shortly.

Reciprocal Divisors

We can arrange the list of divisors D_r for each integer in such a way that pairs of divisors can be created that, when the members of the pairs are multiplied together, yield the integer *r*. We will refer to such pairs of divisors as *reciprocal divisors* in this presentation. The formulae relevant to reciprocal divisor pairs are:

$$d \cdot d' = r; \qquad d = r \,/\, d'$$

By these definitions, a reciprocal divisor *d'* can be determined by using the ratio of the base *r* to the divisor *d*. The notion of reciprocal divisors and RDPs is key to the process described in this presentation.

The reciprocal divisor pair {1, *r*} is an element of the set of divisors D_r for every integer *r*. This pair is called the "unity-identity" pair of divisors.

<u>Octal</u> features two reciprocal divisor pairs, {{1, 8}, {2, 4}}

<u>Decimal</u> has two reciprocal divisor pairs, {{1, 10}, {2, 5}}.

<u>Dozenal</u> has three RDPs, {{1, 12}, {2, 6}, {3, 4}}.

<u>Hexadecimal</u> has 2½ RDPs {{1, 16}, {2, 8}, {4}}. Four multiplied by itself yields sixteen. The divisor set {4} actually represents four multiplied by itself.

<u>Sexagesimal</u> features six RDPs, {1-60, 2-30, 3-20, 4-15, 5-12, 6-10}.

Totatives

Each integer *r* greater than one possesses a set of integers T_r which are lesser than *r* that are relatively prime to *r*. Two integers are said to be relatively prime when the lesser integer *t* does not divide the greater integer *r* to yield an integer quotient. Relatively prime digits are referred to as *totatives*. The least common multiple of an integer *r* and its totative *t* is $(r \cdot t)$. Because of this, totative digits in base *r* do not reach a multiple of *r* until they are multiplied by *r*. The totatives are the "weird numbers" that are not covered by the multiplicative permutations of the prime factors Π_r of *r*. The number of totatives of a given integer is given by the Euler totient function Φ_r. A smaller totient function value indicates a base *r* that features more factors which are divisors and factors which are products of one or more divisors. Prime numbers Π have a totient function value Π - 1.

Every integer possesses a pair of totatives $T_0 = \{1, (r - 1)\}$. These totatives exhibit patterns in the multiplication table M_r of *r* which are relatively simple to understand.

The integer 1 as a factor is very special. Note that the totative 1 of every integer *r* is also a divisor of every integer *r*. Thus, 1 is the only divisor of any integer base *r* which is also a totative. Like all divisors *d* of base *r*, 1 exhibits a cycle of products which repeat within one multiple of *r*. Like all totatives *t* of base *r*, 1 yields a multiple of *r* as a product only when 1 is multiplied by *r*.

Figure 1E lists the totatives of the integers considered in this section. Note that the totatives also exhibit symmetry. The lists place the totatives lesser than *r* / 2 above those greater than *r* / 2. The smaller totatives run left to right, while the greater run right to left. This method of listing totatives places the totatives in pairs whose sums are *r*. The first pair, read vertically, is the totative pair $T_0 = \{1, (r - 1)\}$, which is a subset of the totatives T_r of every integer base *r*. This symmetry is evident in the multiplication tables for their respective bases, and is an important tool.

Eight possesses four totatives {1, 3, 5, 7}, which are all the odd octal digits, because 8 is a power of the prime factor 2. The totient function Φ for 8 is 4. The totative ratio is Φ_r / r = 4/8 or one half.

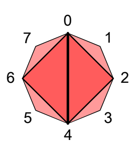
Ten has four totatives {1, 3, 7, 9}. The totient function of ten is 4; the totative ratio is 0.4 or 2/5.

Twelve also has four totatives {1, 5, 7, 11}. The totient function of twelve is 4; the totative ratio is 0.333… or 1/3.

Sixteen has eight totatives, which are the odd digits, because it is a pure power of a prime number. Its totient function is 8, its totative ratio is 0.5 or ½.

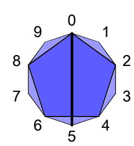
Sixty has 16 totatives {1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59}; its totient function is 16. Sixty's totative ratio is 0.266… or 4/15.

This study shows that dozenal resolves two thirds of its digits, while decimal resolves 60% of its digits. Sexagesimal resolves 46 of its digits, or more than 73%.



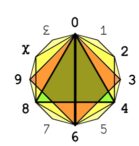
8	1	3
	7	5

Octal (Base 8)



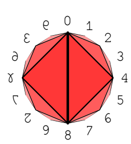
10	1	3
	9	7

Decimal (Base 10)



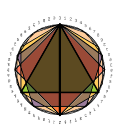
12	1	5
	11	7

Duodecimal (Base 12)



16	1	3	5	7
	15	13	11	9

Hexadecimal (Base 16)

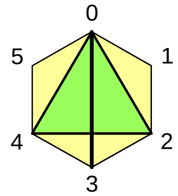


60	1	7	11	13	17	19	23	29
	59	53	49	47	43	41	37	31

Sexagesimal (Base 60)

Figure 1E. *Totatives of certain bases.*

Table 1B • Summary of the Properties of Selected Integers • 6, 8, 10, 12



Senal

$2 \cdot 3$

6

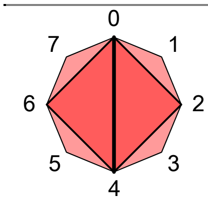
1 2
6 3

DIVISORS

1
5

TOTATIVES

$r^2 = 36$	$P_M = 21$	$\sigma_0 = 4$	$\sigma_1 = 12$	$\Phi = 2$
36 / 100	21 / 55	4 / 8	5 / 6	2 / 6
36%	38.2%	50%	83.3%	33.3%
$r^2 / 10^2$	$P_{M(r)} / P_{M(10)}$	σ_0 / r	$[\sigma_1 - (r+1)]/r$	Φ / r



Octal

2^3

8

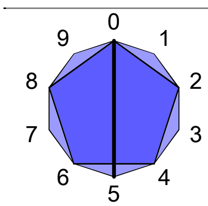
1 2
8 4

DIVISORS

1 3
9 7

TOTATIVES

$r^2 = 64$	$P_M = 36$	$\sigma_0 = 4$	$\sigma_1 = 15$	$\Phi = 4$
64 / 100	36 / 55	4 / 8	6 / 8	4 / 8
64%	65.5%	50%	75%	50%
$r^2 / 10^2$	$P_{M(r)} / P_{M(10)}$	σ_0 / r	$[\sigma_1 - (r+1)]/r$	Φ / r



Decimal

$2 \cdot 5$

10

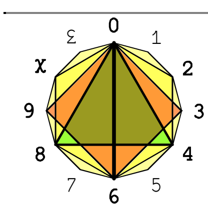
1 2
10 5

DIVISORS

1 3
9 7

TOTATIVES

$r^2 = 100$	$P_M = 55$	$\sigma_0 = 4$	$\sigma_1 = 18$	$\Phi = 4$
100 / 100	55 / 55	4 / 10	7 / 10	4 / 10
100%	100%	40%	70%	40%
$r^2 / 10^2$	$P_{M(r)} / P_{M(10)}$	σ_0 / r	$[\sigma_1 - (r+1)]/r$	Φ / r



Dozenal

$2^2 \cdot 3$

12

1 2 3
12 6 4

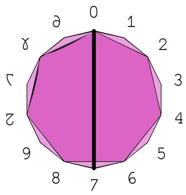
DIVISORS

1 5
11 7

TOTATIVES

$r^2 = 144$	$P_M = 78$	$\sigma_0 = 6$	$\sigma_1 = 28$	$\Phi = 4$
144 / 100	78 / 55	6 / 12	17 / 12	4 / 12
144%	141.8%	50%	141.7%	33.3%
$r^2 / 10^2$	$P_{M(r)} / P_{M(10)}$	σ_0 / r	$[\sigma_1 - (r+1)]/r$	Φ / r

Table 1C • Summary of the Properties of Selected Integers • 14, 16, 20, 60

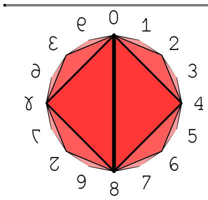


Tetradecimal — $2 \cdot 7$

14

Divisors			Totatives		
1	2		1	3	5
14	7		13	11	9

$r^2 = 196$	$P_M = 105$	$\sigma_0 = 4$	$\sigma_1 = 24$	$\Phi = 6$
196 / 100	105 / 55	4 / 14	10 / 14	6 / 14
196%	190.9%	28.6%	71.4%	42.9%
$r^2/ 10^2$	$P_{M(r)}/ P_{M(10)}$	σ_0/r	$[\sigma_1 - (r+1)]/r$	Φ/r

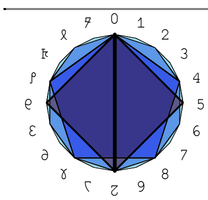


Hexadecimal — 2^4

16

Divisors			Totatives			
1	2	4	1	3	5	7
16	8		15	13	11	9

$r^2 = 256$	$P_M = 136$	$\sigma_0 = 5$	$\sigma_1 = 31$	$\Phi = 8$
256 / 100	136 / 55	5 / 16	14 / 16	8 / 16
256%	247.3%	31.25%	87.5%	50%
$r^2/ 10^2$	$P_{M(r)}/ P_{M(10)}$	σ_0/r	$[\sigma_1 - (r+1)]/r$	Φ/r

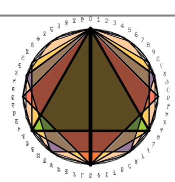


Vigesimal — $2^2 \cdot 5$

20

Divisors			Totatives			
1	2	4	1	3	7	9
20	10	5	19	17	13	11

$r^2 = 400$	$P_M = 210$	$\sigma_0 = 6$	$\sigma_1 = 42$	$\Phi = 8$
400 / 100	210 / 55	6 / 20	21 / 20	8 / 20
400%	381.8%	30%	105%	40%
$r^2/ 10^2$	$P_{M(r)}/ P_{M(10)}$	σ_0/r	$[\sigma_1 - (r+1)]/r$	Φ/r



Sexagesimal — $2^2 \cdot 3 \cdot 5$

60

Divisors						Totatives							
1	2	3	4	5	6	1	7	11	13	17	19	23	29
60	30	20	15	12	10	59	53	49	47	43	41	37	31

$r^2 = 3600$	$P_M=1830$	$\sigma_0 = 12$	$\sigma_1 = 168$	$\Phi = 16$
3600 / 100	1830 / 55	12 / 60	107 / 60	16 / 60
3600%	3327%	20%	178%	26.7%
$r^2/ 10^2$	$P_{M(r)}/ P_{M(10)}$	σ_0/r	$[\sigma_1 - (r+1)]/r$	Φ/r

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	**4**	6	8	χ	10	12	14	16	18	1χ	20
3	6	**9**	10	13	16	19	20	23	26	29	30
4	8	10	**14**	18	20	24	28	30	34	38	40
5	χ	13	18	**21**	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	**30**	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	**41**	48	53	5χ	65	70
8	14	20	28	34	40	48	**54**	60	68	74	80
9	16	23	30	39	46	53	60	**69**	76	83	90
χ	18	26	34	42	50	5χ	68	76	**84**	92	χ0
Ɛ	1χ	29	38	47	56	65	74	83	92	**χ1**	Ɛ0
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	**100**

Figure 2. *The dozenal multiplication table.*

Part 2 • The Multiplication Fact Table

Introduction

For each integer r, we can assemble a matrix of values that can be memorized or referred to in order to multiply in that base. The matrix consists of rows and columns that are headed by the factors. The head cells run the full scale of the set of digits F_r of base r. The products $p(f_1, f_2)$ of any two factors f_1 and f_2 appear at the cell in the matrix where the line of products for factor f_1 intersects the line of products for factor f_2. The elements of the matrix represent every product p for each digit F_r if base r. This is the familiar multiplication table we learn in primary school.

Most people are familiar with the decimal multiplication table. Many observe how products in the 9 column (or row) have unit digits that decrease while the group digits (the tens, in the case of decimal) increase, and that the sum of these two digits is always a multiple of nine (for products lesser than or equal to $9 \cdot 10$). Another observation that may be somewhat apparent is how the fives "make ten" for every even factor. This is an exhibition of the cyclical nature of a factor which is also a divisor of the base.

One of the most striking qualities of the dozenal multiplication table is its wonderful rhythmic simplicity. Much of the table appears to "come out". There is a certain rhythm in the table that decimal largely lacks. It is this *cyclical quality* that we can capitalize on to abbreviate the multiplication table.

For the purposes of analyzing the multiplication table, definitions are given below which might facilitate the discussion of "rhythm" and the interrelatedness of the reciprocal divisors of a base.

Reciprocal Divisor Pair. The set of divisors D_r of base r can be arranged so that any divisor d can be multiplied by another divisor d' to yield the product r.

$$r = d \cdot d'; \quad d = r / d'; \quad d' = r / d;$$

$$\{d, d'\} \text{ is a subset of } D_r$$

It is important to note that, for bases r whose square root is integral (that is, for r that is the square of another integer) there exist divisors d which serve as their own reciprocal divisor. Thus, if $r^{1/2}$ is an integer, then there is a $d = r^{1/2}$ for which $\{d, d'\}$ exists, and in this case, $d = d'$.

Multiplication Table. The multiplication table M_r of base r, for the purposes of this study, is the matrix of all the products p of all the positive integer factors f which are lesser than or equal to the base r, written in base r.

Period. A *period* P_r for base r is equal to the integer r. The period is useful in analyzing the rhythms present in a multiplication table; of particular interest is any p for which p/r is an integer.

Cycle. A *cycle* refers to the set of unit digits of products that proceed from one multiple of r to another multiple of r, inclusive of the digit zero, which is the start of the cycle. For 5 in base 10, the cycle $C(5, 10) = \{0, 5\}$.

<u>Cycle Length</u>. The cycle length λ is the number of products in a cycle. The cycle length for five in base ten is two.

<u>Phase</u>. A phase refers to the range determined by set of unit digits of products which ascend or descend between points of inflexion as the co-factor *f'* increases. A phase is a cycle when the points of inflexion falls on a period.

<u>Mate</u>. This term refers to the other divisor in a pair of reciprocal divisors. An example of a "mate" for the dozenal divisor 3 is 4; this is the reciprocal divisor pair {3, 4}, a subset of the divisors of the integer 12, {{1, 12}, {2, 6}, {3, 4}}.

	T	6	4	3	T	2	T	3/2	4/3	6/5	T	1
T	1	2	3	4	5	6	7	8	9	χ	Ɛ	10
6	2	4	6	8	χ	10	12	14	16	18	1χ	20
4	3	6	9	10	13	16	19	20	23	26	29	30
3	4	8	10	14	18	20	24	28	30	34	38	40
T	5	χ	13	18	21	26	2Ɛ	34	39	42	47	50
2	6	10	16	20	26	30	36	40	46	50	56	60
T	7	12	19	24	2Ɛ	36	41	48	53	5χ	65	70
3/2	8	14	20	28	34	40	48	54	60	68	74	80
4/3	9	16	23	30	39	46	53	60	69	76	83	90
6/5	χ	18	26	34	42	50	5χ	68	76	84	92	χ0
T	Ɛ	1χ	29	38	47	56	65	74	83	92	χ1	Ɛ0
1	10	20	30	40	50	60	70	80	90	χ0	Ɛ0	100

Figure 2A. *The dozenal multiplication table, one which features a number of cyclical patterns among products of a given factor, attributable to the high divisibility of twelve. The patterns of both axes are shown in this diagram. Totative phases are ignored.*

Rhythms in the Multiplication Table

The multiplication tables of any integer base *r* feature patterns which are both symmetrical and linked to the divisors D_r and factors F_r of the base *r*. The patterns present in a multiplication table can be used to help determine the ease of working computationally within a base. Highly patterned multiplication tables can mitigate the difficulty memorization may present. More importantly, these patterns are clues to how the multiplication table might be abbreviated.

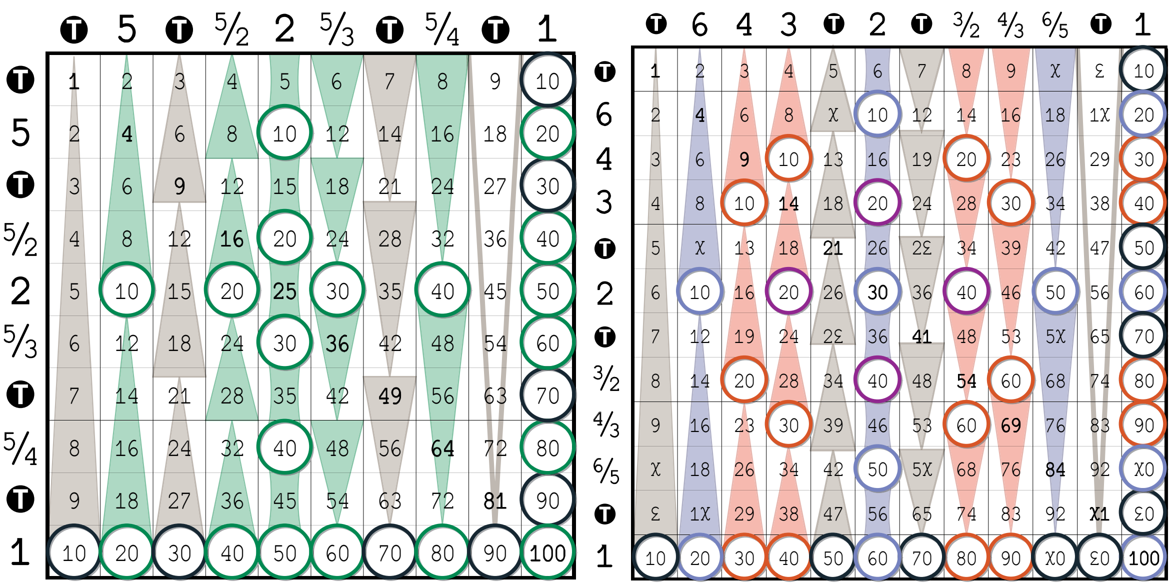


Figure 2B. *Patterns in the decimal (left) and dozenal (right) multiplication table. The color green represents factors governed by the divisors {2, 5}, blue by {2, 6}, and red by {3, 4}. Periods are circled. The totative factors are shown in gray. The totative divisor {1} is shown in solid gray, while the totative {*r *- 1} has a reciprocal cycle that is shown outlined in gray. Other totatives are simply indicated by a gray stroke. Note the presence of "phases" within the cycles of the decimal factors 4 and 6, which are completely absent from dozenal.*

Multiples or First-Rank Digits

All integers *r* reach a multiple of *r* for every co-factor *f'* of the factor $f_r = r$. This is evident in Figure 2B. The products of "10", regardless of which base is in play, yield multiples of ten for each co-factor in the "10"-line. These multiples of *r* are called periods P in the multiplication table. Periods indicate resonances between the factors F_r and *r* itself. Totative factors f_t do not possess products p_P which are periods until $f' = (f_t \cdot r)$.

Any instance of a period among the products of a factor *f* is an indication that factor *f* is either a divisor *d* or a non totative non divisor. These factors are special, because periods in their product lines hint at modularity which can be leveraged in the possible abbreviation of the multiplication table.

It is interesting to observe that the arrangement of periods in the multiplication tables of composite integer bases *r* is symmetrical. The symmetry seems to be organized about the axes *r* / 2, or even about lines which join the squares. Some of this symmetry has to do with the fact that the nonsquare products are stated twice in the traditional square table.

Units or Zero-Rank Digits

Examination of Figure 2B reveals that the unit digits of those factors *f* lesser than *r*/2 increase, while those greater than *r*/2 decrease as the co-factor *f'* increases. This is true for any integer *r*.

Phases.

All factors will generate a range of end digits of products which ascend or descend until they reach a point of inflexion p_I. A point of inflexion lies between the "jump" from the local lowest and highest unit digits for a given factor. See the products 8 and 12 of the factor 4 and the co-factors 2 and 3, respectively. This point lies at a fractional point between the two adjacent products. Phases are not coincident with a period at one or both inflexion points.

The total number of phases per *r* co-factors *f'* is equal to the factor *f* itself.

When the factor *f* is greater than *r* / 2, the phases will appear to descend rather than ascend. This is because the difference between *f* and *r* is now lesser than *r* / 2, and the unit digits seem to "count down" rather than climb toward inflexion.

All factors *f* exhibit a total number of phases in the table equal to the factor *f* itself. Thus all totative factors f_t possess a number of phases equal to the totative factor f_t itself.

The phases appear symmetrical about the factor *r* / 2 for even *r*, or a point at *r* / 2, between two factors for odd *r*. The phases of the factors lesser than *r* / 2 will be seen to progress as the co-factor *f'* increases. The phases of the factors greater than *r* / 2 will seem to regress as the co-factor *f'* increases. In this way, the factors (*r* / 2) – *f* have equal but opposite phases to the factors (*r* / 2) + *f*.

Cycles.

Some factors have phases which terminate at a period or multiple of *r* at products less than ($f \cdot r$). Such factors are not totatives of *r*. Those factors which are divisors of the base feature cycles that are exactly one period in length. Factors which are a multiple *m* of a divisor have cycles which require *m* periods to complete. The totative factors return to a multiple of a base *r* only when they are multiplied by *r*. These three facts are paramount in the abbreviation of a multiplication table.

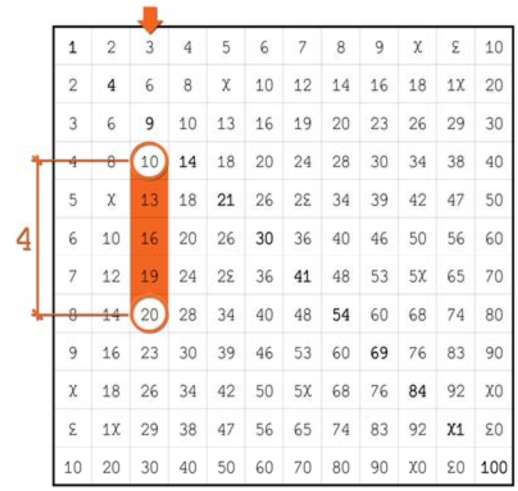
The cycles of some non totative non divisor factors f_n include multiple phases. This pattern happens in many bases but is curiously absent from the dozenal table. The number of products in a cycle, the number of periods per cycle, and other patterns in the multiplication table are clues to how the table might be abbreviated so that multiplication and division can function within the larger bases.

The decimal table, like those of most other bases, feature non totatives non divisor factors which exhibit phases within their cycles.

The dozenal multiplication table is rare in that every factor that is not totative is free of phases within cycles. Only the dozenal totatives possess phases which terminate on a period only when the co-factor is a multiple of the base *r*. This fact perhaps renders the dozenal table that much easier to memorize.

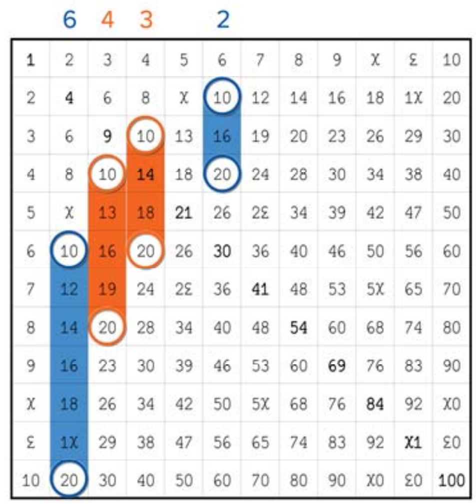
The presence or absence of phases within cycles does not affect the usefulness of non totative non divisor factors, nor the possible abbreviation of the multiplication table.

Let's examine the products of the factor 3 to examine a cycle in depth. After a cycle of 4 products, 3 returns to a multiple of twelve. There are three such cycles within the three line; i.e. $n_C(3, 12) = 3$. This is because 3 cycles of 4 products each equal twelve products. Figure 2C illustrates the cycle as a band of red between two products which terminate periods, which are circled. This shows that the cycle is coterminous with a period.



1	2	3	4	5	6	7	8	9	X	Ɛ	10
2	4	6	8	X	10	12	14	16	18	1X	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	X	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5X	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
X	18	26	34	42	50	5X	68	76	84	92	X0
Ɛ	1X	29	38	47	56	65	74	83	92	X1	Ɛ0
10	20	30	40	50	60	70	80	90	X0	Ɛ0	100

Figure 2C. *An example of a* cycle *in the "three line" of the dozenal multiplication table which runs one* period *in length. Note that this analysis could alternatively be conducted horizontally.*



1	2	3	4	5	6	7	8	9	X	Ɛ	10
2	4	6	8	X	10	12	14	16	18	1X	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	X	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5X	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
X	18	26	34	42	50	5X	68	76	84	92	X0
Ɛ	1X	29	38	47	56	65	74	83	92	X1	Ɛ0
10	20	30	40	50	60	70	80	90	X0	Ɛ0	100

Figure 2D. *Four factors feature cycles that run one period in length. These factors are the divisors of twelve. The divisor pair {2, 6} is indicated in blue, while the pair {3, 4} is shown in red.*

Classification of the Factors f of Any Integer Base r

There are three classes of factors f for all integer bases r:

1. Factors f_d which are divisors d of the integer r. There is a pair of divisors $\{1, r\}$, called the "unity-identity" pair, which is a subset of the set of divisors D_r of every integer r. The "unity-identity" subset of divisors feature simple multiplication rules which do not require a multiplication table. The divisor 1 is also totative. Composite bases have at least one more divisor than the "unity-identity" pair of divisors.

2. Factors f_t which are totatives t of the integer r. There is a pair of totatives $\{1, r\text{-}1\}$ which is a subset of the set of totatives T_r of every positive integer r.

3. Factors f_n which are neither divisors nor totatives of the integer r. These factors f_n can be divided into two subgroups. These factors relate to base r in a ratio f_n / r which when simplified yields a numerator m_n and a denominator d'. The denominator d' is the reciprocal divisor of the related divisor d_s. The factor f_n is an integer multiple of the related divisor d_s, and may also be a multiple of other

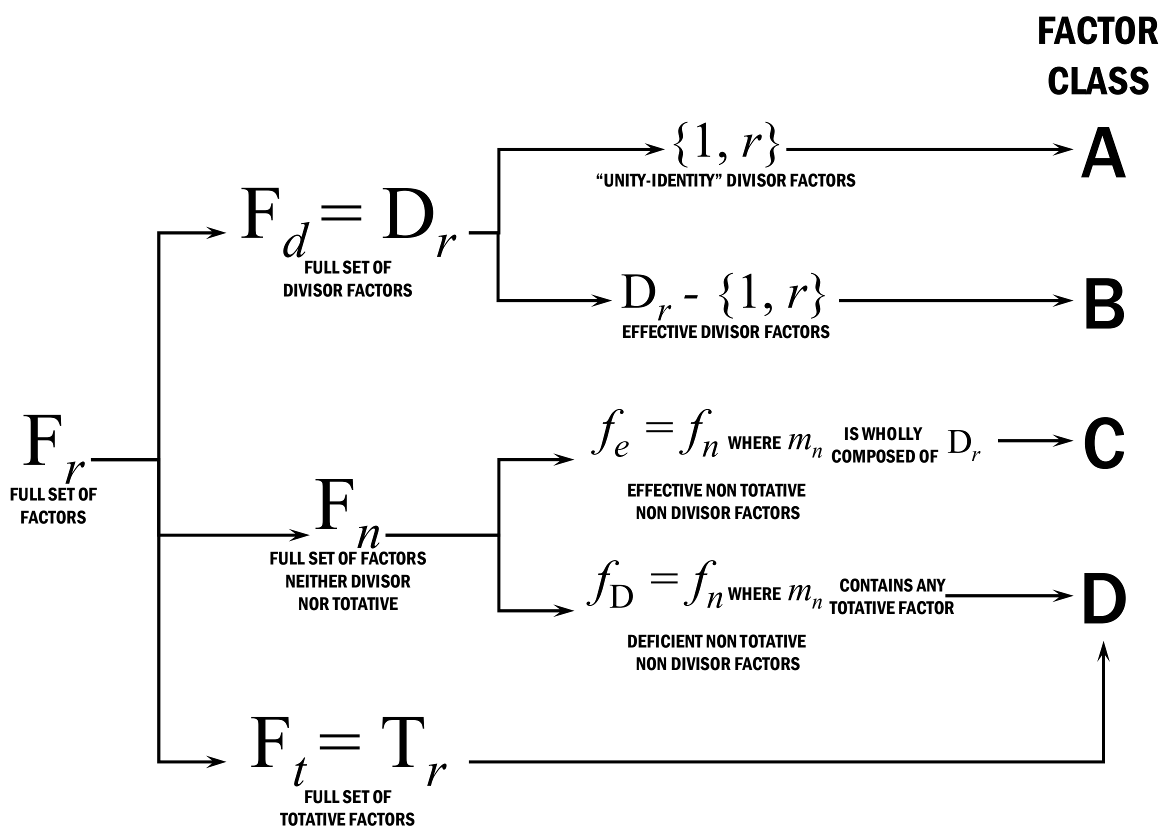
divisors. The relationship of the integer m_n to the base r divides the factors f_n which are neither divisors or totatives of base r.

a. If m_n is also a divisor d of base r, or a multiple composed entirely of divisors d of base r, the factor f_n is termed an "effective factor f_e". The effective factors f_e are powerful tools for multiplication given an abbreviated multiplication table.

b. If m_n is also a totative t of base r, or a multiple that involves any totative t (except the totative 1), the factor f_D is a "deficient factor f_n". These factors f_D are treated in the same manner as the totative factors f_t.

These kinds of factors will be examined in detail in the following subsections. The factors are then placed in classes as shown in Table 2A, which relate to the Operation Classes described in the next section. The point of classifying the factors found in the multiplication table of base r is to help determine which "tool" or method will help solve a given problem.

Table 2A • Classification of Factors



Factors which are Divisors of the Base (f_d)

The factor 3 happens to be a divisor of 12. Further examination of the other factors f_d that are divisors of twelve leads to some useful observations.

Cyclical Unit Digits in Products of f_d

These factors feature products with cyclical unit digits; the number of elements of a set that consists of the range of unit digits observed for any product of the factor defines the cycle length. The cycle length for the factor 3 is 4 elements. Drawing a line from the product *10;* for the factor 3 to the next multiple of one dozen is possible. Thus the cycle of these factors which are divisors will span from one multiple of the base to the next. This is true for the factor 3, as well as any other factor that is a divisor of 12. This means the number of cycles per period for any factor that is also a divisor is always 1. If we count the cycles in the 3 line, we see there are precisely 3 cycles. Checking the 4 line, we see four cycles of three elements each. Thus the number of cycles present in the multiplication table for the factor which is also a divisor equals that divisor itself.

Divisors exhibit cycles which span precisely one period in the multiplication table. This is because the divisor *d* as factor f_d eventually encounters the reciprocal divisor mate *d'*; when that occurs, the product of *d* and *d'* is *r*.

Complementary Relationships within Reciprocal Divisor Pairs

The divisors *d* of a base *r*, as shown previously, can be paired {*d*, *d'*} so that the pair, when multiplied together, yields the base *r*. These mated divisors that inhabit the same reciprocal divisor pair exhibit complimentary cycles. Dozenal possesses two sets of reciprocal divisors which are not unity-identity:

$$\{\{2, 6\}, \{3, 4\}\}$$

Figure 2D illustrates cycles and their lengths for the factors in the dozenal multiplication table which are also divisors. The factors of twelve feature cycles that run one period in length. Each divisor *d* features a cycle which repeats after *d'* products. The factor *d* is indicated by the first number in each column. The reciprocal divisor *d'* of those factors which are divisors appears in color above *d* in the table. Base twelve has two effective divisor pairs; divisor pair {2, 6} is indicated in blue, while the pair {3, 4} is shown in red. The divisor pair {1, 12} is the "unity-identity" pair, which is special, and will be discussed later.

The cycle length for the factor 3 shown in Figure 2C is 4; 4 happens to be the reciprocal divisor "mate" for 3. Figure 2E shows that the cycle length of a factor which is also a divisor of base equals that divisor's reciprocal divisor mate.

Another quality associated with factors f_d which are also divisors of the base is illustrated in Figure 2F. Each factor which is also a divisor of the base exhibits unit digits in their products which ascend as the co-factor increases, until the unit digit is reset when the products reach a multiple of the base. Thus, the unit digits of all the products of factors which are also divisors of the base within a given cycle vary directly with the co-factor.

Table 2B summarizes values associated with factors f_d for the dozenal multiplication table. The inextricable relationship between reciprocal divisor mates should be evident.

Table 2B • Summary of the Properties of Dozenal Divisor Factors

Factor, $f_d = d$	2	3	4	6
Cycle Length, λ	6	4	3	2
Number of Cycles, n_C	2	3	4	6
Reciprocal Divisor, d'	6	4	3	2

The "Unity-Identity" Divisors as Factors

Recall that the divisors of twelve include a pair of divisors {1, 12} which we did not include in the set of "effective divisor pairs" because these were special. This is the "unity-identity" pair of divisors. All integers possess the unity-identity pair of divisors. This pair simply illustrates that, among the set of divisors of an integer *r*, a relationship between one and the integer *r* itself exists.

Factors which are Unity (f_1)

The factor 1 is a special case; 1 is both a divisor and relatively prime, thus a totative of all integers. Because 1 is a divisor, it exhibits the five hallmarks of a divisor described in the previous section. Its cycle repeats in exactly one period. The number of cycles present in the table is equal to the factor 1. The cycle length of the factor 1 is equal to the reciprocal divisor of 1, which is 12. The unit digits of the products of 1 increase in a cycle that is precisely one period in length. The factor 1 is also totative, which means that the factor reaches a multiple of the base only when the factor is multiplied by the base. Thus the factor 1, though it is a divisor, is more importantly a totative, and is classified with the factors which are totative. Totative factors will be covered later. In practical applications, the "one" line is used as an index. Otherwise, the "one" line can be completely ignored, because multiplication of any factor by one yields that factor.

Factors which are "Identity" (f_r)

The factor f_r which is equivalent to the base *r* is also a divisor. In the case of dozenal, the factor f_r equals 12. Like every divisor except 1, f_r is not relatively prime. The factor f_r exhibits four of the five hallmarks of all factors f_d which are divisors. The cycle of f_r is exactly one period in length, because one period P equals f_r. The number of cycles present in the table is equal to f_r. The cycle length of the factor f_r, which is also the divisor *d*, is equal to the reciprocal divisor *d'*, which in the case of f_r is 1.

The factor f_r does not obey the fifth observation for factors f_r which are divisors. The products *p* of the factor f_r do not have unit digits which ascend through the one-period-long cycle. This is because the cycle length is 1; the cycles contain only multiples of *r*, and thus all unit digits are zero. This is what makes f_r a special case of f_d.

In practical applications, the products of factor f_r are easily generated by shifting the digits of any factor leftward, and writing a zero in the units place. This makes the inclusion of the products of f_r unnecessary.

Factor Classes for all f_d

The factors f_d which are divisors d of base r can be divided into two types or classes. The first class, Factor Class A, consists of the set of f_d $\{f_1, f_r\}$, the "unity-identity" pair of divisors, for which all products p can be intuitively computed. The second class, Factor Class B, consists of the "effective divisors" of base r, which include the entire set of divisors except $\{f_1, f_r\}$. These divisors can be leveraged to compute products beyond the first period in the multiplication table, enabling the abbreviation of the table which will be described in the next section. Factor classes will be discussed later as well.

Summary – Divisor Factors

Factors f_d which are divisors d of the base r, having reciprocal divisors d', exhibit the following in a multiplication table M_r of base r:

1. The unit digits υ_p of the products p of the factors f_d that are also divisors d of base r are cyclical. The cycle C of any factor is the set of unit digits of the all the products p of that factor.

2. The number of periods per cycle or m_d for any factor f_d that is also a divisor is always 1. The cycle C will span from one multiple of the base P_n to the next $P_{(n+1)}$.

$$m_d = (P/C)_d = 1$$

3. The number of cycles present in the multiplication table, n_C, for the factor f_d which is also a divisor d of base r equals f_d itself.

$$n_C(f_d) = f_d = d$$

4. The cycle length λ of a factor f which is also a divisor of base r equals that divisor d's reciprocal divisor mate, d'. The cycle length λ is less than r, and r varies in direct integral proportion to the cycle length λ.

$$\lambda_d = d'$$

5. The unit digits υ_p of all the products p in a given cycle $C(f_d)$ increase within a cycle that is precisely one period in length. These unit digits υ_p increase as the product p increases, until the product p is a multiple of the base r. If the factor f is equal to $r/2$, the unit digits will be precisely $r/2$ or 0 mod r, which can be read as neither increasing nor decreasing.

$$\text{All } \upsilon_p \text{ (except for } f_d = r/2) \propto p \text{ within each C}$$

6. The "unity-identity" reciprocal divisor pair $\{1, r\}$ is special. The divisors 1 and r are considered separately from all other divisors; the remaining divisors D_e in the set of divisors D_r of base r comprise the "effective divisors" of base r.

$$D_r = D_e + \{1, r\}$$

7. The divisor 1, a divisor of every base r, is also always a totative of every positive integer r.

8. The products p of the factor f_r do not have unit digits which ascend through the one-period-long cycle. This is because the cycle length is 1; the cycles contain only multiples of r, and thus all unit digits are zero.
9. Factor Class A contains the factors f_1 and f_r, the "unity-identity" divisors $\{1, r\}$. Products involving Factor Class A are computed easily and do not require a multiplication table at all.
10. Factor Class B contains the set of factors f_e which are "effective divisors" D_e. Problems involving Factor Class B can employ the reciprocal divisor "mates" to compute products beyond the range of an abbreviated multiplication table. This process, along with the abbreviation of a multiplication table, will be described in a later section.

Higher Rank Divisors

The reciprocal divisors used in this booklet are merely the first rank divisors of a given base r. That is, they are the divisors of r^1. As seen in Table 1C, the number of divisors in higher ranks compounds. The Reciprocal Divisor Method can be applied using two or more digits at once using higher rank divisor pairs. For example, the factors eight and nine can use Operation Class B directly, recognizing their reciprocal divisors are 07;30: and 06;40: respectively. Figures 2E and 2F show some of the higher rank sexagesimal divisors, and their cycles. These higher rank divisors may aid the division process, especially when the problem's divisor is recognized as one of the resonances of a higher rank divisor. These higher rank divisor pairs are beyond the scope of this booklet.

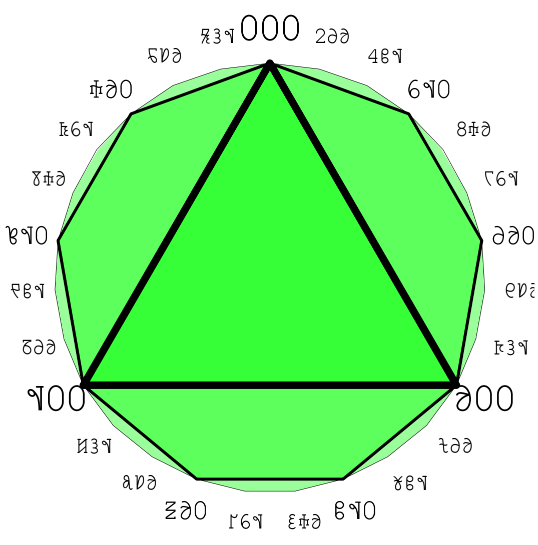


Figure 2E. *The sexagesimal third-rank divisor 27: and its resonances. Ninths and thirds are prominent.*

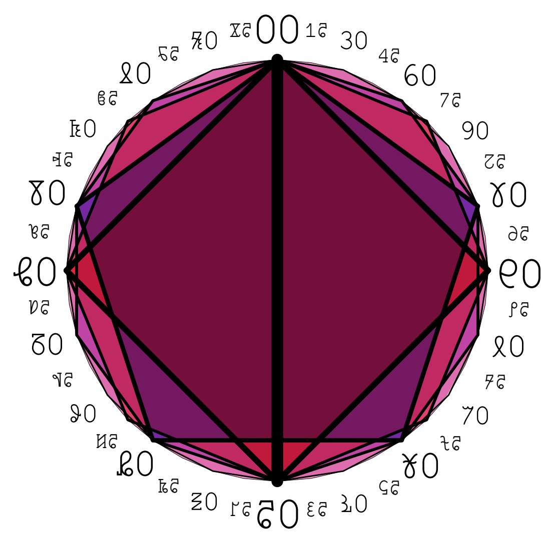


Figure 2F. The sexagesimal second rank divisor 40: and its resonances. Halves, quarters, fifths, eights, tenths, and twentieths are subcycles.

Factors which are both Nondivisors and Nontotative (f_n)

The factors which are divisors of the base account for only part of the rhythm seen in the multiplication table. Using the factor 9 in the dozenal multiplication table as shown by Figure 2H, we can see that its products exhibit a cyclical pattern. This is despite the fact 9 is not among the divisors of 12. Nine is a member of a set of factors which are neither divisors nor totatives of the base, the elements of which we will call f_n. Subtraction of the union of the sets of the divisors and the totatives of twelve from the set of all factors in the dozenal multiplication table yields, by definition, the set of the factors in this class:

$$F_M - (D_r \cup T_r) = F_n$$

$$F_{12} - \{1, 2, 3, 4, 6, 12\} \cup \{1, 5, 7, 11\} = \{8, 9, 10\}$$

Thus, there are three such factors f_n for the dozenal multiplication table: {8, 9, 10} (the elements given in decimal notation.)

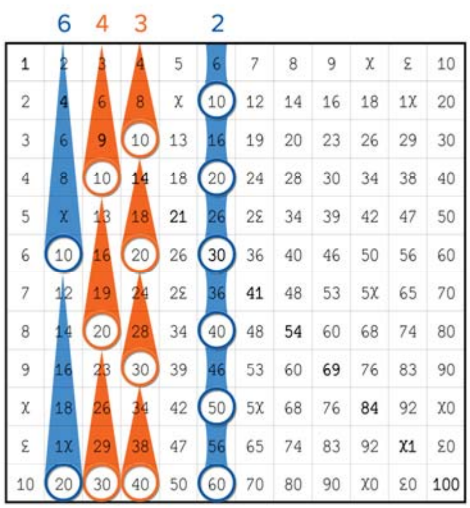


Figure 2G. *All cycles for factors* f_d *which are also divisors are shown. Each cycle for the divisor* f_d *repeats* f_d *times before reaching (*$f_d \cdot r$*) at the end of the line. In this figure, the unit digits of the products in a cycle increase, cycling once per period: this is a hallmark of factors* f_d *which are also divisors.*

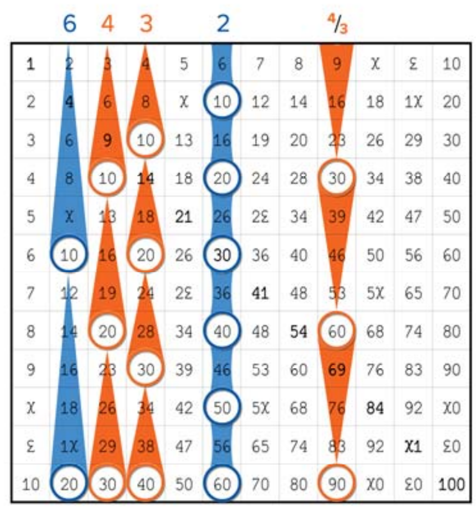


Figure 2H. *Factors* f_n *which are both nondivisors and not totatives feature cycles which repeat given multiple periods. These factors* f_n *are themselves multiples of factors that are divisors* f_d*. The factors* f_n *have cycles which include the number of products equal to the reciprocal divisor mate of* d*,* d'*. The cycles repeat every* m *periods. The number* d' / m *is written above the factor* f_n.

There is an important difference between the cycles of factors which are divisors of the base and factors which are neither divisors nor totatives of the base. The latter factors have cycles that span multiple periods, rather than spanning precisely 1 period. So the cycle for the 9 line spans 3 periods. If we look at the other factors f_n in the multiplication table to confirm from where the 3 originates. The factor 10 (X; in dozenal notation) has a cycle that spans 5 periods. The factor 8 has a 2 period cycle. Nine has an integral relationship to the dozenal divisor 3; it is 3 times the factor 3. This integral relationship can be represented by the multiplier m_n. Thus, the number of periods in a cycle for all factors f_n equal m_n.

Examining the factor 8 in the table, we see that a factor f_n can be related to multiple divisors; 8 is 4(2), and 2(4). The divisor to regard as the "significant divisor" d_s can be found using the following routine. Express the division of the factor f_n by the base r as a vulgar fraction: in the case of the factor 8 in base 12, this is 8/12. Simplify the fraction to 2/3. The denominator of the simplified fraction is the related divisor d_s. The numerator of the same simplified fraction 2/3 is the multiplier m_n.

$$\text{SIMPLIFY}[\, f_n / r \,] = m_n / d_s$$

The factors f_n fall into two camps.

Factor Class D. The factors f_n for which only d_s is a divisor of r, with an m_n that is totative, require the use of the totative m_n to compute products. This limits the efficacy of any method of multiplication table abbreviation that relies on the leverage of the divisors of the base to compute digits beyond the abbreviated table. An example of such a factor f_n in dozenal is the factor 10, for which $m_n / d_s = 5/6$. Here the totative 5 would be required to reduce f_n to d_s. The multiplication or division by 5 would not be supported by an abbreviated multiplication table, so the factor 10, despite the fact that it is neither a divisor nor a totative of 12, must be regarded as part of the same class of factors as the totatives. These factors f_n belong to Factor Class D, which will be discussed later.

Factor Class C. Those factors f_n for which both m_n and d_s are divisors of r offer the ability to leverage two divisors in the computation of products. A dozenal example is the factor 8, for which $m_n / d_s = 2/3$, both factors of 12. Both 2 and 3 can be manipulated so that products beyond the abbreviated table can be computed efficiently. Base twelve is too concise to include a related set of factors f_n for which m_n is not a divisor, but a composite factor that is the product of two or more divisors. An example of such a factor in base 60 is 32, for which $m_n / d_s = 8/15$. The m_n in this case is the product of 4 and 2, both of which are divisors of 60. These comprise a class of factors known as Class C, which will be discussed later.

The cycle length for the factor 9 is 4. The factor nine has a special relationship with the divisor 3, and the reciprocal divisor mate of 3 in base twelve is 4. Note that the unit digits of the products in each cycle decrease as the product increases; this is counter to what the factors which are also divisors exhibit.

Table 2C summarizes values for the factors f_n in the dozenal multiplication table. The inextricable relationship between reciprocal divisor mates should be evident.

Table 2C • Summary of the Properties of Dozenal Non Totative Non Divisor Factors

Factor, f_n	8	9	Χ
Cycle Length, λ	3	4	6
Number of Cycles, n_C	4	3	2
Periods per Cycle, m_n	2	3	5
Related Divisor, d	4	3	2
Reciprocal Divisor, d'	3	4	6

Summary – Nontotative Nondivisor Factors

Factors f_n that is neither a divisor nor a totative of base r exhibit the following in a multiplication table M_r of base r:

1. The unit digits υ_p of the products p of the factors f_n that are neither divisors nor totatives of base r are cyclical. The cycle C of any factor is the set of unit digits of the all the products p of that factor.
2. Determine which divisor d_s' of base r is related to the factor f_n that is neither a divisor nor a totative of base r. (There can be multiple relationships.) Simplify the ratio of the factor f_n divided by the base r, and use the resultant denominator.

   $\text{SIMPLIFY}[f_n / r] = m_n / d_s'$; $d_s = m_n \cdot r / \text{SIMPLIFY}[f_n / r]$: $d_s' = r / d_s$
3. The factor f_n that is neither a divisor nor a totative of base r possesses an integral relationship m_n with a factor that is the reciprocal divisor d_s' of d_s.

   $$f_n = m_n \cdot d_s'; \quad m_n = f_n / d_s'; \quad d_s' = f_n / m_n$$
4. The number of cycles present in the multiplication table, n_C, for the factor f_n that is neither a divisor nor a totative of base r is equal to the integral relationship m_n between factor f_n and the divisor d of base r.

   $$n_C(f_n) = m_n; \quad n_C(f_n) = f_n / d_s$$
5. The cycle length λ_n of any factor f_n that is neither a divisor nor a totative of base r is equal to the reciprocal divisor d_s' of the divisor d_s with which the factor f_n possesses an integral relationship m.

   $$\lambda_n = d_s'; \quad \lambda_n = r / d_s; \quad \lambda_n = m_n\, r / f_n$$
6. The number of periods per cycle or P/C for any factor f_n that is neither a divisor nor a totative of base r is equal to the integral relationship m between factor f_n and the divisor d of base r.

   $$m_n = (\text{P/C})_n; \quad (\text{P/C})_n = f_n / d_s; \quad (\text{C/P})_n = d_s / f_n$$
7. The unit digits υ_p of all the products p in a given cycle $C(f_n)$ may increase or decrease within a cycle that spans multiple periods. These unit digits υ_p may additionally have distinct phases where the trend is reset without reaching the end of the cycle (i.e. the cycle exhibits phases.)
8. The period digits, which indicate the multiples μ_p of base r which will still yield a positive number lesser than r if μ_p is subtracted from the product p, increase for each element of the cycle C after the first element.
9. Factor Class C includes those factors f_n which are neither divisors d_r nor totatives t_r of base r, for which both m_n and d_s are divisors of r. Problems involving Factor Class C can use two factors to compute products beyond the range of the abbreviated multiplication table.
10. Factor Class D includes those factors f_n which are neither divisors d_r nor totatives t_r of base r, for which only d_s is a divisor of r, with an m_n that is totative. Problems involving Factor Class D will require commutation to be solvable.

Totative Factors (f_t)

The factors f_t which are also totatives t of base r comprise the last set of factors to consider. These factors exhibit cycles that require r products between each period. The number of periods per cycle for the factors f_t that are totatives is equal to the totative t. The totative 1 is, as stated previously, easy to calculate because the number 1 multiplied by any factor f equals f itself. Likewise, the number $r – 1$, a totative in any integral base r, can be computed via $(f \cdot r) – f$. The other totatives in the table feature complicated patterns that will not be analyzed here.

Factor Classes for the Totative Factors f_t

All the totative factors f_t of base r fall into Factor Class D. The only means of computing products for totative factors is the commutation of the problem via addition or subtraction, so that a divisor may be employed. This class includes the factors f_n which are not totative nor divisors of base r, for which only d_s is a divisor of r, with an m_n that is totative. These factors f_n are considered “totative” for the purposes of the abbreviation of the multiplication table. The factor classes will be discussed later.

Summary – Totative Factors

1. The number of phases φ_t of the totative factor f_t within the period is equal to the totative factor f_t itself.

$$\varphi_t = f_t$$

2. The factors f_t which are also totatives t of base r exhibit cycle lengths λ_t equivalent to r.

$$\lambda_t = r$$

3. The number of periods per cycle m_t for the factors f_t that are totatives is equal to the totative t.

$$m_t = (P/C)_t = t$$

4. All integral bases r possess totatives T_0 at negative and positive 1 modulus r.

$$T_0 = \pm 1 \bmod r$$

5. The products of the totative 1 equal the co-factor f.

$$f \cdot 1 = f$$

6. The products of the totative $(r – 1)$ equal the base r minus the co-factor f.

$$f \cdot (r - 1) = (f \cdot r) - f$$

Factor Classes

Four factor classes have been created out of the three types of factors described previously. These factor classes sort the factors according to which methods can be used to compute their products. These methods will be explained in a later section. Table 2D summarizes these.

Table 2D • Summary of the Factor Classes for Any Integer Base r

	Divisor Factors f_d	Non Divisor Non Totative Factors f_n	Totative Factors f_t
Factor Class A	Unity-Identity Divisors $\{1, r\}$	$\emptyset$	$\emptyset$
Factor Class B	Effective Divisors $D_e = D_r - \{1, r\}$	$\emptyset$	$\emptyset$
Factor Class C	$\emptyset$	Effective Non Divisor Non Totative Factors where $m_n \in D_r$	$\emptyset$
Factor Class D	$\emptyset$	Non Divisor Non Totative Factors where $m_n \in T_r$	All Totative Factors Except $\{1\}$

Factor Class A: The Unity-Identity Divisors

This class consists of the factors 1 and r. The multiplicative identity property states that multiplication of any factor f by 1 yields the product f. Multiplication by r is accomplished by shifting the digits of the factor f leftward one place, and writing a zero in the vacated unit place. Thus these computations involving these factors do not require a table to compute their products. For base twelve, this includes the factors {1, 12}

Factor Class B: The Effective Divisors

The set of divisors D_r of base r which are neither 1 nor r comprise the set of effective divisors D_e. For base twelve, this includes the factors {2, 3, 4, 6}. The cycles of these factors allows the abbreviation of the multiplication table to include only the first period.

Factor Class C: The Effective Non Divisor Non Totatives

This class contains all those factors f_n which are neither totative nor a divisor of the base r which are products of a "significant" divisor d_s and a multiplier m_n that is also a divisor or composed itself entirely of divisors of base r. In the case of dozenal, these are {8, 9}. The sexagesimal factor 32: is also an effective non divisor non totative of base 60:

$$\text{SIMPLIFY}[f_n / r] = m_n / d_s; \quad \text{SIMPLIFY}[32/60] = m_n / d_s; \quad 8/15 = m_n / d_s$$

$$m_n = 8; \quad 8 = 2 \cdot 4; \quad \{2, 4\} \in \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

Class C factors can use the associative property of multiplication to render the multiplication problem one that involves a divisor. {10} is not included, because one would need to use 5 to render 10 a divisor, and 5 is not a divisor of twelve.

Factor Class D: Totatives and Ineffective Nontotative Nondivisors

This set includes all totatives t_r of the base r except 1, and those factors f_n for which only d_s is a divisor of r, with an m_n that is totative. The factors of this class require the use of totatives to compute products. Totatives do not have cycles shorter than r itself, by definition. This means that abbreviation of the multiplication table is not possible for totatives unless we use the distributive property of multiplication to involve a divisor of base r. In base twelve, this set includes {5, 7, 10, 11}.

The following tables summarize factor classes for dozenal and sexagesimal.

Table 2E • Classification of Dozenal Factors

f	Class	Totative?	Divisor?	Relation to Base
0	A	-	Yes (12)	= 1
1	A	Yes	Yes	= 1/12, Relatively Prime
2	B	-	Yes	= 1/6
3	B	-	Yes	= 1/4
4	B	-	Yes	= 1/3
5	D	Yes	-	Relatively Prime
6	B	-	Yes	= 1/2
7	D	Yes	-	Relatively Prime
8	C	-	-	= 2/3, both numerator and denominator are divisors
9	C	-	-	= 3/4, both numerator and denominator are divisors
Χ	D	-	-	= 5/6, the numerator is not a divisor
Ɛ	D	Yes	-	Relatively Prime

Table 2F • Classification of Sexagesimal Factors

f	Class	Divisor?	Totative?	m_n	*f*	Class	Divisor?	Totative?	m_n
00	A	Yes 60	-	-	30	B	Yes	-	-
01	A	Yes	Yes	-	31	D	-	Yes	-
02	B	Yes	-	-	32	C	-	-	D
03	B	Yes	-	-	33	D	-	-	T
04	B	Yes	-	-	34	D	-	-	T
05	B	Yes	-	-	35	D	-	-	T
06	B	Yes	-	-	36	C	-	-	D
07	D	-	Yes	-	37	D	-	Yes	-
08	C	-	-	D	38	D	-	-	T
09	C	-	-	D	39	D	-	-	T
10	B	Yes	-	-	40	C	-	-	D
11	D	-	Yes	-	41	D	-	Yes	-
12	B	Yes	-	-	42	D	-	-	T
13	D	-	Yes	-	43	D	-	Yes	-
14	D	-	-	T	44	D	-	-	T
15	B	Yes	-	-	45	C	-	-	D
16	C	-	-	D	46	D	-	-	T
17	D	-	Yes	-	47	D	-	Yes	-
18	C	-	-	D	48	C	-	-	D
19	D	-	Yes	-	49	D	-	Yes	-
20	B	Yes	-	-	50	C	-	-	D
21	D	-	-	T	51	D	-	-	T
22	D	-	-	T	52	D	-	-	T
23	D	-	Yes	-	53	D	-	Yes	-
24	C	-	-	D	54	C	-	-	D
25	C	-	-	D	55	D	-	-	T
26	D	-	-	T	56	D	-	-	T
27	C	-	-	D	57	D	-	-	T
28	D	-	-	T	58	D	-	-	T
29	D	-	Yes	-	59	D	-	Yes	-

Summary

The patterns present in the dozenal multiplication table are summarized by Figure 2J. The products which are multiples of 12, representing exact periods in the table, are circled. Because twelve is highly composite, many combinations of factors will result in a product that is divisible by twelve. Because twelve is so highly composite, we can capitalize on the profound cyclical quality exhibited by the periods of twelve, and thereby abbreviate the multiplication table.

There are three principal classes of factor for any base.

Divisors. These are the factors f_d which are divisors d_r, the factors f_t which are totatives t_r, and the factors f_n which are neither divisors nor totatives of the base r. Among the factors f_d, which are the divisors d_r, there is a special pair of divisors {1, r} called the "unity-identity" pair. The number 1 is a divisor of all bases. The divisor 1, along with the totative r – 1, are totatives for every positive integer r. Thus the divisor 1 is always totative, and is the only totative divisor of any integral base r. The behavior of the divisor 1 in the multiplication table is totative and unlike the other divisors. The remaining f_d shall be considered the "effective divisors" of base r. This means that prime bases r possess no effective divisors.

Non Divisor Non Totatives. The factors f_n which are at the same time not divisors d_r nor totatives t_r of the base r possess an integral relationship m_n to one or more divisors d. The significant relationship is that divisor which is the denominator of the simplified fraction which is the ratio f_n / r. The numerator of this same simplified fraction, m, represents the number of periods P per cycle C which the products p of the factor f_n exhibit in the multiplication table. The factors f_n can be divided into two kinds. The first kind include factors f_n which feature an m_n which is totative; these f_n are classified with the factors f_t which are totative. The second kind are "effective factors f_n" which include those factors f_n for which both the numerator m_n and the denominator d are divisors of base r.

Totatives. The factors f_t which are totatives t_r of base r are by definition relatively prime to r. They feature cycles which repeat only after the entire span of r has passed. Because of this, the problems involving totative factors must be commutated so that the divisors of the base can be employed to solve the problem.

Factor Classes. There are four factor classes. Factor Class A consists of the "unity-identity" divisors, that is, the pair of divisors {1, r}. Factor Class B includes all divisors of r except {1, r}; this is the class which includes the "effective divisors" of base r. Factor Class C includes the factors f_n which are neither divisors d_r nor totatives t_r of base r, for which both m_n and d_s are divisors of r. Factor Class D includes the totative factors f_t and the factors f_n for which only d_s is a divisor of r, with an m_n that is totative. These factor classes will govern which techniques to use, given an abbreviated multiplication table for base r. Factor classes and the abbreviation of the multiplication table will be covered in ensuing sections.

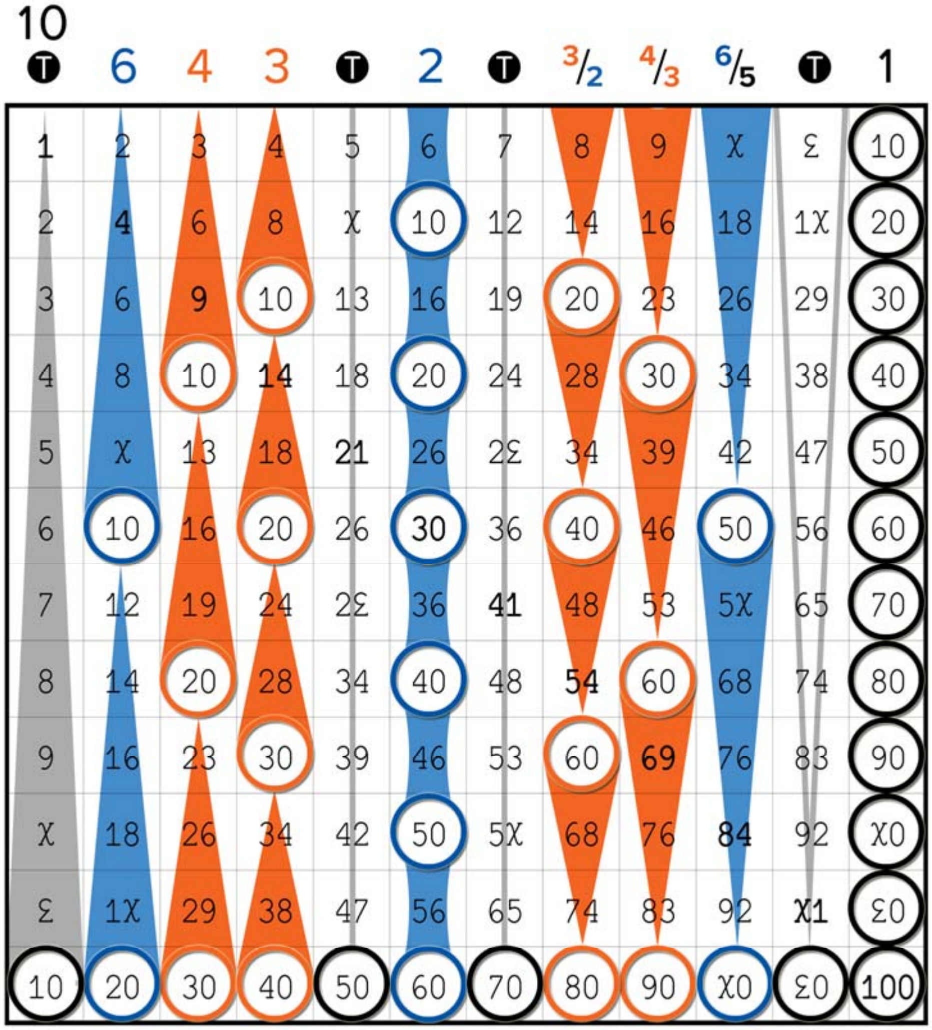


Figure 2J. *This table illustrates all the patterns present in the dozenal multiplication fact table.*

In Figure 2J above, effective divisor pairs {2, 6} and {3, 4} are indicated by blue and red, respectively. The cycle lengths λ are indicated by triangles which increase in width as the terminal digits of the products of factors f_d that are divisors increase, or decrease with the terminal digits of products of factors f_n which are nontotative nondivisors. The terminal digit of any factor $r/2$ (6, in the case of dozenal) is either 0 or $r/2$. Periods are circled. The reciprocal divisors d' appear above their corresponding divisor-factors f_d. The ratio of the reciprocal divisor mate d' of the related divisor d to the number of periods per cycle m_n is written above those factors f_n which are nontotative nondivisors. The four factor columns which do not feature a pattern are the totatives t of twelve; a T in a black circle marks these. It is interesting to note that, of the divisors of 12, only the divisor 1 is totative. This is in fact true for all integer bases. Also interesting is the fact that all integer bases r possess totatives at ±1 mod r.

Part 3 • Abbreviation of Multiplication Fact Tables

Introduction

We have demonstrated that the divisors of a base have products that repeat in cycles within one period. We have also seen that the remaining factors which are not totatives are cyclical in multiple periods. These demonstrations suggest that the entire multiplication table does not need to be memorized, provided there is some way to use the relationships between reciprocal divisors to generate the product. In the case of decimal, dozenal, and perhaps hexadecimal, the abbreviation of the multiplication table is unnecessary. These tables are concise enough to be entirely memorized. In the case of sexagesimal, these observations, coupled with the greater number of reciprocal divisor pairs, significantly abbreviates the multiplication table.

The Full Multiplication Table

The number of products that populate the traditional square layout of the multiplication table of base *r* is given by r^2. The sexagesimal multiplication table includes 3600 values when it is presented in the “square” manner which the decimal and dozenal tables were presented. This traditional table represents factor combinations twice, where the factor combinations involve unequal factors. The number of products of unique factor combinations of the multiplication table M_r of base *r* is given by adding *r* to the square of *r*, then dividing that quantity by 2. When we limit the table to include products of unique factor pairs, we arrive at 1830 figures (the “triangle” of 60). Both values lie beyond the ability of most people to memorize and recall for general computations. Using reciprocal divisor pairs, we are able to abbreviate the table to a manageable size. Let’s return to the dozenal multiplication table to study how the method will work.

1	2	3	4	5	6	7	8	9	X	Ɛ	10
2	4	6	8	X	10	12	14	16	18	1X	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	X	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5X	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
X	18	26	34	42	50	5X	68	76	84	92	X0
Ɛ	1X	29	38	47	56	65	74	83	92	X1	Ɛ0
10	20	30	40	50	60	70	80	90	X0	Ɛ0	100

Figure 3A. *The full, traditional form of the dozenal multiplication table.*

		6	4	3		2		3/2	4/3	6/5		1
	1	2	3	4	5	6	7	8	9	X	Ɛ	10
6	2	4	6	8	X	10	12	14	16	18	1X	20
4	3	6	9	10	13	16	19	20	23	26	29	30
3	4	8	10	14	18	20	24	28	30	34	38	40
	5	X	13	18	21	26	2Ɛ	34	39	42	47	50
2	6	10	16	20	26	30	36	40	46	50	56	60
	7	12	19	24	2Ɛ	36	41	48	53	5X	65	70
3/2	8	14	20	28	34	40	48	54	60	68	74	80
4/3	9	16	23	30	39	46	53	60	69	76	83	90
6/5	X	18	26	34	42	50	5X	68	76	84	92	X0
	Ɛ	1X	29	38	47	56	65	74	83	92	X1	Ɛ0
1	10	20	30	40	50	60	70	80	90	X0	Ɛ0	100

Figure 3B. *The dozenal multiplication table abbreviated to the first period.*

Truncation of the Table to the First Period

It is possible to cut the multiplication table to the first period for each of the factors in the table. This is because we can use the reciprocal divisor pairs present within the set of divisors of base r to compute products greater than r. The divisors as factors f_d exhibit cycles that repeat every period. Thus, we can find each instance of "10" in the table, and ignore figures greater than this. We can leave the "1" line so that it serves as an index of factors. This truncation is shown by Figure 3B, with the traditional dozenal multiplication table appearing in Figure 3A. Since the "1" line serves as an index, we do not need to write anything in the "1" line greater than or "south" of the point where the "2" line hits its first period for even bases. (This is optional for odd bases.) This further truncation appears in Figure 3C.

		6	4	3		2						
	1	2	3	4	5	6	7	8	9	X	Ɛ	10
6	2	4	6	8	X	10	12	14	16	18	1X	20
4	3	6	9	10	13	16	19	20	23	26	29	30
3	4	8	10	14	18	20	24	28	30	34	38	40
	5	X	13	18	21	26	2Ɛ	34	39	42	47	50
2	6	10	16	20	26	30	36	40	46	50	56	60
	7	12	19	24	2Ɛ	36	41	48	53	5X	65	70
	8	14	20	28	34	40	48	54	60	68	74	80
	9	16	23	30	39	46	53	60	69	76	83	90
	X	18	26	34	42	50	5X	68	76	84	92	X0
	Ɛ	1X	29	38	47	56	65	74	83	92	X1	Ɛ0
	10	20	30	40	50	60	70	80	90	X0	Ɛ0	100

Figure 3C. *Truncation of the "ones" column beyond the first period in the "twos" column.*

	1	2	3	4	5	6	7	8	9	X	Ɛ	10
6	2	4	6	8	X	10	12	14	16	18	1X	20
4	3	6	9	10	13	16	19	20	23	26	29	30
3	4	8	10	14	18	20	24	28	30	34	38	40
	5	X	13	18	21	26	2Ɛ	34	39	42	47	50
2	6	10	16	20	26	30	36	40	46	50	56	60
	7	12	19	24	2Ɛ	36	41	48	53	5X	65	70
	8	14	20	28	34	40	48	54	60	68	74	80
	9	16	23	30	39	46	53	60	69	76	83	90
	X	18	26	34	42	50	5X	68	76	84	92	X0
	Ɛ	1X	29	38	47	56	65	74	83	92	X1	Ɛ0
	10	20	30	40	50	60	70	80	90	X0	Ɛ0	100

Figure 3D. *Further truncation beyond the products which are square. This fully abbreviated table includes only unique facts. There are thirteen facts in the abbreviated table. The full table included 78 unique facts.*

Reduction to Eliminate Restatements

The Minimal Abbreviated Table

In order to pare down a large multiplication table to something more manageable, we can ignore everything "northeast" of the diagonal line of squares. For dozenal, this step leads to a table of thirteen values, which is the minimum required using the reciprocal divisor method. This reduced table is ¼ of the minimum triangular table of 78 values, and is illustrated by Figure 3D.

The Crossing Abbreviated Table

Perhaps for ease of use, we might restate some products "northeast" of the square line, at that square that is the largest square that is smaller than the integer that is the radix. This second suggestion restates three products so that one can use the table as a matrix. In this second case, the 144-value square table was pared down to 1/9 its size. The resultant abbreviated table shown in Figure 3E is all we need in order to employ the reciprocal divisor method.

The "crossing" table situates the smaller half of the set of effective reciprocal divisors to the left, and the larger half at the top. The products that lie at the crossing of the rows bearing the smaller reciprocal divisors and the columns bearing the larger reciprocal divisors equal *r*. This placement of reciprocal divisors accentuates the location of the factors f_d in the abbreviated table which are also divisors of base *r*, while reminding the user of the abbreviated table of the reciprocal divisors *d'* of these factors f_d.

The abbreviated table thus facilitates the multiplication in base *r* using the reciprocal divisor method that will be described in the next section.

		6	4									
	1	2	3	4	5	6	7	8	9	X	Ɛ	10
	2	**4**	6	8	X	10	12	14	16	18	1X	20
	3	6	**9**	10	13	16	19	20	23	26	29	30
3	4	8	10	14	18	20	24	28	30	34	38	40
	5	X	13	18	21	26	2Ɛ	34	39	42	47	50
2	6	10	16	20	26	30	36	40	46	50	56	60
	7	12	19	24	2Ɛ	36	41	48	53	5X	65	70
	8	14	20	28	34	40	48	54	60	68	74	80
	9	16	23	30	39	46	53	60	69	76	83	90
	X	18	26	34	42	50	5X	68	76	84	92	X0
	Ɛ	1X	29	38	47	56	65	74	83	92	X1	Ɛ0
	10	20	30	40	50	60	70	80	90	X0	Ɛ0	100

1	2	3	4	5	6	7	8	9	X	Ɛ	10
2	**4**	6	8	X	10	12	14	16	18	1X	20
3	6	**9**	10	13	16	19	20	23	26	29	30
4	8	10	**14**	18	20	24	28	30	34	38	40
5	X	13	18	**21**	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	**30**	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	**41**	48	53	5X	65	70
8	14	20	28	34	40	48	**54**	60	68	74	80
9	16	23	30	39	46	53	60	**69**	76	83	90
X	18	26	34	42	50	5X	68	76	**84**	92	X0
Ɛ	1X	29	38	47	56	65	74	83	92	**X1**	Ɛ0
10	20	30	40	50	60	70	80	90	X0	Ɛ0	**100**

Figure 3E. *An abbreviated table which includes full columns from one through the period or whichever number is lesser than a full period is perhaps more handy. The divisors of the base then can be split evenly, with the greater half on the short axis, and the lesser half on the long axis. This arrangement then allows the reciprocal divisor pairs to meet at the full periods. Note above that the 2 and 6 written outside of the table cross at "10"; the 3 and 4 written outside the table does the same thing in a different location. All Class A operations take place within the limits of the abbreviated table.*

Abbreviated Tables

The objective of the abbreviation of the multiplication table is to minimize memorization required to multiply and divide using a given base. The reduction of the number of products in the table is helpful, but does not eliminate the need to memorize. Some memorization is still required if fluency in computation in the base is desired. This booklet features a sexagesimal table where the digits have been expressed in one of two fashions. One method is traditional, while the other attempts to completely eliminate decimal thinking from computation in higher bases, at least as far as multiplication and the identity of integers are concerned.

Traditional Sexagesimal Notation

The traditional method of expressing a sexagesimal place with a decimal number is found in the expression of time. The version of this notation used in this booklet writes a pair of digits for each sexagesimal place. Technically speaking, the first of these two digits is base-6, the second base-10. This expression of sexagesimal by two digits renders sexagesimal a mixed radix, where every other digit is written in the same sub-base. Most people consider this technicality unimportant.

Pairs of digits which express the value of a sexagesimal place are separated by another character to demarcate the sexagesimal place. In this booklet, a semicolon (;) separates places above or below the radix point. The radix point, akin to the decimal point in decimal notation, is represented by the colon (:). Thus, the quantity 96 is represented by this system as 01;36:. Examples of this notation appear in table 3A.

Table 3A • Sexagesimal Mixed Radix versus Pure Radix Notation

Decimal	*Traditional Sexagesimal*	*Argam Sexagesimal*	*Decimal*	*Traditional Sexagesimal*	*Argam Sexagesimal*
15	15:	℮	2.71818	02:43;05;49	2.Ɒ5ᖷ
40	40:	ᑫ	3.14159	03:08;29;44	3.8ᖶᖰ
75	01;15:	1℮	1/2	00:30;	0.ᘔ
81	01;21:	17	1/3	00:20;	0.Ɣ
96	01;36:	1ℓ	3/4	00:45;	0.ℓ
100	01;40:	1ᑫ	2/5	00:24;	0.ɣ
144	02;24:	2ɣ	5/6	00:50;	0.Ƨ
225	03;45:	3ℓ	0.3	00:20;	0.∂
360	06;00:	60	5/12	00:25;	0.ς
441	07;21:	77	13/16	00:48;45;	0.Ɣℓ
576	09;36:	9ℓ	1/27	00:02;13;20;	0.2∂∂
729	12;09:	Ɣ9	2%	00:01;12;	0.1Ɣ
1728	28;48:	ƐƔ	1/144	00:00;25;	0.0ς
2007	33;27:	Ƶᒽ	1/1728	00:00;02;05;	0.025
2520	42;00:	Ꝺ0	99.44%	00:59;39;50;24	0.ᖵ϶Ƨɣ
100,000	27;46;40:	ᒽɞᑫ	7¾%	00:04;39	0.4϶
1,000,000	04;37;46;40:	4Ꙟɞᑫ	1 ppm	:00;00;00;12;57;36	0.000Ɣ₹ℓ

Argam Notation

A second method of notation designed for bases greater than ten is the Argam notation. The Argam system of numerals is an extension of the Hindu Arabic numerals. The Argam numerals, individually called "ragam" or "argam" plural, build on the identities of the ten established integer symbols to produce a vast array of new numeral characters. Composite argam take on the graphic and nominal qualities of their progenitor argam. The identity of an integer in the Argam system is derived from the integer's divisors or prime factorization, and not by addition or any decimal reference. The goal of Argam is to furnish an explorer of the higher bases a tool through which a purer expression of quantities in a higher base can be attained. References to decimal or any other system of numeration are minimized.

		30:	20:	15:	12:	10:	
	01:	02:	03:	04:	05:	06:	07:
	02:	**04:**	06:	08:	10:	12:	14:
	03:	06:	**09:**	12:	15:	18:	21:
	04:	08:	12:	**16:**	20:	24:	28:
	05:	10:	15:	20:	**25:**	30:	35:
	06:	12:	18:	24:	30:	**36:**	42:
	07:	14:	21:	28:	35:	42:	**49:**
	08:	16:	24:	32:	40:	48:	56:
	09:	18:	27:	36:	45:	54:	
06:	10:	20:	30:	40:	50:	01;00:	
	11:	22:	33:	44:	55:		
05:	12:	24:	36:	48:	01;00:		
	13:	26:	39:	52:			
	14:	28:	42:	56:			
04:	15:	30:	45:	01;00:			
	16:	32:	48:				
	17:	34:	51:				
	18:	36:	54:				
	19:	38:	57:				
03:	20:	40:	01;00:				
	21:	42:					
	22:	44:					
	23:	46:					
	24:	48:					
	25:	50:					
	26:	52:					
	27:	54:					
	28:	56:					
	29:	58:					
02:	30:	01;00:					

2 ↔ 30
3 ↔ 20
4 ↔ 15
5 ↔ 12
6 ↔ 10

Figure 3F. *The abbreviated table for sexagesimal, written with 6-on-10 notation, or decimal as a sub-base.*

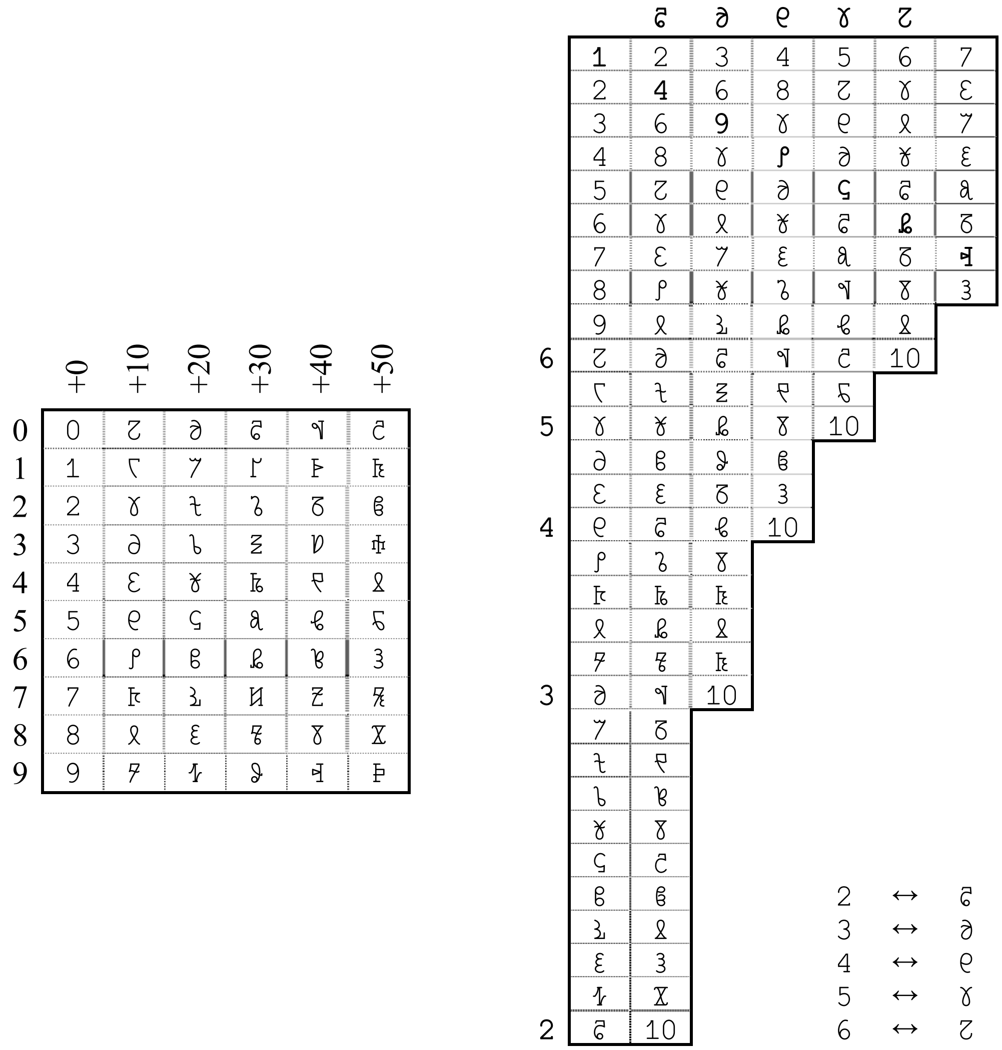


Figure 3G. *The abbreviated table for sexagesimal, using a purely sexagesimal notation. This notation represents each digit within one place, and features a unique symbol for each of the sixty digits.*

The Argam tables and notation is used in this booklet for two reasons. The first reason is Argam furnishes a purer representation of digits in a base higher than decimal, especially those higher than the Latin alphabet mounted on the Hindu Arabic numerals. Secondly, the expression of numbers written in sexagesimal and higher bases is needlessly complicated by the use of decimal sub-bases. These higher-base numbers appear more complex than they really are because two or more decimal digits are spent on each place in the higher-base number. At the expense of appearing alien, the Argam digits are a toolset to simply express numbers written in bases significantly higher than ten.

The names of the first 120 argam are included in the appendix. The Argam system is employed by all of the author's explorations into transdecimal bases. The argam which appear in this booklet are provided in the spirit of facilitating your own forays "into the mountains" of the higher bases.

Summary

The abbreviated multiplication table is intended to reduce the number of products which need memorizing. For maximum fluency, the user of a large base should be well acquainted with the products in the abbreviated multiplication table. Any integer below the base *r* should be keenly known. All the reciprocal divisor pairs should be familiar. Recognition of effective factors and totatives is useful. With usage and time, perhaps a larger portion of the full table can be used directly in class A operations.

1. The traditional full multiplication table M_r of base *r* is square; the number of products P_{full} in the full tables is equal to the square of *r*.
$$P_{full} = r^2$$
2. The multiplication table M_r which displays only the products *p* of unique combinations of the set of factors F of base *r* is triangular; the number of products P_M in these tables is equal to the quantity of *r* plus the square of *r*, the quantity divided by two.
$$P_M = r\,(r + 1) / 2$$
3. The first truncation involves the elimination of all products *p* greater than the radix *r*.
4. The second truncation, more applicable to even bases, involves ignoring all unity products greater than the first period of the products of the factor 2. This is optional in the case of odd radixes.
5. The final truncation involves ignoring products which are repeated on one side of the diagonal line of square products. The elimination of either side is acceptable.
6. The “Crossing” Abbreviated Table. The abbreviated table may be more legible if products in the lines of products whose square *p* is less than the radix *r* are allowed to appear on the side of the line of squares where the products were eliminated in 5 above.
7. Notation. Reciprocal divisors *d’* can be written next to factors f_d which are divisors. The reciprocal divisors which are greater than $r^{1/2}$ can be written above the corresponding factors f_d of the shorter axis of the table, while the balance can be written next to the factors f_d of the long axis of the table.

Part 4 • Operation Classes

When we multiply or divide in decimal or dozenal, we mentally refer to the memorized multiplication table; this is second nature to everyone out of primary school. These operations have analogs in bases for which abbreviated tables provide a means to multiply and divide. Obviously, for larger multiplication tables which lie beyond the human ability or desire for memorization, the number of products we need to memorize should be reduced. This section describes the four operation classes which apply to an abbreviated multiplication table. The section also studies the range of these operation classes. The dozenal system (base 12) will be used as an example. The operation classes will then be applied to sexagesimal numbers (those in base 60).

Know Thy Base's Divisor Pairs

Abbreviation of the multiplication tables of large bases depends crucially upon the use of effective reciprocal divisor pairs of a given base. The abbreviated multiplication table and a short table of effective reciprocal divisor pairs go hand in hand.

Since the abbreviated multiplication table is a resource for study, noting the reciprocal divisors *d'* next to the factors f_d which are divisors *d* helps to remind the user of the reciprocal divisors of the base. Absent of this notation, the reciprocal divisors of any base can be determined by finding every occurrence of products which equal *r*, then noting the factors that occur in the index (1 line) above and to the left of each product. Every occurrence of "10" will mark a pair of reciprocal divisors. However, before conducting any operations, one should know keenly the effective divisor pairs for the base in use.

The dozenal effective divisors are: {{2, 6}, {3, 4}}.

Operation Class A: Products within the Abbreviated Table

This class includes Factor Class A, along with any product in the abbreviated table. The unity-identity divisors {1, *r*}, as stated previously, have simple rules by which a product can be computed, eliminating the need for these to populate a multiplication table. The abbreviated multiplication table contains all products *p* which are lesser than or equal to the radix *r*. This is a vastly reduced table of factors, when compared to the full multiplication table. A limit imposed by the human capacity to memorize the table is reached perhaps when the table reaches a size comparable to the full multiplication tables of the "human scale bases", somewhere between 55 to maybe 220 figures. This interesting limit is not very distinctly defined, and has not been tested for this study. These figures correspond to minimal abbreviated tables for base 30 or 32 on the low end to that of base 120 on the high end. The sexagesimal minimal abbreviated table includes 104 unique values, with the "crossing" abbreviated table containing 125 values.

In order to employ Class A, one simply recalls the memorized product given two factors. For problems involving the factor 1, the multiplicative identity rule can be applied. This rule states that any co-factor *f'* multiplied by 1 equals that co-factor *f'*. For problems involving the factor *r*, the co-factor's digits can be shifted leftward one place, and a zero can be written in the vacated unit place. Class A operations exist for all bases; as the bases increase in size, multiplication table abbreviation becomes handy and makes practical bases that have tables beyond most people's abilities to perform Class A operations for every product.

The dozenal abbreviated table at right presents 13 of 78 products of unique factor combinations that populate the full table. The other 65 must be computed in other ways. A significant number of the products that do not lie within the table involve the factors 1 and *r*, which in this case is one dozen. These products are easy to compute, and no memorization is necessary.

Bear in mind that the dozenal abbreviated table is presented only as an example. The application of the abbreviated table to dozenal multiplication is actually inefficient because dozenal is a human scale radix. This means that all dozenal multiplication can be more easily accommodated by memorization of its full multiplication table.

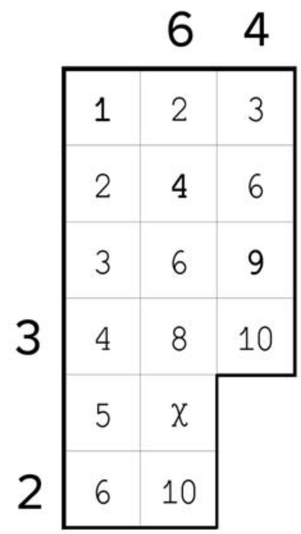


Figure 4A. *The dozenal "crossing" abbreviated multiplication table, showing reciprocal divisors next to their corresponding factors in the table.*

Computation Process

In order to compute a Class A product, follow these steps:

1. For problems involving the factor 1: use the multiplicative identity rule: any co-factor *f'* multiplied by 1 equals that co-factor *f'*.

   $$1 \cdot f' = f'$$

2. For problems involving the factor *r*: the product of any co-factor *f'* multiplied by the factor *r* can be generated by shifting the digits of the product leftward one place, and writing a zero in the unit digit place.
3. The abbreviated multiplication table will yield products lesser than *r* for some factors lesser than *r*/2. If the abbreviated table has been memorized, intuition may reveal whether a product lies within or outside the abbreviated table.
4. If the product is determined to lie outside the table, use another operation class.

Dozenal Examples

Examples for Operation Class A are relatively straightforward. These examples are sufficient to illustrate Operation Class A for whichever base *r* is used.

Table 4A • Operation Class B • Dozenal Examples

2 · 5 = χƐ	Per Step 3 above. The product lies within the abbreviated table.
828Ɛ · 10 = 828Ɛ0	Per Step 2 above. The problem involves *r*.
Ɛ · 1 = Ɛ	Per Step 1 above. The problem involves 1.
4 · 8 = ?	We need to use another operation class.
7 · 8 = ?	We need to use another operation class.

Operation Class B: Effective Divisor Factors

This operation class includes problems where at least one of the factors is member of Factor Class B, that is, a factor f_d which is a divisor d of the base r. The reciprocal divisor d' of the factor f_d can be used to yield any product p which lies outside the abbreviated multiplication table. Figure 4B illustrates a class B operation. Operation Class B is available to all integer bases r which have more than 2 divisors; that is, to all composite bases. Use the following procedure to determine products off the abbreviated table.

1. Determine the reciprocal divisor d' for the divisor d which is a factor f_d in the problem. The co-factor can be any factor f'.

$$d' = r / f_d$$

2. Divide the co-factor f' by the reciprocal divisor d'. Keep the integer quotient q_I and the remainder q_r separate.

$$f' / d' = (q_I + q_r)$$

3. Take the integer part of the quotient which is the result of the division in step 2. This quotient will be carried to the place or digit one order of magnitude greater than the factor f.

4. Multiply the remainder of division q_r by the original f_d. This product will occupy the digit of the product which corresponds to the place in operation.

5. The full formula for the computation of a product p involving one factor f_d which is a divisor d of base r appears below:

$$p = (q_I \cdot r) + q_r; \quad p = (\text{INTEGER}[f' / d'] \cdot r) + \text{REMAINDER}[f' / d']$$

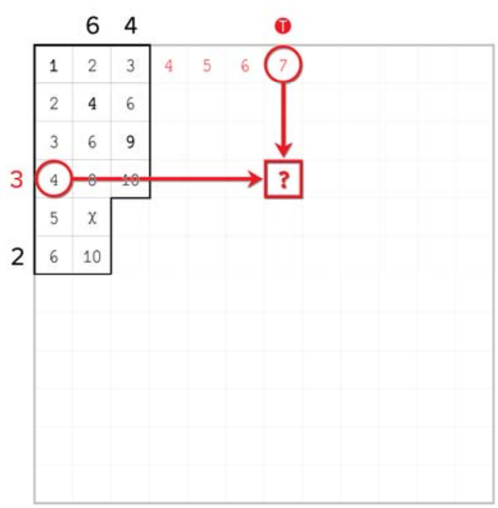
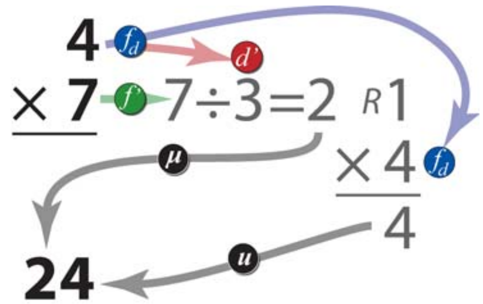


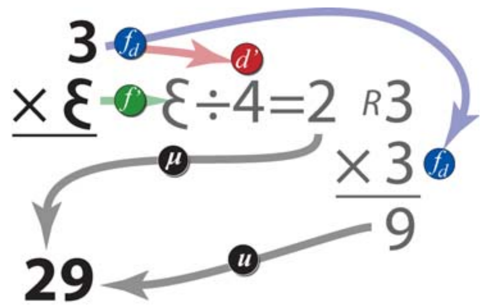
Figure 4B. *An example of a Class B operation, where a product which involves at least one factor that is a divisor occurs beyond the limits of the abbreviated table.*

Operation Class B • Dozenal Examples

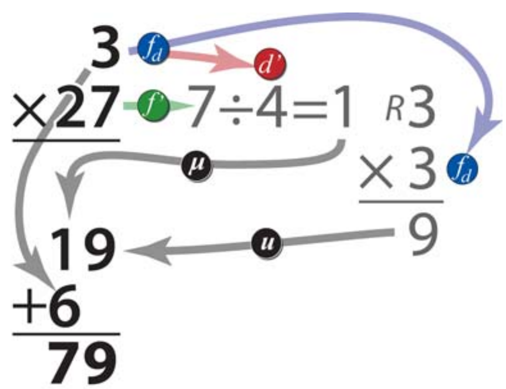
1. Multiply 4 by 8. The divisor involved in this problem is 4. The reciprocal divisor mate of 4 is 3. Dividing the co-factor 8 by the reciprocal divisor 3, we obtain 2 remainder 2. The integer 2 will be carried. The remainder 2 times the original factor 4 equals 8. Thus the product is 2 dozen 8.



2. Multiply 3 by Ɛ. The divisor involved in this problem is 3. The reciprocal divisor mate of 3 is 4. Dividing the co-factor Ɛ by the reciprocal divisor 4, we obtain 2 remainder 3. The integer 2 will be carried. The remainder 3 times the original factor 3 equals 9. Thus the product is 2 dozen 9.



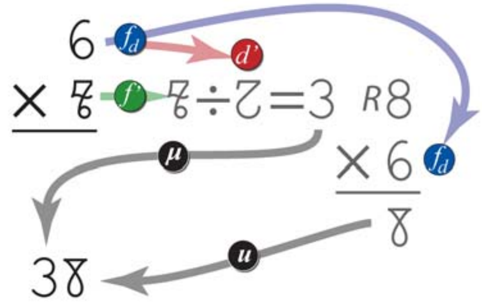
3. Multiply 27; by 3. The divisor involved in the problem is 3. The reciprocal divisor mate of 3 is 4. Dividing the co-factor 7 by the reciprocal divisor 4 yields 1 remainder 3. The integer 1 is carried. The remainder 3 multiplied by the original factor 3 is 9. The unit digit for the product is 9. Look at the next digit in the co-factor, 2. The product of 2 and 3 appears in the abbreviated table; it is 6. Add the 1 which we carried from the last operation to the 6 to yield 7 for the "dozens" or 12^1 place. Thus the product of 3 and 27; is 79;.



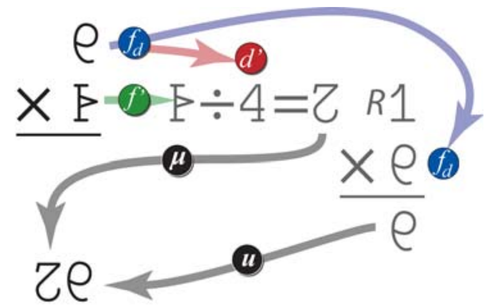
Operation Class B • Sexagesimal Examples

The images that accompany the following examples use a set of sexagesimal digits. A table that describes the digits appears in the Appendix. In the description, the 6-on-10 or decimal sub-base digits are used. The sexagesimal digits are employed here to illustrate the method does not depend on a decimal operation. In the decimal sub-base representation, each sexagesimal digit is represented by two sub-base digits. These digits are separated by the character ";". The radix point used here in a sub-base notation is ":".

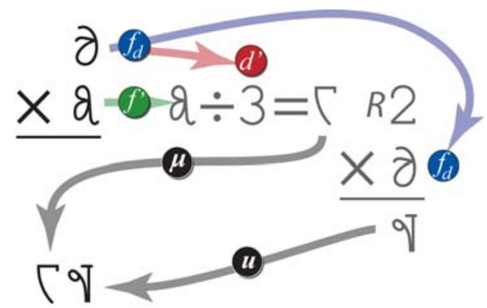
1. Multiply 06: by 38: The divisor involved in this problem is 06:. The reciprocal divisor mate of 06: is 10:. Dividing the co-factor 38: by the reciprocal divisor 10:, we obtain 03: remainder 08:. The integer 03: is carried to the next place. The remainder 08: times the original factor 06: equals 48:. Thus the product is 03;48:.



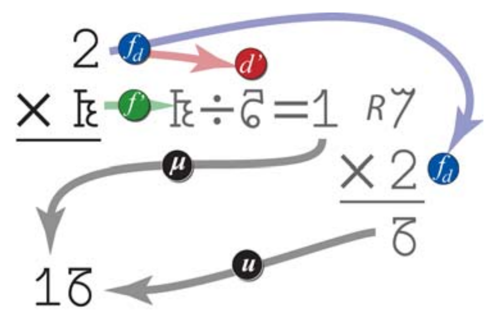
2. Multiply 15: by 41: The divisor involved in this problem is 15:. The reciprocal divisor mate of 15: is 04:. Dividing the co-factor 41: by the reciprocal divisor 04:, we obtain 10: remainder 01:. The integer 10: is carried. The remainder 01: times the original factor 15: equals 15:. Thus the product is 10;15:.



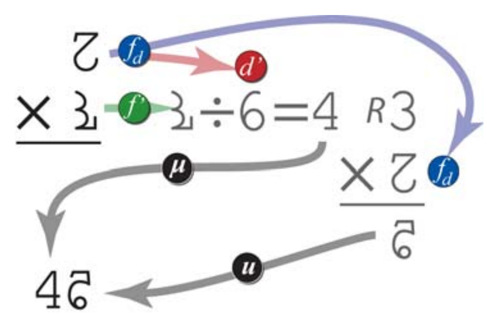
3. Multiply 20: by 35: The divisor involved in this problem is 20:. The reciprocal divisor mate of 20: is 03:. Dividing the co-factor 35: by the reciprocal divisor 03:, we obtain 11: remainder 02:. The integer 11: is carried. The remainder 02: times the original factor 20: equals 40:. Thus the product is 11;40:.



4. Multiply 51: by 02: The divisor involved in this problem is 02:. The reciprocal divisor mate of 02: is 30:. Dividing the co-factor 51: by the reciprocal divisor 30:, we obtain 01: remainder 21:. The integer 01: is carried. The remainder 21: times the original factor 02: equals 42:. Thus the product is 01;42:.



5. Multiply 27: by 10: The divisor involved in this problem is 10:. The reciprocal divisor mate of 10: is 06:. Dividing the co-factor 27: by the reciprocal divisor 06:, we obtain 04: remainder 03:. The integer 04: is carried. The remainder 03: times the original factor 10: equals 30:. Thus the product is 04;30:.



Operation Class C: Effective Non Divisor Non Totative Factors

This operation class includes problems where at least one of the factors is member of Factor Class C. This factor class includes all those factors f_n which are neither totative nor a divisor of the base r which are products of a "significant" divisor d_s and a multiplier m_n that is also a divisor or composed itself entirely of divisors of base r. Solution of the problem involving the effective f_n begins by extracting m_n from f_n to yield d_s. Once d_s is known, we can use the reciprocal divisor pair { d_s , d_s' } in a class B operation as described above. The factor m_n must be recorded and applied at the end of the Operation Class B process. When the m_n is itself a divisor, the application of m_n becomes simply a second Operation Class B process. In cases where m_n is a composite of divisors, the m_n may be broken down in an Operation Class C.

Operation Class C can be regarded as an extraction of the m_n multiplier so that Operation Class B can be applied to the co-factor *f'*. This class of operation is available to all bases *r* which have 5 or more divisors and many which have 4 or more divisors. Bases which have a diverse set of prime factors may feature many avenues open to Operation Class C. Prime bases can not use Operation Class C. Figure 4C illustrates an Operation Class C problem. Use the following procedure to determine Operation Class C products which lie beyond the abbreviated table.

1. Obtain both m_n and d_s from the simplified ratio f_n / *r*. The numerator of the simplified ratio is m_n while the denominator is equal to *r* / d_s.
2. Divide the factor f_n by m_n to obtain d_s; make a note of m_n (this is equivalent to replacing the factor f_n in the problem with d_s and noting m_n elsewhere).
3. Carry out a class B operation.
4. Examine m_n to see if it is itself a divisor of *r*. Proceed with the next step if this is true. If m_n is composed entirely of divisors, begin a new second application of a class C operation, with a divisor that comprises m_n serving as "d_s" in Step 2, and the remaining portion of m_n as "m_n" which is to be noted. Operation Class C will need repetition until all the divisors except one which compose the original m_n have been exhausted.
5. Obtain the final product *p* by multiplying the product obtained in Step 3 (the class B operation) by (the original) m_n.

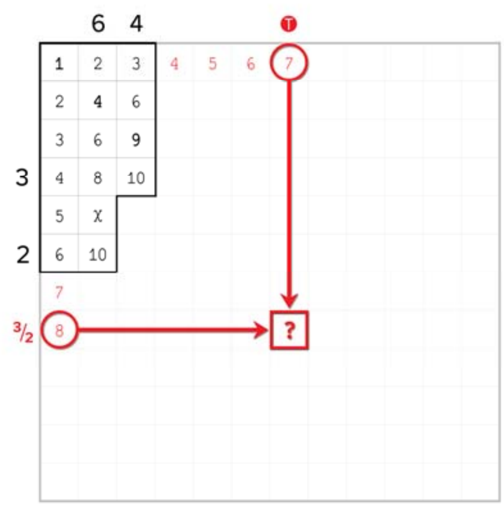
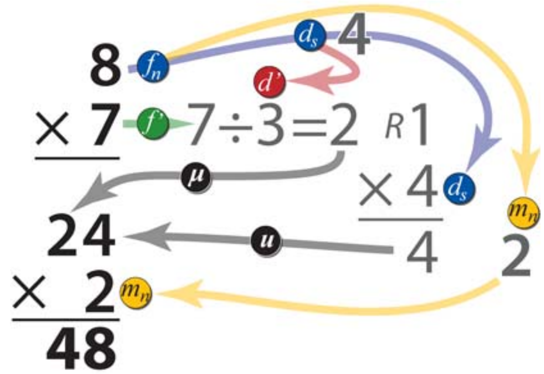


Figure 4C. *An example of a Class C operation which involves a nontotative nondivisor.*

Operation Class C • Dozenal Examples

1. Multiply 7 by 8. The factor 8 has a ratio with the base 12 that simplifies to 2/3. Thus, 2 is the multiplier m_n and 3 is the related divisor's reciprocal mate d_s'. The related divisor itself, d_s, is 4. The problem is split into two stages. The first stage is a class B operation. Thus, the co-factor f' = 7 is divided by d_s' = 3, yielding 2 remainder 1. The integer 2 is carried. The remainder 1 is multiplied by d_s = 4, yielding 4. Thus Operation Class B yields the result 24;. This result requires multiplication by m_n = 2, yielding the answer, 4 dozen 8.



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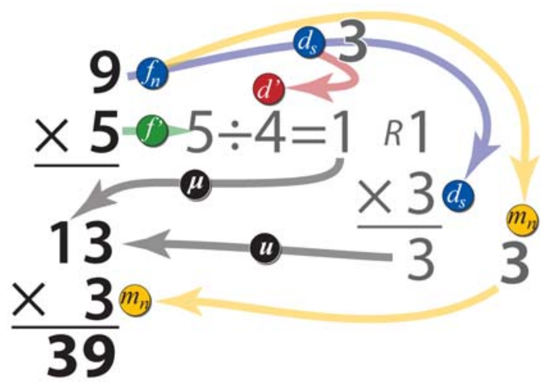
2. Multiply 9 by 5. The factor 9 has a ratio with the base 12 that simplifies to 3/4. Thus, 3 is the multiplier m_n and 4 is the related divisor's reciprocal mate d_s'. The related divisor itself, d_s, is 3. The problem is split into two stages. The first stage is a class B operation. Thus, the co-factor f' = 5 is divided by d_s' = 4, yielding 1 remainder 1. The integer 1 is carried. The remainder 1 is multiplied by d_s = 3, yielding 3. Thus Operation Class B yields the result 13;. This result requires multiplication by m_n = 3, yielding the answer, 3 dozen 9.



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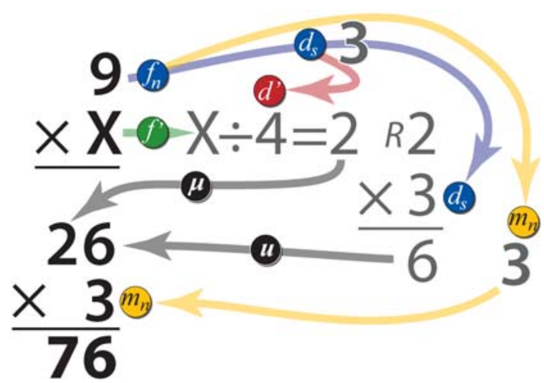
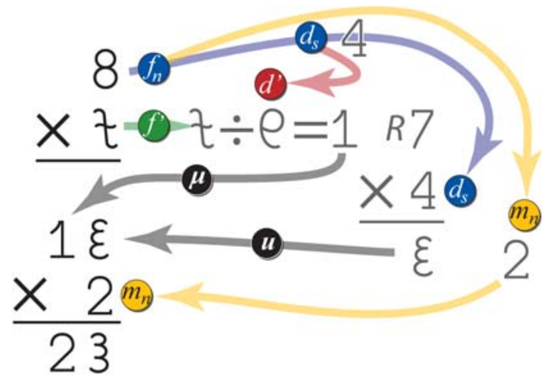
3. Multiply 9 by X. The factor 9 has a ratio with the base 12 that simplifies to 3/4. Thus, 3 is the multiplier m_n and 4 is the related divisor's reciprocal mate d_s'. The related divisor itself, d_s, is 3. The problem is split into two stages. The first stage is a class B operation. Thus, the co-factor f' = X is divided by d_s' = 4, yielding 2 remainder 2. The integer 2 is carried. The remainder 2 is multiplied by d_s = 3, yielding 6. Thus Operation Class B yields the result 26;. This result requires multiplication by m_n = 3, yielding the answer, 7 dozen 6.

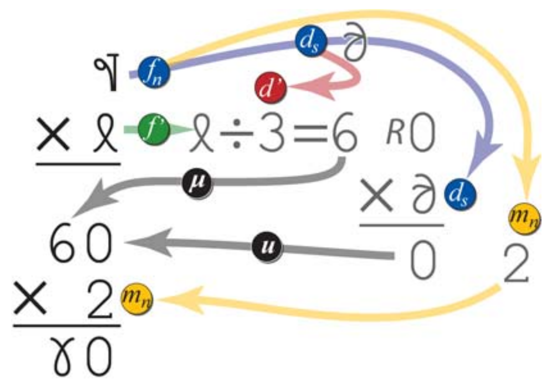


Operation Class C • Sexagesimal Examples

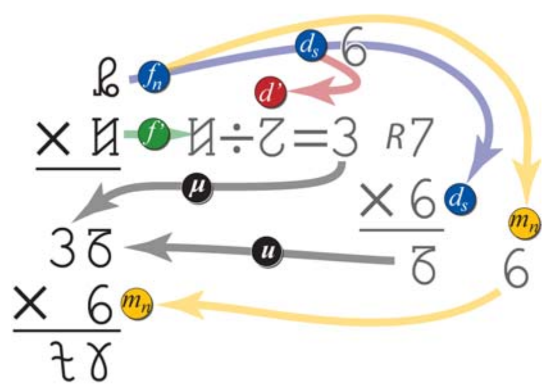
1. Multiply 22: by 08:. The factor 08: has a ratio with the base 60 that simplifies to 02:/15:. Thus, 02: is the multiplier m_n and 15: is the related divisor's reciprocal mate d_s'. The related divisor itself, d_s, is 04:. The problem is split into two stages. The first stage is a class B operation. Thus, the co-factor f' = 22: is divided by d_s' = 15:, yielding 01: remainder 07:. The integer 01: is carried. The remainder 07: is multiplied by d_s = 04:, yielding 28:. Thus Operation Class B yields the result 01;28:. This result requires multiplication by m_n = 02:, yielding the answer, 02;56:.



2. Multiply 40: by 18:. The factor 40: has a ratio with the base 60 that simplifies to 02:/03:. Thus, 02: is the multiplier m_n and 03: is the related divisor's reciprocal mate d_s'. The related divisor itself, d_s, is 20:. The problem is split into two stages. The first stage is a class B operation. Thus, the co-factor f' = 18: is divided by d_s' = 03:, yielding 06: without remainder. Thus Operation Class B yields the result 06;00. This result requires multiplication by m_n = 02:, yielding the answer, 12;00:. Note that this problem might have employed 18: via the divisors {2, 3, 3}. This would require three rather than two phases for 40: = {2, 20}. Thus 40: presents a more efficient process than 18:, so 40: is preferable.



3. Multiply 37: by 36:. The factor 36: has a ratio with the base 60 that simplifies to 06:/10:. Thus, 06: is the multiplier m_n and 10: is the related divisor's reciprocal mate d_s'. The related divisor itself, d_s, is 06:. The problem is split into two stages. The first stage is a class B operation. Thus, the co-factor f' = 37: is divided by d_s' = 10:, yielding 03: remainder 07:. The integer 01: is carried. The remainder 07: is multiplied by d_s = 06:, yielding 42:. Thus Operation Class B yields the result 03;42:. This result requires multiplication by m_n = 06:, via another full class C operation, ultimately yielding the answer, 22;12:.



Method for Determining Effective Factors f_e

The effective factor f_e is the basic tool of Operation Class C. In order to reduce the problem to one which can be resolved using reciprocal divisors, a multiple m_n can be extracted from the non totative non divisor factor f_n. This multiple is effective if it is the product of divisors or itself a divisor of the base. If the multiple involves a totative, the factor f_n is technically unusable in a class C operation, because this particular factor f_n would require multiplication by a totative. It is possible to transact on an "ineffective factor" f_n. The dozenal digit χ can be broken into $(5 \cdot 2)$; the 5 could use a class D operation (yet to be described). This is more involved than necessary because the dozenal digit χ itself warrants a class D operation.

The determination of a base's effective factors is important because these factors enable Operation Class C. Table 4B illustrates one method of determining the factors f_n which are non totative non divisors of base r that have an m_n which is entirely composed of divisors. In effect, the table is simply a miniature multiplication table of divisors. We are only interested in single-digit results.

1. Construct the table by multiplying all the effective divisors. Use the base's notation rather than decimal. In the case of 6-on-10 notation in sexagesimal, decimal can be used, but care should be taken in recognizing that the products are *not* decimal products, but sexagesimal digits. Results written in decimal figures greater than 59 are mistakes.
2. Ignore any product which is already accounted for in the set of effective divisors.
3. Any product not among the set of effective divisors is an effective factor f_e.

Thus, using the dozenal table, the factors 8 and 9 are "new", so these are effective factors.

Table 4B • Effective Factor Tables for Several Bases

	2	5
2	4	10
5	10	

Decimal

	2	3	4	6
2	4	6	8	10
3	6	9	10	
4	8	10		
6	10			

Dozenal

	2	3	4	6	8	ɣ
2	4	6	8	ɣ	ʃ	10
3	6	9	ɣ	ꭓ	10	
4	8	ɣ	ʃ	10		
6	ɣ	ꭓ	10			
8	ʃ	10				
ɣ	10					

Quadrovigesimal (Base 24)

	2	3	4	5	6	Ƨ	ɣ	ϱ	∂	30
2	4	6	8	Ƨ	ɣ	∂	ɤ	ɞ	ɋ	10
3	6	9	ɣ	ϱ	ꭓ	ɞ	ß	ɮ	10	
4	8	ɣ	ʃ	∂	ɤ	ɋ	Ɣ	10		
5	Ƨ	ϱ	∂	ϛ	ɞ	Ↄ	10			
6	ɣ	ꭓ	ɤ	ɞ	ß	10				
Ƨ	∂	ɞ	ɋ	Ↄ	10					
ɣ	ɤ	ß	Ɣ	10						
ϱ	ɞ	ɮ	10							
∂	ɋ	10								
ɞ	10									

Sexagesimal (Base 60)

Operation Class D: Totatives and f_n Involving Totatives

This class of operations includes all remaining problems. This ends up meaning all problems which involve a factor belonging to Factor Class D. This factor class includes all totatives except the totative {1}, and all the factors f_n for which only d_s is a divisor of r, with an m_n that is totative. In short, if both factors are either totative or involve a totative as the numerator of the simplified ratio f_n / r, and neither factor is 1, then the problem is governed by Operation Class D.

The class D operation renders the most difficult problems amenable by selecting one of the Factor Class D factors f_D, adding or subtracting a small integer n from them, splitting the problem into two problems. The problem, originally of the form $f_{D1} \cdot f_{D2}$, becomes $(f_{\text{modified}} \cdot f_{D2}) + (n \cdot f_{D2})$. The objective of this dissociation of one of the class D factors is to create one, preferably two class A or B factors, in order to minimize operations. In bases abundant with divisors, this is not so difficult. For factors greater than r / 2 or those less than r / 2 at an inconvenient distance from these factor classes, transferal to a class C factor is helpful. Base 60 class D operations are all the more facilitated by the unbroken wall of divisors covering 1 through 6; any totative can be brought down if they are within 6 units from any class C factor. Bases that are not so well entrained at times may not be able to transfer the problem to class C without dividing the number into three or four parts, thereby rendering the method far less efficacious. This is why the reciprocal divisor methods are suitable for highly divisible bases, and fail to support diprimes or squares that may be far smaller than some of these highly composite bases.

Like Operation Class C, Operation Class D is simply a preparation applied to a problem that commutes the problem to one that involves a higher grade operation class. Ideally, the application of Operation Class D shifts the problem to class A or B, reducing the number of steps in the solution of the problem. There are reasons why a class B operation is not available from a class D problem. Certain bases aren't as well-entrained as others; these tend to be the less highly divisible bases. Bases which are diprimes are initially adequate candidates for the reciprocal divisor method. As r becomes greater, diprimes offer relatively few avenues for Operation Class B, so many class D problems lie far from this tool.

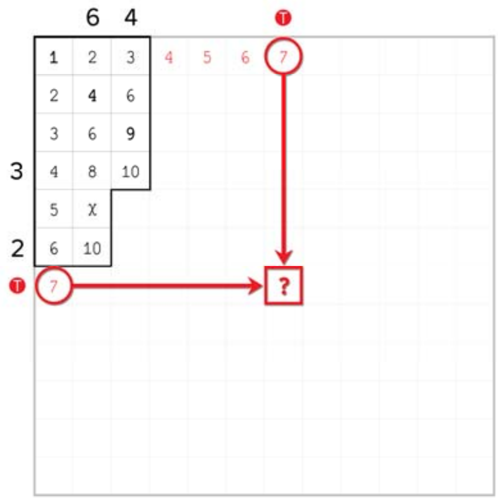


Figure 4D. *An example of a Class D operation, this example involving two totative factors.*

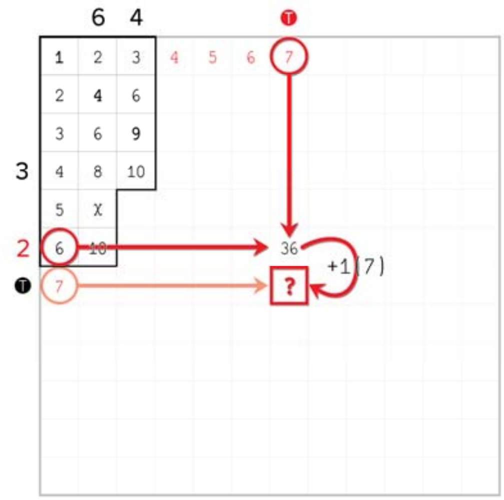


Figure 4E. *An illustration of the method by which a product that involves two totative factors can be commuted so that the user can employ the divisors of the base to solve the problem.*

Figure 4D illustrates an Operation Class C problem. Use the following procedure to determine Operation Class C products which lie beyond the abbreviated table.

1. Attempt to divide the problem $f_{D1} \cdot f_{D2}$ involving two class D factors into two problems using the associative property of addition. Split one factor f_D into a factor $f_{modified}$ of a higher factor class and a small integer n, preferably within Factor Class B. Several integers n_x may be required for bases r having relatively few divisors; this will require a term $(n_x \cdot f_{D2})$ for each n_x. The "split" can either be an addition of a positive or negative n. There may be no "right answer" as to which factor of a higher class is the one to use. The resultant problem is now of the form:

$$f_{D1} \cdot f_{D2} = (f_{modified} \cdot f_{D2}) + (n \cdot f_{D2})$$

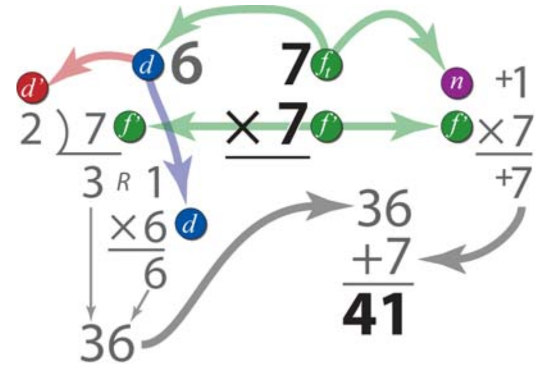
2. There is a special form of Operation Class D which applies to problems featuring a factor f_D which is of the form $(r - d)$. This problem can be divided into $(r \cdot f') - (d \cdot f') = p$. The first term is class A, and the second is class B.

$$(r - d) \cdot f' = (r \cdot f') - (d \cdot f') = p$$

3. Use Operation Classes A, B, or C as necessary to resolve both $(f_{modified} \cdot f_{D2})$ and $(n \cdot f_{D2})$.
4. Reunite the problem using addition of all the terms to determine the final product.

Operation Class D • Dozenal Examples

1. Multiply 7 by 7. The problem can be divided into (6 + 1) · 7 = (6 · 7) + (1 · 7). Solving the first term, a class B operation, the factor 6 is a divisor of 12, possessing a reciprocal divisor mate *d'* = 2. The co-factor *f'* = 7 is divided by *d'*, yielding 3 remainder 1. The integer 3 is carried. The remainder 1 is multiplied by 6, yielding 6. Thus Operation Class B yields the result 36;. The second term (1 · 7) has a simple solution, 7. The sum of 36; and 7 is 41;.



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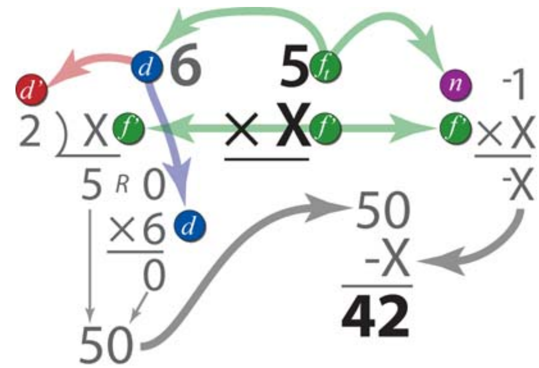
2. Multiply 5 by ↊. The problem can be divided into (6 - 1) · ↊ = (6 · ↊) + (-1 · ↊). Solving the first term, a class B operation, the factor 6 is a divisor of 12, possessing a reciprocal divisor mate *d'* = 2. The co-factor *f'* = ↊ is divided by *d'*, yielding 5, which is carried to the next place, generating a class B operation result of 50;. The second term (-1 · ↊) has a simple solution, -↊. The sum of 50; and -↊ is 42;.



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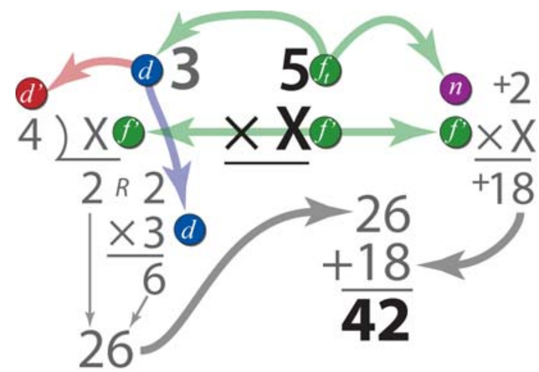
3. Multiply 5 by ↊. The problem can be alternatively divided into (3 + 2) · ↊ = (3 · ↊) + (2 · ↊). Solving the first term using a class B operation yields the result 26;. The second term (2 · ↊) yields 18;. The sum of 26; and 18; is 42;. Operation Class D problems may present several viable solution options. Some options are more efficacious than others.



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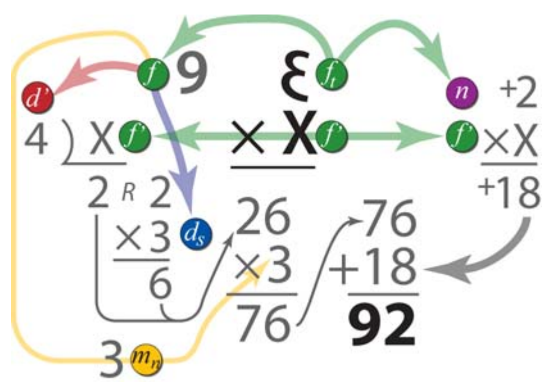
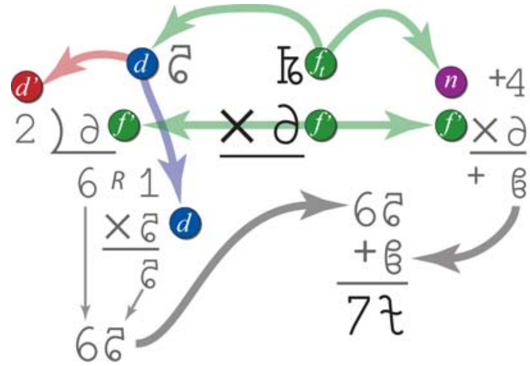
4. Multiply ↋ by ↊. The problem can be divided into (9 + 2) · ↊ = (9 · ↊) + (2 · ↊). Solving the first term using a class C operation yields a result of 76;. The second term (2 · ↊) yields 18;. The sum of 76; and 18; is 92;. This problem could have used ↋ = (3 + 4 + 4), [3 + 2(4)], etc. Another viable process would involve ↋ · ↊ = (↋ · 4) + (↋ · 6). The problem could have used the factors *r* and *d*: ↊ = (10 - 2).

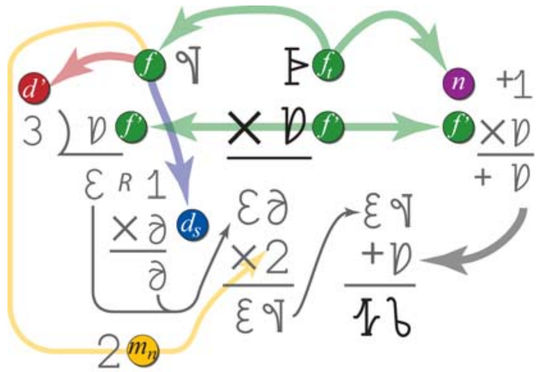


Operation Class D • Sexagesimal Examples

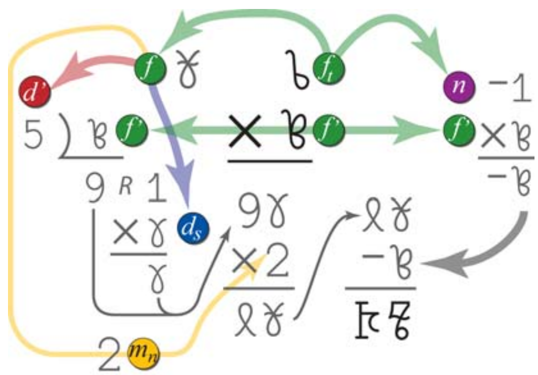
1. Multiply 34: by 13:. The problem can be divided into (30: + 04:) · 13: = (30: · 13:) + (04: · 13:). Solving the first term, a class B operation, the factor 30: is a divisor of 60, possessing a reciprocal divisor mate *d’* = 02:. The co-factor *f’* = 13: is divided by *d’*, yielding 06: remainder 01:. The integer 06: is carried. The remainder 01: is multiplied by 30:, yielding 30:. Thus Operation Class B yields the result 06;30:. The second term (04: · 13:) has a class A solution, 52:. The sum of 06;30: and 52: is 07;22:.



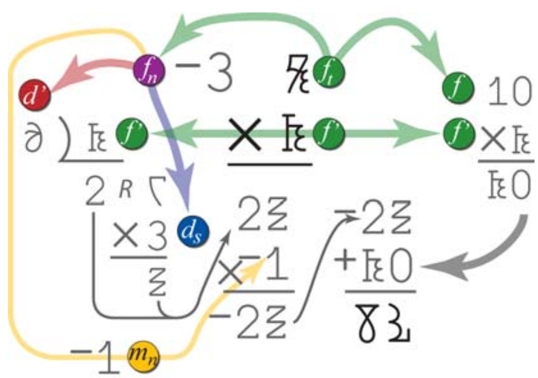
2. Multiply 41: by 43:. The problem can be divided into (40: + 01:) · 43: = (40: · 43:) + (43:). Solving the first term, a class C operation, results in the product 28;40:. Adding 43: to 28;40: yields the end product 29;23:. This problem could have used 41: = (45: - 04:), which would have necessitated a class B operation on the second term (04: · 43:). The use of 41: = (40: + 01:) is more efficient.



3. Multiply 23: by 46:. The problem can be divided into (24: - 01:) · 46: = (24: · 46:) + (-46:). Solving the first term, a class C operation, results in the product 18;24:. Adding 46: to -18;24: yields the end product 17;38:. This problem could have used 46: = (45: + 01:), 23: = (20: + 03:), 46: = (40: + 06:). Operation Class D processes involving *n* = 1 or -1 usually are most efficient.



4. Multiply 57: by 51:. The problem can be divided into (-03: + 01;00:) · 51: = (-03: · 51:) + (01;00:). Solving the first term, a class C operation with an m_n = -1, yields a result -02;33:. The second term is a simple Operation Class A problem yielding 51;00: as its result. The sum of these terms is 48;27:. The factor *r* can be a powerful ally in the solution of problems involving a factor *f* = (*r* – *d*).



Multiplication Processes

Thus far multiplication has been covered. There are four operation classes. Operation Class A resolves problems involving 1 or r or any product on the abbreviated table. Operation Class B involves a divisor of the base; the reciprocal divisor is leveraged to yield a product beyond the abbreviated table. Operation Class C simply extracts a multiplier which is a divisor or wholly composed of divisors so that Operation Class B can work. The multiplier is applied to the result of the class B operation to obtain the product of the class C operation. Operation Class D splits the problem into two separate problems, preferably both of class A or B, but C is also possible. The results of these two separate sub-problems are added, yielding the product of the class D operation. Figure 4F is a flow chart appropriate for multiplication.

Several items should be handy or present in mind when operating in base r. The purpose of constructing an abbreviated multiplication table is to facilitate its memorization. The abbreviated table is intended to be concise enough to be kept in mind. It is your judgment how large a table can be handled. Since the Reciprocal Divisor Method leverages the reciprocal divisor pairs of base r, it is essential to know these keenly. The set of totatives for base r is also handy but not essential. Knowing the list of effective factors decreases guesswork using class C. The class C and D operations normally have several avenues toward solution, so precise knowledge of totatives and effective factors is not necessary. Memorization is the key to computational fluency.

There are certain factors which, despite their status as Class C, may be easily resolved using a class D operation. An example of this is the sexagesimal digit 16:. This is very easily ($15f' + f'$). The class C interpretation uses $4(4f')$. Thus, Operation Classes C and D involve some measure of creativity. In a well-entrained base, this "wiggle room" is tremendously advantageous, because precise knowledge of which factors are totative or effective, etc. is unnecessary.

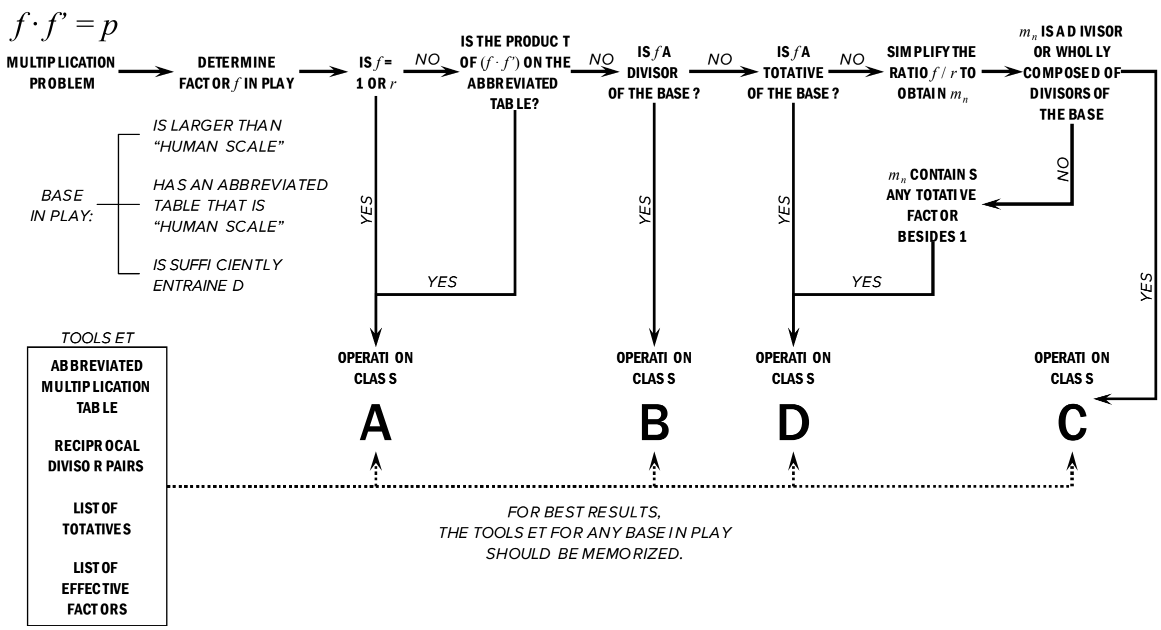


Figure 4F. *The multiplication process.*

Division Processes

Division using the Reciprocal Divisor Method involves the same processes used to maneuver the multiplication problem to one which can be solved using the reciprocal divisor pairs of a highly divisible integer. The problem-divisor (referred to as such to avoid confusion with a divisor of the base) is assessed, and the appropriate operation class is selected.

Division is the reversal of a multiplication process; thus the divisor (denominator) is a former factor f and the dividend (numerator) a former product p. The quotient to be found is another factor f'.

$$p / f = f'$$

The factor classes by which each digit of figures involved in a multiplication problem are classified do apply to the numerator and denominator of a division problem. Operations involving totatives are notably more difficult under division. This is because the process of dividing by totatives involves an empirical and iterative process. The following subsections describe division processes.

Operation Class A: Dividends within the Abbreviated Table

Like the multiplicative class A operation, the division operation of the same class is easiest. There are two principal applications of Operation Class A on division problems. The first involves problems where the problem-divisor f is either 1 or r or an integer power of r. This simply involves doing nothing, or shifting the radix point right 1 or more places in the dividend. The second application involves a dividend p and problem-divisor f which are both lesser than r. For dividends that are directly represented on the abbreviated table, the answer is given by finding the problem divisor on the table and locating the quotient on the other axis. It goes without saying that division by 0 is not defined. If the resonances of higher rank divisors of base r are known, these may also be leveraged: these resonances are not covered in this booklet.

Computation Process

In order to compute a Class A quotient, follow these steps:

1. Problems involving a problem-divisor or denominator of 0 are undefined.
2. For problems involving the problem divisor 1: the quotient f' is equal to the dividend p.

$$p / 1 = p$$

3. For problems involving the problem divisor or denominator which is an integer power of r: the quotient of any dividend f divided by r^n can be generated by shifting the digits of the dividend rightward n places.
4. The abbreviated multiplication table will feature composite dividends lesser than r for some problem divisors.
5. If the product is determined to lie outside the table, use another operation class.

Operation Class B: Effective Divisor Problem-Divisors

This operation class includes division problems where the problem divisor f_d is a member of Factor Class B, that is, a factor f_d which is a divisor d of the base r. The reciprocal divisor d' of the factor f_d can be used to yield any product p which lies outside the abbreviated multiplication table. Figure 4B illustrates a class B operation. Operation Class B is available to all integer bases r which have more than 2 divisors; that is, to all composite bases. Use the following procedure to determine products off the abbreviated table.

1. Verify that the entire denominator (problem divisor) f is an element in the set of divisors D_r of base r.

   In the problem $p / f = f'$, $f \in D_r$ is true; f is a divisor factor f_d

2. Begin with the highest rank digit of the dividend p, and iterate for each digit in the dividend p. The digits of the dividend p are referred to in the steps below as $\{ x_0, x_1, \ldots \}$ where x_0 is the most significant digit of the dividend p.

   a. Divide the digit x_0 by the problem-divisor f_d which is a divisor d of base r.

   b. Take the integer part of the result and retain this as a digit of the problem's quotient f'. These integer parts will accumulate, assembling the problem's quotient f' from most to least significant digit.

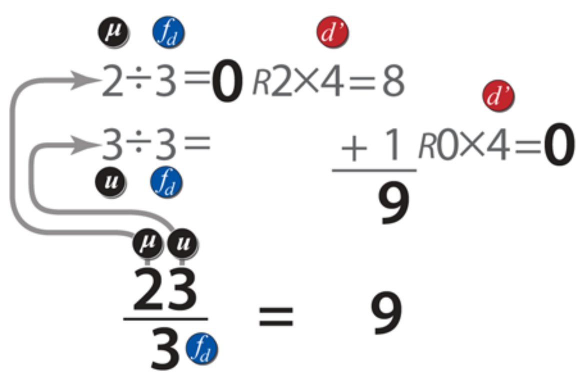
   c. Multiply the remainder of the result by the reciprocal divisor d' of f_d.

   d. If there is a digit in the dividend p which is lesser in significance (i.e. to the right of the digit just played), add the result of the remainder operation to the integer part of the next digit x_0's step 2a result. The sum of these will be retained as the next digit in the problem quotient f'.

3. Continue process until no further digits x_n are available. The accumulated integers are the digits of the problem's quotient f'.

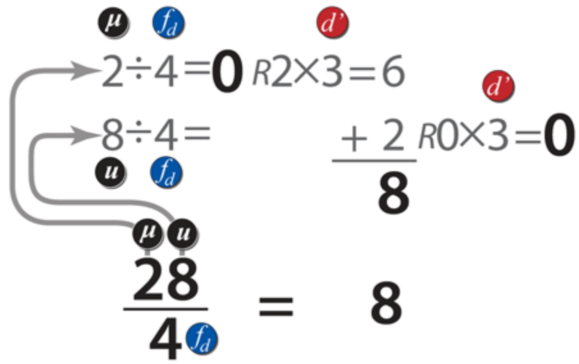
Operation Class B • Dozenal Examples

1. Divide 23; by 3. Starting with the highest-rank digit of 23;, divide the digit 2 by the problem-divisor f_d, which is 3. The result is 0 remainder 2. Multiply the remainder 2 by the reciprocal divisor 4 to yield 8. Using the digit 3 of 23;, divide by 3 again. The result is 1, which will be added to the result of the remainder operation from the last step to get 9. There is no remainder for this step. Collect the integer part from the initial step, and the sums of the integer parts and remainder operation results of the ensuing steps.



The answer to the problem is 09.0 or simply the number 9; 2 dozen 3 divided by 3 is 9.

2. Divide 28; by 4. Beginning with the highest-rank digit of 28;, divide the digit 2 by the problem-divisor f_d, which is 4. The result is 0 remainder 2. Multiply the remainder 2 by the reciprocal divisor 3 yielding 6. Using the digit 8 from 28;, divide by 4 again. The result is 2, with no remainder. Add the 6 from the previous step to the integer 2 from this step to obtain 8. Collect the results of all steps to obtain the quotient: 2 dozen 8 divided by 4 yields a quotient of 8



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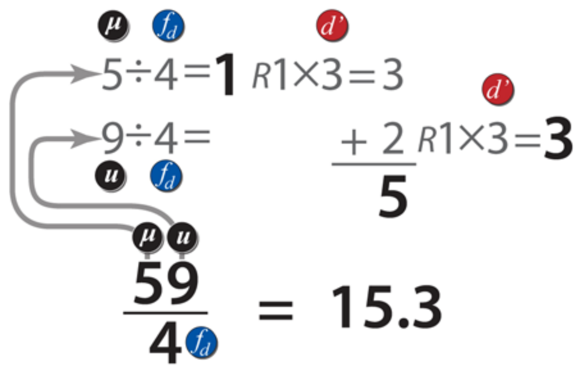
3. Divide 5 dozen 9 by 4. Dividing the first digit of the dividend by the problem divisor 4 yields 1 remainder 1. The remainder 1 times the reciprocal divisor 3 yields 3. Dividing the next digit 9 by 4 yields 2 remainder 1. Adding the remainder operation result from the last step, 3, to this integer part 2 sums to 5. The remainder 1 times the reciprocal divisor 3 yields 3. Thus 5 dozen 9 divided by 4 equals 1 dozen 5 and one quarter.



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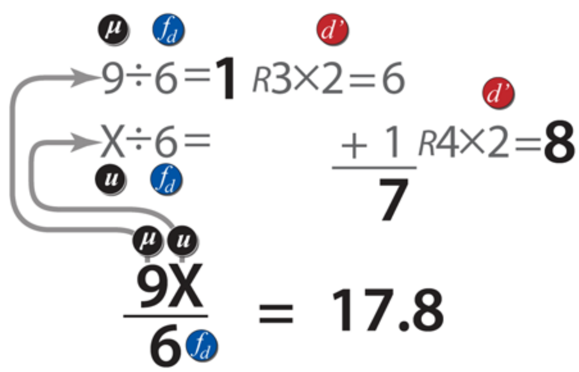
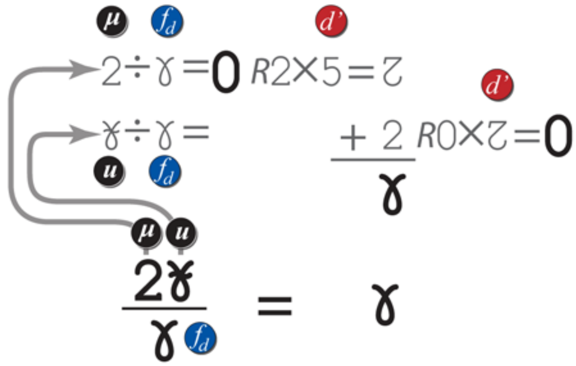
4. Divide 9 dozen ten by 6. Nine divided by 6 yields 1 remainder 3. Multiplying the remainder by the reciprocal divisor 2 yields 6. The next digit in the dividend, ten, divided by 6 yields 1 remainder 4. Adding 6 and 1 totals 7. The remainder 4 times the reciprocal divisor 2 yields 8. Thus, nine dozen ten divided by six equals one dozen seven and two thirds.

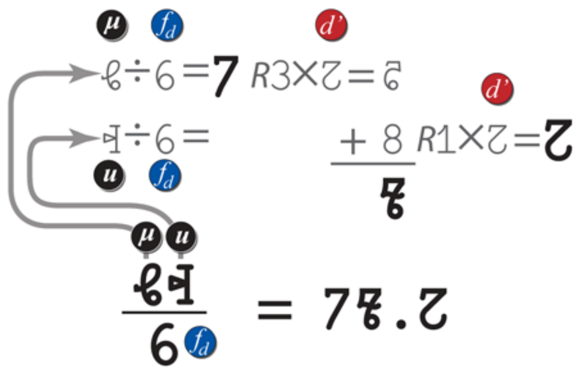


Operation Class B • Sexagesimal Examples

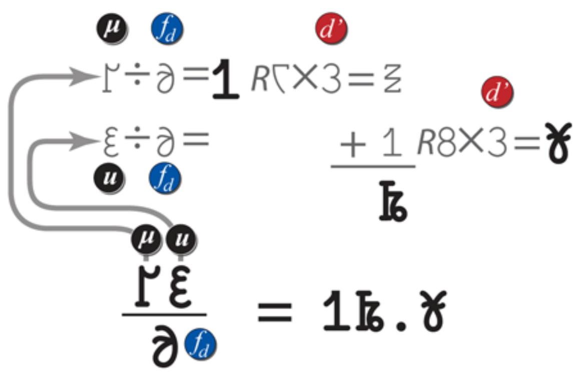
1. Divide 02;24: by 12:. Divide the digits of the dividend by the problem-divisor 12:, then multiply the remainders of these operations by the reciprocal divisor 05:. Sum the products of the remainder operation and the integer part of the digit greater than the current digit in play. Collect the sums to build the quotient 12;. (144 / 12 = 12).



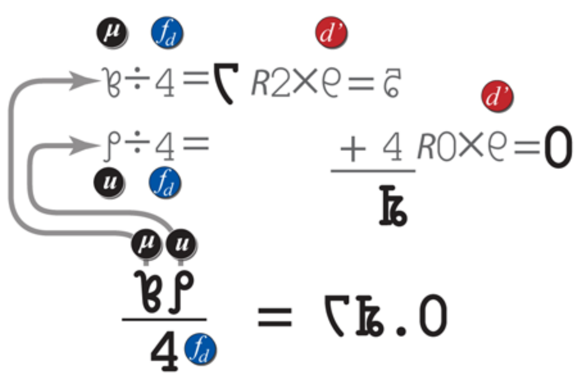
2. Divide 45;49: by 06:. Divide the digits of the dividend by the problem-divisor 06:, then multiply the remainders of these operations by the reciprocal divisor 10:. Sum the products of the remainder operation and the integer part of the digit greater than the current digit in play. Collect the sums to build the quotient 07;38:10;. (2,749 / 6 = 458+1/6).



3. Divide 31;28: by 20:. Divide the digits of the dividend by the problem-divisor 20:, then multiply the remainders of these operations by the reciprocal divisor 03:. Sum the products of the remainder operation and the integer part of the digit greater than the current digit in play. Collect the sums to build the quotient 01;34:24;. (1888 / 20 = 94.4).



3. Divide 46;16: by 04:. Divide the digits of the dividend by the problem-divisor 04:, then multiply the remainders of these operations by the reciprocal divisor 15:. Sum the products of the remainder operation and the integer part of the digit greater than the current digit in play. Collect the sums to build the quotient 11;34:. (2776 / 4 = 694).



Operation Class C: Non Totative Non Divisor Problem-Divisors

The class C operation involves division problems where the problem-divisor f_n is member of Factor Class C. This factor class includes all those factors f_n which are neither totative nor a divisor of the base r which are products of a “significant” divisor d_s and a multiplier m_n that is also a divisor or composed itself entirely of divisors of base r.

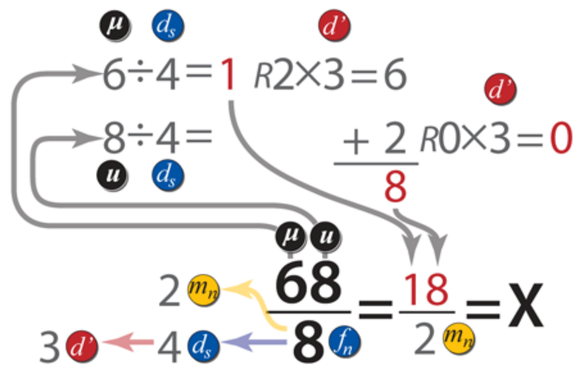
Solution of the problem involving the effective f_n begins by extracting m_n from f_n to yield d_s. Once d_s is known, we can use the reciprocal divisor pair { d_s , d_s' } in a class B operation as described above. The factor m_n must be recorded and applied at the end of the Operation Class B process. When the m_n is itself a divisor, the application of m_n becomes simply a second Operation Class B process. In cases where m_n is a composite of divisors, the m_n may be broken down in an Operation Class C.

Operation Class C can be regarded as an extraction of the m_n multiplier so that Operation Class B can be applied to the co-factor f'. This class of operation is available to all bases r which have 5 or more divisors and many which have 4 or more divisors. Bases which have a diverse set of prime factors may feature many avenues open to Operation Class C. Prime bases can not use Operation Class C. Figure 4C illustrates an Operation Class C problem for multiplication: in division, the product “?” is known, but one or the other factors is unknown. Use the following procedure given a dividend p beyond the abbreviated table, and a class C factor as problem-divisor f.

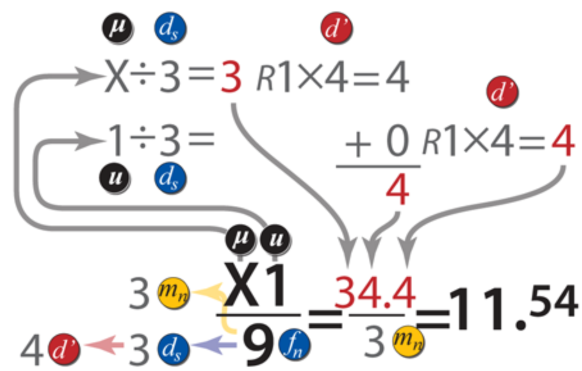
1. Verify that the entire denominator (problem divisor) f is neither a totative t nor divisor d of base r. Further, simplify the ratio f / r to obtain m_n / d_s'. The numerator m_n must be fully composed of divisors d of base r, thus no totative t is a part of m_n. Since d_s' is known, the related divisor d_s can be identified.
2. For each digit of the dividend p, beginning with the highest rank digit μ, divide μ by the related divisor d_s to obtain an integer part and a remainder.
3. Multiply each remainder by the reciprocal divisor d_s', and add the product to the integer part from the digit immediately to the left or of immediately greater rank. Each of these sums will serve as a digit in the subquotient.
4. Collect the sums to build the subquotient, and divide the subquotient by m_n. Division by m_n may necessitate another class C or higher (A or B) operation. This is the final quotient of the problem

Operation Class C • Dozenal Examples

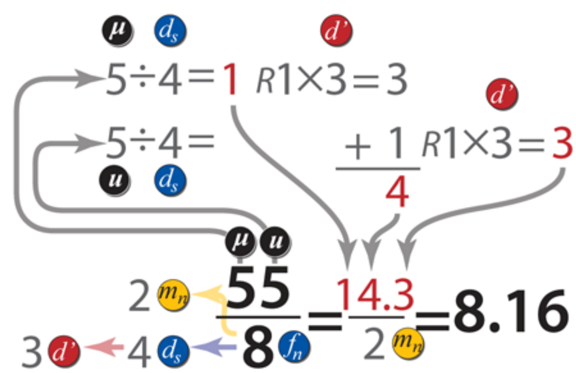
1. Divide 68; by 8. The simplified ratio 8/12 supplies m_n = 2, d_s' = 3, and d_s = 4. Divide the digits of the dividend by the related divisor 4. Multiply the remainders of these quotients by the reciprocal divisor 3. Collect the results from each step to assemble the sub-quotient 18;. Divide the subquotient by the m_n 2 to yield the final quotient, ten (dek). (Six dozen eight is decimal 80).



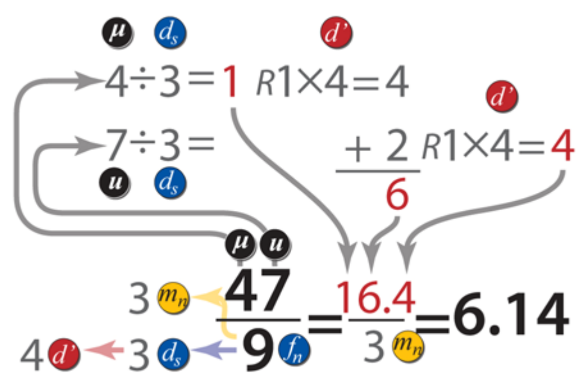
2. Divide ten dozen one by nine. The simplified ratio 9/12 supplies m_n = 3, d_s' = 4, and d_s = 3. Divide the digits of the dividend by the related divisor 3. Multiply the remainders of these quotients by the reciprocal divisor 4. Collect the digits of the subquotient to obtain 34;3. Reapply m_n by dividing 34;3 by 3. The quotient 11;54 is the result of dividing ten dozen one by nine. (121 / 9 = 13+4/9).



3. Divide 55; by 8. Dividing the digits of the dividend by the related divisor 4, then multiplying the remainders of these operations by the reciprocal divisor 3 yields the subquotient 14;3. Dividing the subquotient 14;3 by the m_n, 2 yields 8;16. Thus five dozen five divided by 8 yields the quotient 8;16 or eight and one eighth. (Decimal 65 / 8 = 8.125).

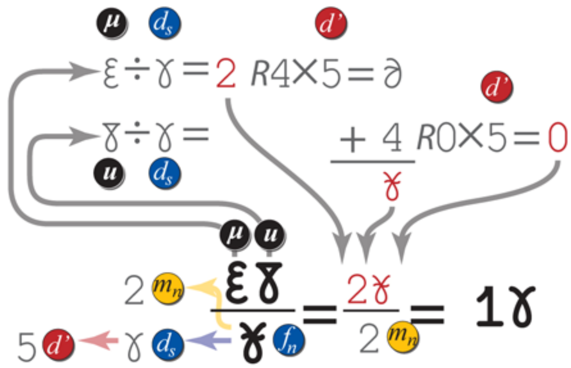


4. Divide 47; by 9. Dividing the digits of the dividend by the related divisor 3, then multiplying the remainders of these operations by the reciprocal divisor 4 yields the subquotient 16;4. Dividing the subquotient 16;4 by the m_n, 3 yields 6;14. Thus four dozen seven divided by nine yields the quotient 6;14 or six and one ninth. (Decimal 55 / 9 = 6+1/9).

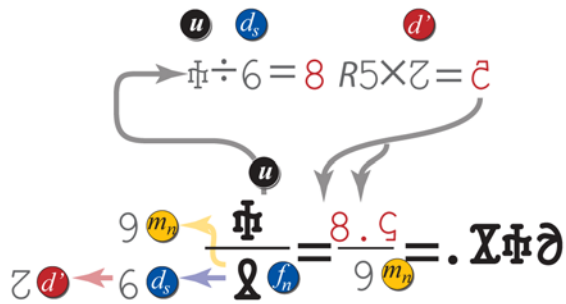


Operation Class C • Sexagesimal Examples

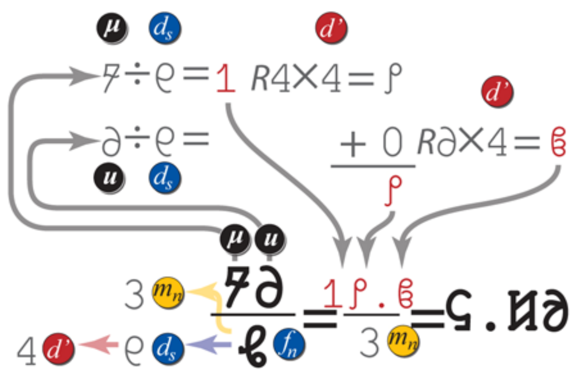
1. Divide 28;48: by 24:. The simplified ratio 24/60 supplies m_n = 02:, d_s' = 05:, and d_s = 12:. Divide the digits of the dividend by the related divisor 12:. Multiply the remainders of these quotients by the reciprocal divisor 05:. Collect the digits of the subquotient to obtain 02;24:. Reapply m_n by dividing 02;24: by 02:. The quotient is 01;12:. (1728 / 24 = 72).



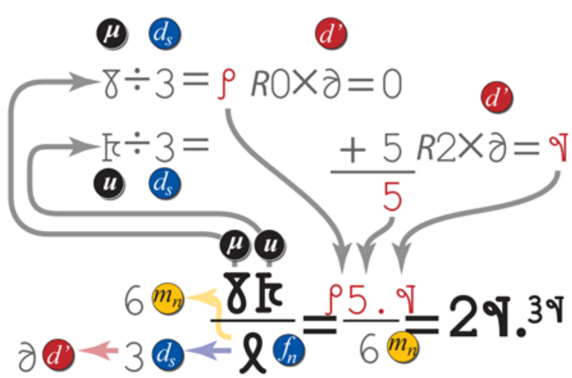
2. Divide 53: by 54:. The simplified ratio 54/60 supplies m_n = 09:, d_s' = 10:, and d_s = 06:. Divide the digits of the dividend by the related divisor 06:. Multiply the remainders of these quotients by the reciprocal divisor 10:. Collect the digits of the subquotient to obtain 08:25. Reapply m_n by dividing 08:25; by 09:. The quotient is 00:58;53;20;. (53 / 54 = 0.9814814…).



3. Divide 19;13: by 45:. The simplified ratio 45/60 supplies m_n = 03:, d_s' = 04:, and d_s = 15:. Divide the digits of the dividend by the related divisor 15:. Multiply the remainders of these quotients by the reciprocal divisor 04:. Collect the digits of the subquotient to obtain 01;16:52;. Reapply m_n by dividing 01;16:52; by 03:. The quotient is 25:37;20;. (1153 / 45 = 25.6222…).



4. Divide 48;17: by 18:. The simplified ratio 18/60 supplies m_n = 06:, d_s' = 20:, and d_s = 03:. Divide the digits of the dividend by the related divisor 03:. Multiply the remainders of these quotients by the reciprocal divisor 20:. Collect the digits of the subquotient to obtain 16;05:40;. Reapply m_n by dividing 16;05:40; by 06:. The quotient is 02;40:56;40;. (2897 / 18 = 160.9444…).



Operation Class D: Totative Problem-Divisors

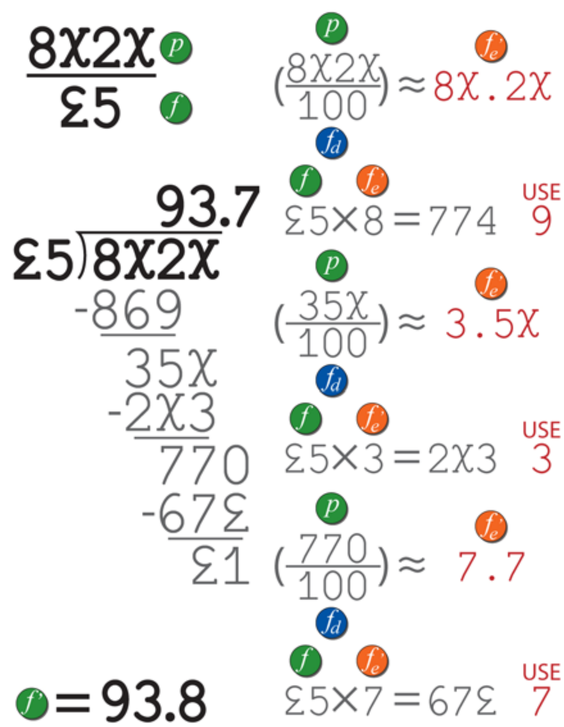
Division problems which involve a problem-divisor that is a member of Factor Class D may use the class D operation. This factor class includes all totatives except the totative {1}, and all the factors f_n for which only d_s is a divisor of r, with an m_n that is totative. In short, if both factors are either totative or involve a totative as the numerator of the simplified ratio f_n / r, and neither factor is 1, then the problem is governed by Operation Class D.

The Operation Class D for division is similar to decimal long division. It employs an empirical and iterative process to produce the quotient. The class D operation requires an estimation of which multiple of the problem divisor might be subtracted from a set of digits of the dividend. Because the abbreviated multiplication table does not contain every product for the totatives by definition, this estimation process is more involved. It is noticeably more tedious than the other division operation methods described in this booklet.

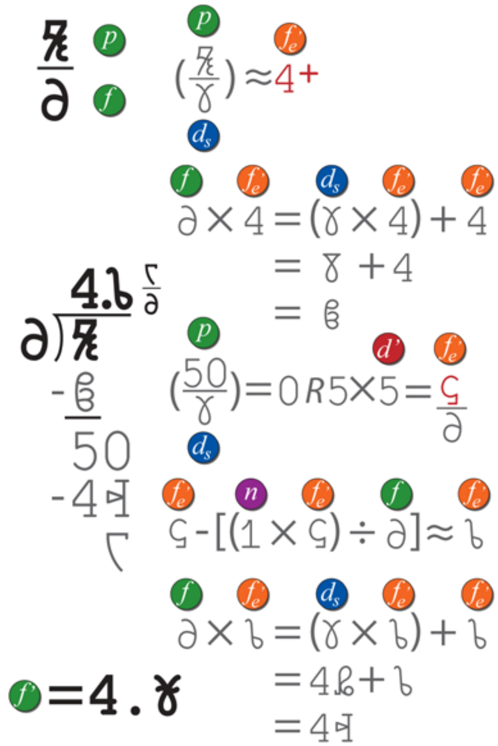
Figure 4G summarizes the process of division under the Reciprocal Divisor Method.

Operation Class D • Examples

1. Divide dozenal 8Χ2Χ; by Ɛ5; to 3 significant digits. Dividing the dividend 8Χ2Χ; by a surrogate divisor 100; obtains an estimated quotient of 8Χ;2Χ. A trial multiplication of the problem divisor Ɛ5; and the first digit of this surrogate, 8; , yields 774;. When 774; is subtracted from 8Χ2; it is too low. Using 9; instead of 8; proves better. The difference, 35Χ;, divided by the surrogate suggests 3; as the next quotient digit. This proves correct. The difference, 770;, divided by the surrogate suggests 7; as the final digit to compute. The difference, Ɛ1; is relatively high and certainly more than half of the problem divisor, so the quotient is rounded up to nine dozen three and two thirds.



2. Divide sexagesimal 57: by 13: to two significant digits. Division of the dividend 57: by a surrogate divisor 12: suggests 04: as a possible quotient digit. The multiplication of 13: and 04: is a class A operation. The difference of 50: divided by the surrogate 12: suggests 25:, but multiplication of 13: by 25: proves too large. Subtracting 50: from the product of 13: and 25:, which is 05;25: suggests the number 23: is the correct quotient digit. A remainder of 11: suggests the two-place figure be rounded up to 04;24:.



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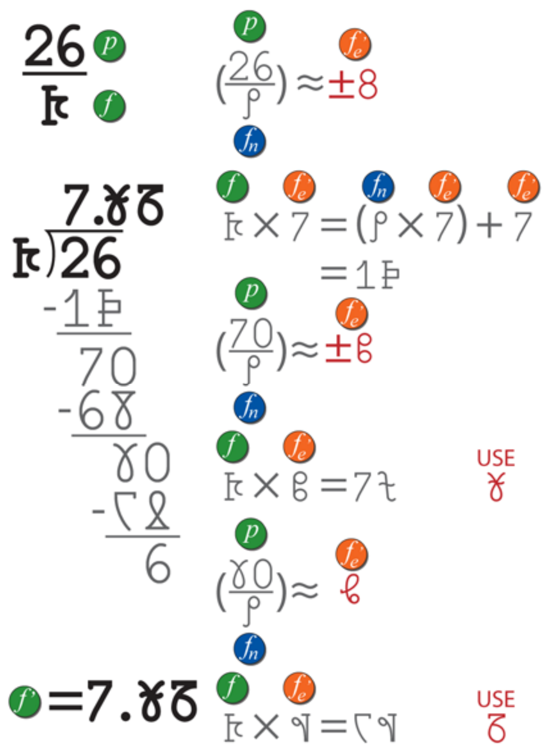
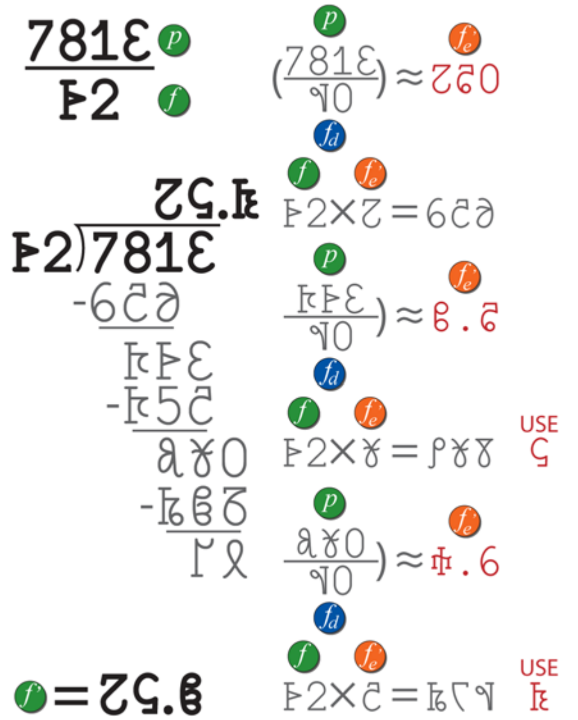
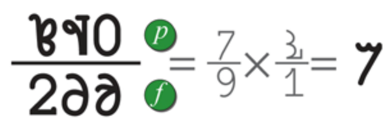
3. Divide sexagesimal 02;06: by 17:. Division of the dividend 02;06: by a non divisor non totative 16: suggests 08:. This proves too high, so 07: is used as the quotient digit, and proves correct. The difference, 07;00:, is divided by the surrogate 16: to yield 26:. This figure is likewise too large, and 24; is indicated. The difference 12;00: divided by the surrogate 16: suggests 45:. This is too large, so 42: is used. The three-place answer is 07:24;42;



4. Divide sexagesimal 07;08;01;14: by 41;02:. Applying the surrogate 40;00: to the dividend suggests 10: as the first digit of the quotient. This proves correct. The difference 17;41;14: divided by the surrogate suggests 26:, which proves too large. The figure 25: proves to be the next digit in the quotient. Division of the difference 35;24;00: by the surrogate suggests that the last digit is 53:. This proves too large. The digit that proves to be correct is 51:. Thus the quotient is 10;25:52; via rounding up.



5. Divide 46;40;00: by 02;13;20:. This is perhaps a trick problem. If the third rank divisors are familiar, the fact that 02;13;20: is the reciprocal divisor to 27;, and the figure 46;40;00: is 21; / 27; may be more evident. This particular problem can be resolved entirely with fraction simplification.



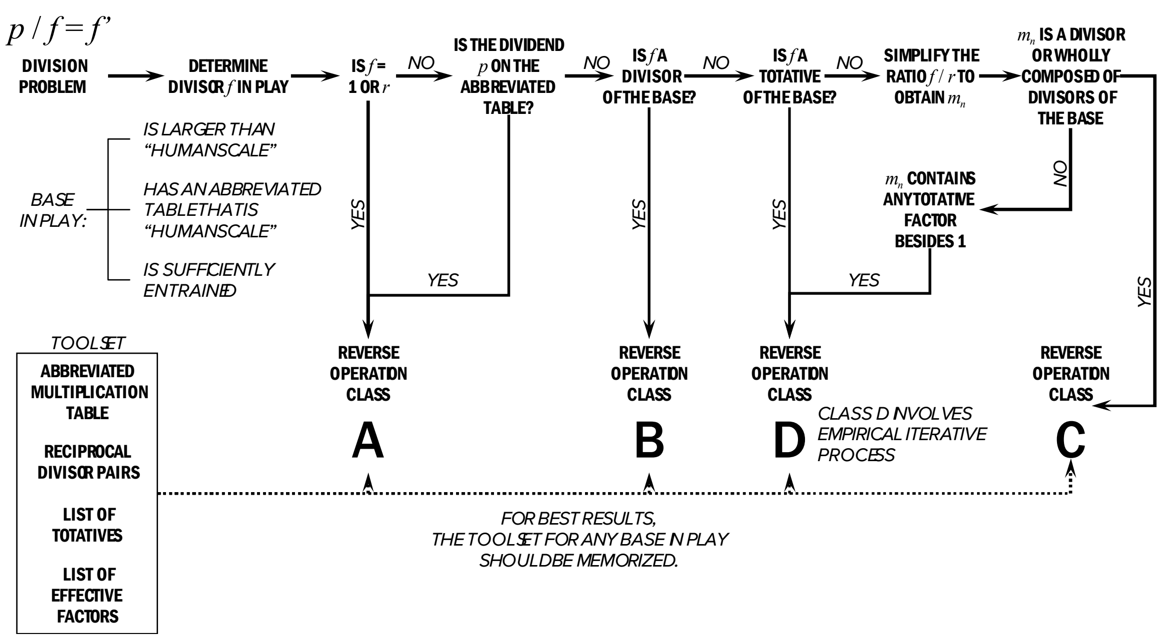


Figure 4G. *The division process.*

Range of Effectiveness for the RDM

Let's examine the range of bases for which the Reciprocal Divisor Method might be a useful toolset.

Range Limited by Human Ability

The two principal factors which govern the usefulness of a base *r* for general human computation are the properties of integers and the capacity of the human mind. These two principal factors thus generate a spectrum of integral bases which run from the simple bases, which are generally less than about 6 or 7. This is the range where human perception of the quantity of objects present in a group nearly never fails to be accurate. The "human scale" bases include integers larger than the simple bases but smaller than a size where the human memory has difficulty learning and practicing the full multiplication table within a period of training time comparable to today's decimal training time. These integers perhaps range between approximately 6 or 7 through approximately 12 through 16, possibly including the two integers 18 and 20, which possess the same number of divisors as twelve, and therefore should be aided by rhythms in their tables. Bases greater than the human scale bases, running up to 36 are the inner large bases, which are representable using the numerals and the letters of the Latin alphabet. Bases greater than the inner large bases to around 60 or 64 comprise the middle large bases. All bases greater than these are simply the large bases. This is neither a scientifically studied nor defined scale.

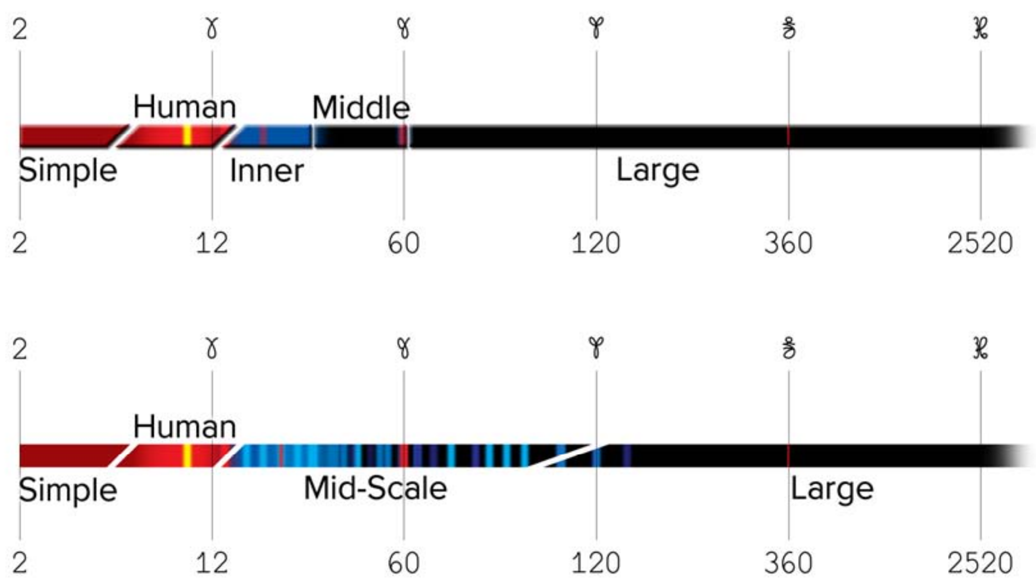


Figure 4H. *Perhaps the RDM facilitates "exploration" of certain "mid-scale" bases. The most significant of these "mid-scale" bases is sexagesimal.*

Range Determined by Mathematical Properties of Integers

The RDM yields useful abbreviations for composite bases that range between hexadecimal through sexagesimal. Above sexagesimal, the RDM yields useful abbreviations for bases having more than 5 divisors perhaps through base 120. The size of the abbreviated multiplication table for base 120 includes 246 products of unique factor combinations, with the "crossing" table including 301 elements. Thus the abbreviated table for r = 120 is comparable to the full traditional hexadecimal multiplication table, which is perhaps the upper limit of the "human scale bases". No empirical testing for these ranges has been conducted to date, to the knowledge of the author.

Entrainment

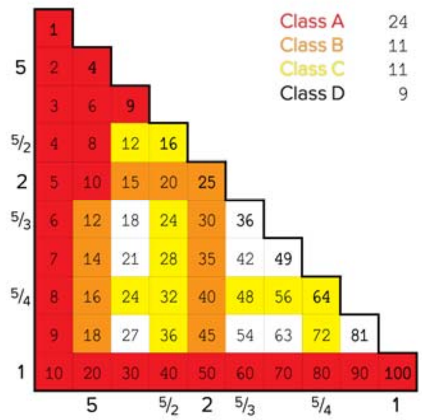
The operation classes function fully for all bases which have more than 5 divisors. Such bases can be called "entrained". This means that the following operations are possible:

- The Class A operations among the factors on the abbreviated multiplication table and those which involve 1 and r,
- The Class B operations which extend the factors f_d,
- The Class C operations between any factor f and a factor f_d, via commutation of m which is a divisor d of the base r, and
- The Class D technique that commutes one of two totative or ineffective nontotative nondivisor factors to a factor f_d so Class B can be applied.

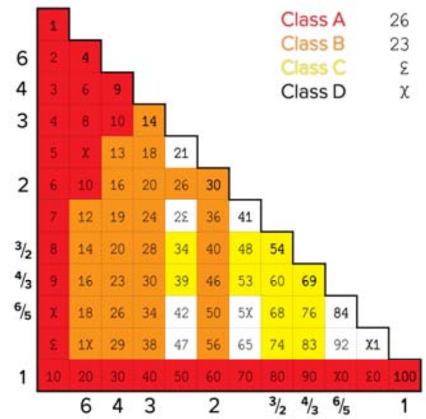
Table 4C • Effectiveness of Operation Classes versus Number of Divisors of Base r

	Class A	Class B	Class C	Class D
2 divisors	●			
3 divisors	●	●		●
4 divisors	●	●	limited	●
> 5 divisors	●	●	●	●

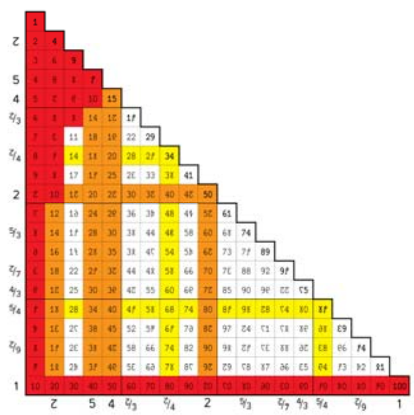
All the operations classes function for these entrained bases because two or more reciprocal divisor pairs can cross-interact. For instance, the dozenal nontotative nondivisor factors f_n include two factors {8, 9} which are multiples m of the divisors {4, 3} that are themselves divisors of twelve {2, 3}. The dozenal nontotative nondivisor factor 10 cannot technically employ the Class C because the multiplier m is 5, which is a totative of twelve. At the scale of the dozenal system, the fact that 10 is the product of a divisor and a totative is perhaps not insurmountable. But at scales on the par of bases 60, 72, or 120, using a factor that is the product of a divisor and a totative is prohibitively impractical. Figure 4J illustrates the entrainment of decimal and dozenal, which are too small for RDM to be effective. The range of integers between 20 and 25 better illustrate entrainment. The highly entrained sexagesimal table appears at the end of the study.



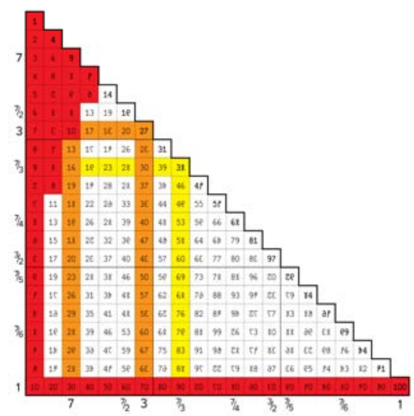
Base 10, D_r = {1, 2, 5, 10}



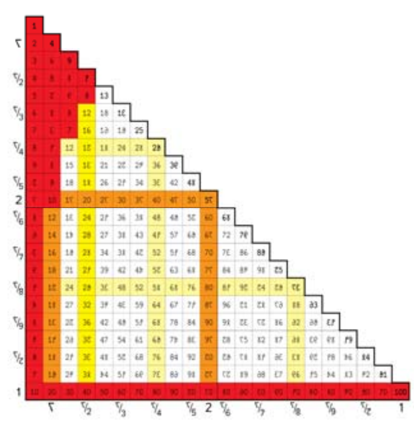
Base 12, D_r = {1, 2, 3, 4, 6, 12}



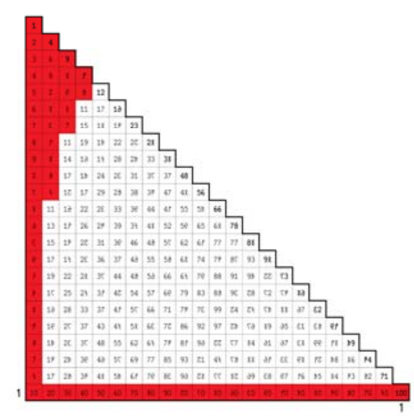
Base 20, D_r *= {1, 2, 4, 5, 10, 20}*



Base 21, D_r *= {1,3, 7, 21}*

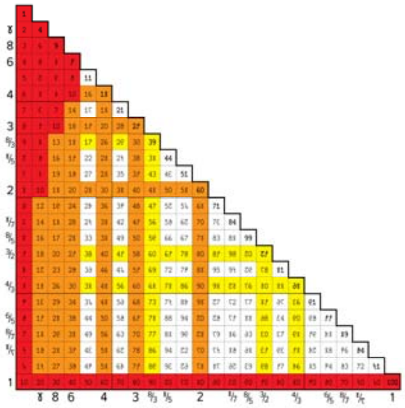


Base 22, D_r *= {1, 2, 11, 22}*

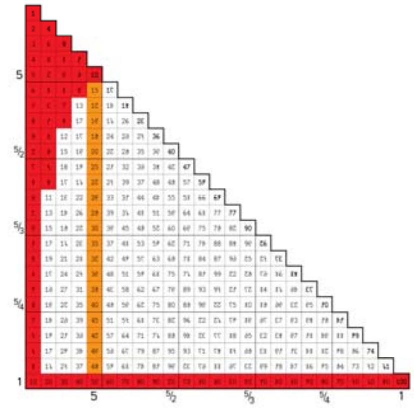


Base 23, D_r *= {1, 23}*

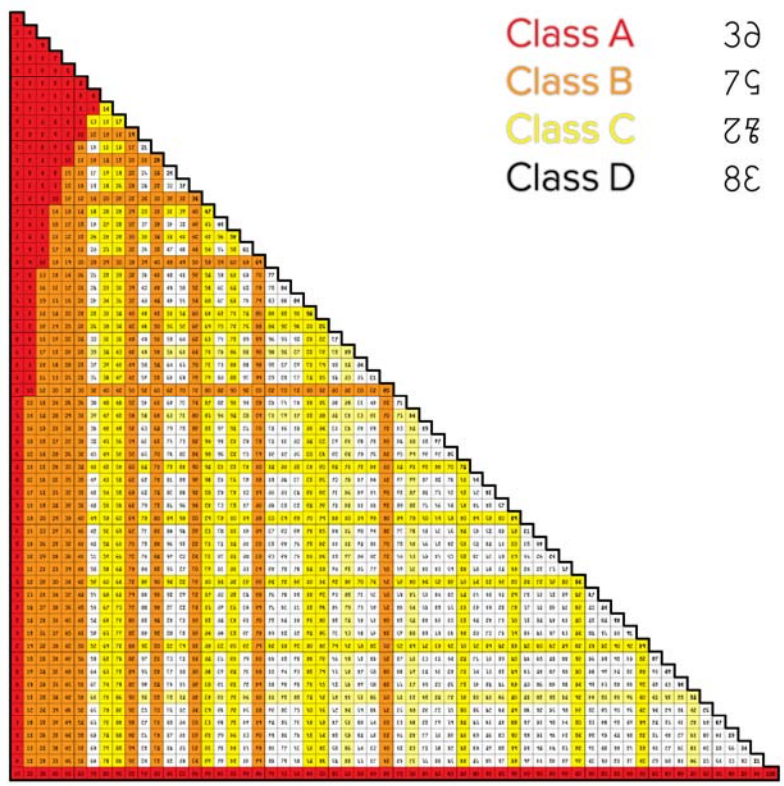
Figure 4J. *An illustration of the entrainment of several bases.*



Base 24, D$_r$ = *{1, 2, 3, 4, 6, 8, 12, 24}*



Base 25, D$_r$ = *{1, 5, 25}*



Base 60, D$_r$ = *{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}*

Figure 4J continued.

Conclusion

This booklet illustrates how the reciprocal divisor pairs of bases with 3 or more divisors can be used to abbreviate a large multiplication table, thereby facilitating computation in that base. The patterns present in the multiplication table of every integer base indicate that abbreviation to the first period is possible, dramatically reducing the quantity of products that must be memorized. The reciprocal divisors can be used in multiplication and division to compute products beyond the abbreviated table, or to find quotients that involve elements not found in the table. The computations are more circuitous than direct computation for "human scale" or simple bases whose multiplication tables are easily memorized. However, the methods do render computation possible in bases between "human scale" and the range where the abbreviated table rivals the size of the largest "human scale" bases.

The three types of factors found in a multiplication table have been described and classified into four "factor classes". Four kinds of "operation classes" have been developed to handle each of these classes of factors. The first class handles problems where one factor or dividend or problem-divisor follows simple rules, or is represented on the abbreviated table. The second class leverages the reciprocal divisor of a factor or problem-divisor which is also a divisor of the base. Two of the four classes merely adjust the problem so that reciprocal divisors can be used to solve the problem. The third splits the problem into one which can be resolved by a reciprocal divisor (i.e. class B), and a multiplier operation. The fourth splits the problem into the sum of a class B operation and a smaller operation. Examples of the multiplication and division under bases twelve and sixty are given.

The Reciprocal Divisor Method functions more efficiently for bases with more than 5 divisors. Bases which possess relatively many divisors are said to have "entrained" multiplication tables. There are many avenues toward solution under such bases.

The intent of this booklet is to share "climbing equipment and techniques" with others who enjoy "climbing the mountains" of the higher bases. The equipment and techniques are perhaps imperfect, but they are complete enough and do work, allowing our safe ascent. There are those that would debate the equipment and techniques, but in my mind, the climb and the views are far more important than these. If enough of us ascend these heights, an "industry" might arise, and provide proper standards. This is perhaps only the beginning, and there is a lot to see up here.

I believe that dozenal is the optimum base for general human computation. I do recognize that sexagesimal and the other superabundant bases 120, 360, 2520, etc. offer even more powerful opportunities to resolve nature's order.

This booklet and a second describing the Argam transdecimal numerals represent the culmination of twenty five years of preoccupation with transdecimal number bases. The effort that has been invested in this booklet is an attempt to ascend to these summits and survey the universe. The patterns from these heights are as beautiful as the Colorado vistas. I do invite fellow climbers to see the land and sky from these mountains, even though we may not inhabit them. We can still enjoy the climb and the view. Perhaps someone might even build an observatory up here. Happy climbing!

References

The following resources contributed to this work. It is difficult to completely pinpoint which parts of these references led to a given idea in this booklet in many cases. This is because this work is the result of leisurely exploration over the course of twenty years.

Initially, the only resource for alternative bases was the *CRC Standard Mathematical Tables*, 28th Edition, Editor William H. Beyer, Ph.D., Boca Raton, Florida, 1987. Octal and hexadecimal were extensively covered. Of interest are the basic Octal Multiplication Table (p. 69), and Hexadecimal Multiplication Table on page 85. The *CRC Standard Mathematical Tables*, 28th Edition also serves as a resource for a list of primes ("Primes", pp. 86-93), factorization of integers (Factors and Primes, pp. 94-104), and the divisor and totative properties of integers (Totient Function, pp. 105-109). The *Tables* were my only exposure to totatives; it took a few years to realize their significance.

My interest in dozenal arose in middle school, around 1981-1982, long before I had contact with the Dozenal Societies of Great Britain and America. I was interested in cryptography, borrowing Greek and Russian letters to encrypt notes. Later this led to learning Russian and inventing languages.

The Argam, covered in a separate work, was devised between 1987 and 2007. This system of numeral symbols and names developed organically, beginning with the dozenal transdecimal numerals in 1982. It was expanded to cover the hexadecimal system that year. In 1987, it grew to represent vigesimal. The symbols for 10 through 19 are as they were then, but the names were different. In 1992, the set was expanded to reach bases 60, then 100. The first 60 symbols are relatively as they were in 1992. The book *Number Words and Number Symbols* by Karl Menninger, published in 1992 by Dover, Mineola, New York (Original, 1969) was a breathtaking resource into the symbology and nomenclature of numbers. Some of the current Argam names (ismarragam) such as "score", "shock", and "hund" derive from his work. The notion of "rank" also derives from Menninger. In 1996 I attempted to replace the dozenal notation of days which I had used since the early eighties with a sexagesimal notation. This did not take hold, because I could not wield base 60; it lay out of reach, like the summit of a mountain. I could visit the mountain but not live up there.

While searching the internet for anything dozenal or related to alternate bases, I stumbled upon two momentous resources in 2002. The first is William Lauritzen's *Versatile Numbers Versatile Economics*, [Electronic version] from http://www.earth360.com/math-versatile.html. This work introduced the concept of highly composite, superabundant numbers. This is a seminal work for those interested in how to determine the optimum base for general human computation. This led to The On-Line Encyclopedia of Integer Sequences, a handy reference on the internet. Specifically, the "Colossally abundant numbers" (A004490), retrieved from http://www.research.att.com/~njas/sequences/A004490, and "Superior highly composite numbers" (A002201), retrieved from http://www.research.att.com/~njas/sequences/ A002201.

In 2002, through the same searches, I exchanged electronic mail with Mr. Shaun Ferguson of the Dozenal Society of Great Britain. This led to visiting the site and plenty of conversation. The Argam symbol for ten, ↊, happens to be the same symbol the British society employs. Shaun introduced me to the work of the Dozenal Societies of America and Great Britain. I became a member of the Dozenal Society of America in 2003, and was then exposed to a vast amount of work in base twelve. Chief among these are listed below:

Manual of the Dozen System, Duodecimal Society of America, Inc., Garden City, New York, 1960. General reference on the properties of the dozenal system, including symbology and nomenclature, systems of measure.

F. Emerson Andrews, *New Numbers*, 2nd Edition, Essential Books, New York, 1944. The symbols used for ten (dek, ☓) and eleven (el, Ɛ) were devised by Dwiggins. This booklet employs these symbols in its dozenal representations.

F. Emerson Andrews, *An Excursion in Numbers*, [Electronic version], Retrieved from http://www.dozenal.org/files/an%20excursion%20in %20numbers.pdf.

The Dozenal Society of Great Britain. Shaun Ferguson. “Words” [Electronic Version]. Retrieved June 2005 from http://www.dozenalsociety.org.uk/roses/shauns.html.

Through invitation by Shaun, I visited the Dozens Online internet forum on 11 April 2006, with the username “icarus”. This board is found at http://s13.invisionfree.com/DozensOnline/index.php. The discussion on this board, as well as with Shaun Ferguson, guided the consideration of other bases and inspired my drive for a “case for dozenal.” The contributions by Bryan Parry, “ruthe”, “dan”, “Listerine”/”leopold plumtree”, and especially “uaxactum” are particularly relevant. Any discussion of octal or hexadecimal was also of high interest, driving me to question the dozenal system, then reinforce the case for dozenal. Sites such as Intuitor, particularly the “Hex Headquarters”, by Tom Rogers, et al., from http://www.intuitor.com/hex/ also played an important role in the creation of a case for dozenal.

I visited Long Island for the Dozenal Society of America’s general meeting in October 2006, meeting Professors Gene Zirkel, Jay Schiffman, Alice Berridge, and Christina D’Aiello. At this meeting I was given most of the back issues of the *Dozenal Bulletin*. This has been not only a resource but an inspiration. The following articles were helpful in the formulation of this booklet:

The Duodecimal Bulletin, Volume 3, Number 4, December 1947; p. 19, Lecture Chart, Paul Van Buskirk. This is a good concise presentation board which perhaps is the forerunner of some of this work.

The Duodecimal Bulletin, Volume 3, Number 4, December 1947; p. 5, “What is the Best Base?” in “A Plea for the Duodecimal System”, H. G. G. Robertson. A short description of the “ingredients” of a good base, which serves as part of the case for dozenal.

The Duodecimal Bulletin, Volume 1, Number 1, January – March 1945; p. ☓, On Multiplication Tables, Kingsland Camp. Mr. Camp analyzed the dozenal multiplication table; it was this analysis that led to my own analyses of several multiplication tables to establish trends. His analysis was perhaps more aesthetic, noting symmetry.

The Duodecimal Bulletin, Volume 6, Number 2, January 1950; p. 35, Cyclic Sequences, George S.Terry. The current integer graphs I produce were created in early 2001 and 2002, a study of resonances within an integer. There are plenty of similar plays on the geometry associated with the dozen. This is one set of studies that strengthened my own graphs. The other resources were the logos of the Dozenal Societies and geometric graphics in the most current *Bulletins*.

The Duodecimal Bulletin, Volume 7, Number 1, October 1951; p. 1Ɛ-20, The Man with Twelve Fingers, F. H. Ames, Jr.

In January 2007, the applications Font Studio 5 and Wolfram Mathematica 5.1 were acquired. The Argam was expanded into the hundreds. The digits necessary to represent sexagesimal directly were created in Adobe Illustrator, modeled on Courier from scratch. A sexagesimal multiplication table was assembled in Microsoft Excel and passed via PDF format to Illustrator for additional graphic treatments. Meanwhile I studied the properties of sexagesimal on scrap paper during evenings at home and mornings in the coffee shop. These were synthesized in a leather book bought in Gubbio, Italy back in 2005. This leather book became the synopsis of all my work relating to bases.

February through August 2007 was an intense period in the office, with more business than ever before. The sketches in sexagesimal were a respite from the all-nighters and solid weeks of production. In March 2007 I visited a few websites that helped me see the light. These were aids in understanding how the Sumerians and Babylonians used sexagesimal on a daily basis.

Duncan J. Melville. Reciprocals and Reciprocal Algorithms in Mesopotamian Mathematics, [Electronic version]. Retrieved 21 September 2007, from http://it.stlawu.edu/~dmelvill/mesomath/Recip.pdf. This work suggested the notion of leveraging reciprocal divisors to compute. It seems the forefathers maintained a set of multiplication tables for reference, but actually computed by doubling or halving.

Duncan J. Melville. Old Babylonian Multiplication Tables, [Electronic version]. Retrieved February 2007, http://it.stlawu.edu/~dmelvill/mesomath/multiply.html.

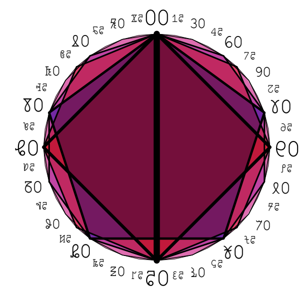
Duncan J. Melville. Mock Reciprocal Table, [Electronic version]. Retrieved February 2007, http://it.stlawu.edu/~dmelvill/mesomath/reciprocal.html.

David E. Joyce. Sexagesimal Reciprocals, [Electronic version]. Retrieved June 2004 from http://aleph0.clarku.edu/~djoyce/ma105/reciprocals3600.html.

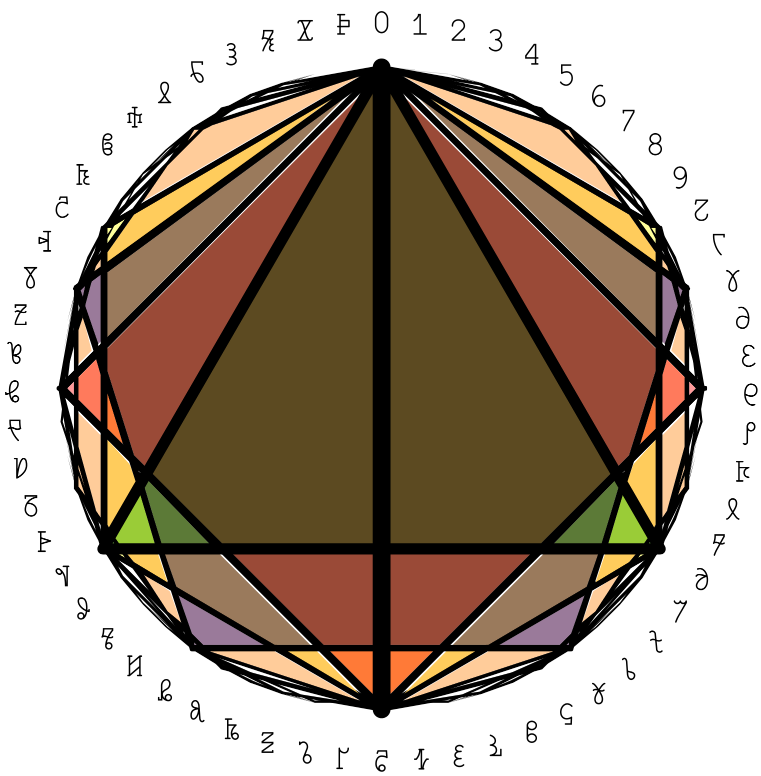
John Harris. Time and Tide; Babylonian Mathematics and Sexagesimal Notation, [Electronic version]. Retrieved October 2003, from http://www.spirasolaris.ca/sbb1sup1.html.

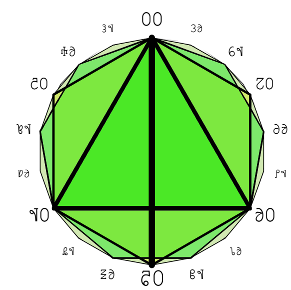
In late April 2007, The idea of abbreviating the full sexagesimal multiplication table came across at the Starbucks at Kingshighway and Chippewa in St. Louis on an early morning. If the first "module" of the divisors was included, and the other factors truncated where they were just under sixty, the reciprocals allowed computation beyond that first "module". Exercising this between April and May proved successful, so that any application which required multiplication and division was possible. I began computing time spent on projects fully in sexagesimal, tallying my work and that of my intern. Fees and project estimates could be calculated more reliably.

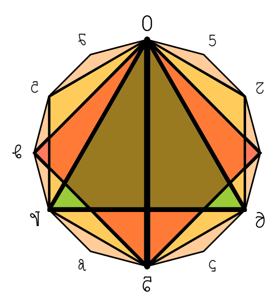
The result of these studies is this summary of what was learned. This work came together in the shadow of a cancelled project in September 2007.

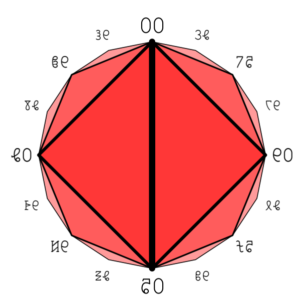


Appendices









Index of Plates in the Appendices

The booklet "Argam: a Transdecimal Numeral System" by Michael De Vlieger, will be finalized in 2008.

	Prime Factorization	σ_0, Total Number of Divisors	σ_1, Divisors Summed	Φ, Euler Totient Function	Divisors in Argam Notation
0	0	0	0	0	0
1	1	1 1.000	1 1.000	1 1.000	1 1
2	2	2 1.000	3 1.500	1 0.500	2 1 2
3	3	2 0.667	4 1.333	2 0.667	3 1 3
4	2^2	3 0.750	7 1.750	2 0.500	4 1 2 4
5	5	2 0.400	6 1.200	4 0.800	5 1 5
6	2 · 3	4 0.667	12 2.000	2 0.333	6 1 2 6 3
7	7	2 0.286	8 1.143	6 0.857	7 1 7
8	2^3	4 0.500	15 1.875	4 0.500	8 1 2 8 4
9	3^2	3 0.333	13 1.444	6 0.667	9 1 3 9
10	2 · 5	4 0.400	18 1.800	4 0.400	Ƨ 1 2 Ƨ 5
11	11	2 0.182	12 1.091	10 0.909	ᑕ 1 ᑕ
12	$2^2 \cdot 3$	6 0.500	28 2.333	4 0.333	ɣ 1 2 3 ɣ 6 4
13	13	2 0.154	14 1.077	12 0.923	∂ 1 ∂
14	2 · 7	4 0.286	24 1.714	6 0.429	ɛ 1 2 ɛ 7
15	3 · 5	4 0.267	24 1.600	8 0.533	ɘ 1 3 ɘ 5
16	2^4	5 0.313	31 1.938	8 0.500	ρ 1 2 4 ρ 8
17	17	2 0.118	18 1.059	16 0.941	Ꝁ 1 Ꝁ
18	$2 \cdot 3^2$	6 0.333	39 2.167	6 0.333	ℓ 1 2 3 ℓ 9 6
19	19	2 0.105	20 1.053	18 0.947	ƻ 1 ƻ
20	$2^2 \cdot 5$	6 0.300	42 2.100	8 0.400	ð 1 2 4 ð Ƨ 5
21	3 · 7	4 0.190	32 1.524	12 0.571	ㄱ 1 3 ㄱ 7
22	2 · 11	4 0.182	36 1.636	10 0.455	ɬ 1 1 ɬ ᑕ
23	23	2 0.087	24 1.043	22 0.957	ƅ 1 ƅ
24	$2^3 \cdot 3$	8 0.333	60 2.500	8 0.333	ɤ 1 2 3 4 ɤ ɣ 8 6
25	5^2	3 0.120	31 1.240	20 0.800	ς 1 5 ς
26	2 · 13	4 0.154	42 1.615	12 0.462	ɞ 1 2 ɞ ∂
27	3^3	4 0.148	40 1.481	18 0.667	ȝ 1 3 ȝ 9
28	$2^2 \cdot 7$	6 0.214	56 2.000	12 0.429	Ɛ 1 2 4 Ɛ 7 ɛ
29	$2^3 \cdot 3$	2 0.069	30 1.034	28 0.966	ɏ 1 ɏ
30	2 · 3 · 5	8 0.267	72 2.400	8 0.267	ɕ 1 2 3 5 ɕ ɘ Ƨ 6
31	31	2 0.065	32 1.032	30 0.968	ɼ 1 ɼ
32	2^5	6 0.188	63 1.969	16 0.500	ʒ 1 2 4 ʒ ρ 8
33	3 · 11	4 0.121	48 1.455	20 0.606	ƺ 1 3 ƺ ᑕ
34	2 · 17	4 0.118	54 1.588	16 0.471	ɮ 1 2 ɮ Ꝁ
35	5 · 7	4 0.114	48 1.371	24 0.686	ɑ 1 5 ɑ 7
36	$2^2 \cdot 3^2$	9 0.250	91 2.528	12 0.333	ȣ 1 2 3 4 6 ȣ ℓ ɣ 9
37	37	2 0.054	38 1.027	36 0.973	И 1 И
38	2 · 19	4 0.105	60 1.579	18 0.474	ꝫ 1 2 ꝫ ƻ
39	3 · 13	4 0.103	56 1.436	24 0.615	ϑ 1 3 ϑ ∂
40	$2^3 \cdot 5$	8 0.200	90 2.250	16 0.400	ꟻ 1 2 4 5 ꟻ ð Ƨ 8
41	41	2 0.049	42 1.024	40 0.976	Ⱶ 1 Ⱶ
42	2 · 3 · 7	8 0.190	96 2.286	12 0.286	δ 1 2 3 6 δ ㄱ ɛ 7
43	43	2 0.047	44 1.023	42 0.977	ʋ 1 ʋ
44	$2^2 \cdot 11$	6 0.136	84 1.909	20 0.455	ҁ 1 2 4 ҁ ɬ ᑕ
45	$3^2 \cdot 5$	6 0.133	78 1.733	24 0.533	ƍ 1 3 5 ƍ ɘ 9
46	2 · 23	4 0.087	72 1.565	22 0.478	ƾ 1 2 ƾ ƅ
47	47	2 0.043	48 1.021	46 0.979	Ƶ 1 Ƶ

n	Prime Factorization	σ_0, Total Number of Divisors	σ_1, Divisors Summed	Φ, Euler Totient Function	Divisors in Argam Notation
48	$2^4 \cdot 3$	10 0.208	124 2.583	16 0.333	1 2 3 4 6
49	7^2	3 0.061	57 1.163	42 0.857	1 7
50	$2 \cdot 5^2$	6 0.120	93 1.860	20 0.400	1 2 5
51	$3 \cdot 17$	4 0.078	72 1.412	32 0.627	1 3
52	$2^2 \cdot 13$	6 0.115	98 1.885	24 0.462	1 2 4
53	53	2 0.038	54 1.019	52 0.981	1
54	$2 \cdot 3^3$	8 0.148	120 2.222	18 0.333	1 2 3 6
55	$5 \cdot 11$	4 0.073	72 1.309	40 0.727	1 5
56	$2^3 \cdot 7$	8 0.143	120 2.143	24 0.429	1 2 4 7
57	$3 \cdot 19$	4 0.070	80 1.404	36 0.632	1 3
58	$2 \cdot 29$	4 0.069	90 1.552	28 0.483	1 2
59	59	2 0.034	60 1.017	58 0.983	1
60	$2^2 \cdot 3 \cdot 5$	12 0.200	168 2.800	16 0.267	1 2 3 4 5 6
61	61	2 0.033	62 1.016	60 0.984	1
62	$2 \cdot 31$	4 0.065	96 1.548	30 0.484	1 2
63	$3^2 \cdot 7$	6 0.095	104 1.651	36 0.571	1 3 7
64	2^6	7 0.109	127 1.984	32 0.500	1 2 4 8
65	$5 \cdot 13$	4 0.062	84 1.292	48 0.738	1 5
66	$2 \cdot 3 \cdot 11$	8 0.121	144 2.182	20 0.303	1 2 3 6
67	67	2 0.030	68 1.015	66 0.985	1
68	$2^2 \cdot 17$	6 0.088	126 1.853	32 0.471	1 2 4
69	$3 \cdot 23$	4 0.058	96 1.391	44 0.638	1 3
70	$2 \cdot 5 \cdot 7$	8 0.114	144 2.057	24 0.343	1 2 5 7
71	71	2 0.028	72 1.014	70 0.986	1
72	$2^3 \cdot 3^2$	12 0.167	195 2.708	24 0.333	1 2 3 4 6 8
73	73	2 0.027	74 1.014	72 0.986	1
74	$2 \cdot 37$	4 0.054	114 1.541	36 0.486	1 2
75	$3 \cdot 5^2$	6 0.080	124 1.653	40 0.533	1 3 5
76	$2^2 \cdot 19$	6 0.079	140 1.842	36 0.474	1 2 4
77	$7 \cdot 11$	4 0.052	96 1.247	60 0.779	1 7
78	$2 \cdot 3 \cdot 13$	8 0.103	168 2.154	24 0.308	1 2 3 6
79	79	2 0.025	80 1.013	78 0.987	1
80	$2^4 \cdot 5$	10 0.125	186 2.325	32 0.400	1 2 4 5 8
81	3^4	5 0.062	121 1.494	54 0.667	1 3 9
82	$2 \cdot 41$	4 0.049	126 1.537	40 0.488	1 2
83	83	2 0.024	84 1.012	82 0.988	1
84	$2^2 \cdot 3 \cdot 7$	12 0.143	224 2.667	24 0.286	1 2 3 4 6 7
85	$5 \cdot 17$	4 0.047	108 1.271	64 0.753	1 5
86	$2 \cdot 43$	4 0.047	132 1.535	42 0.488	1 2
87	$3 \cdot 29$	4 0.046	120 1.379	56 0.644	1 3
88	$2^3 \cdot 11$	8 0.091	180 2.045	40 0.455	1 2 4 8
89	89	2 0.022	90 1.011	88 0.989	1
90	$2 \cdot 3^2 \cdot 5$	12 0.133	234 2.600	24 0.267	1 2 3 5 6 9
91	$7 \cdot 13$	4 0.044	112 1.231	72 0.791	1 7
92	$2^2 \cdot 23$	6 0.065	168 1.826	44 0.478	1 2 4
93	$3 \cdot 31$	4 0.043	128 1.376	60 0.645	1 3
94	$2 \cdot 47$	4 0.043	144 1.532	46 0.489	1 2
95	$5 \cdot 19$	4 0.042	120 1.263	72 0.758	1 5

n	Prime Factorization	σ_0, Total Number of Divisors	σ_1, Divisors Summed	Φ, Euler Totient Function	Divisors in Argam Notation
96	$2^5 \cdot 3$	12 0.125	252 2.625	32 0.333	1 2 3 4 6 8
97	97	2 0.021	98 1.010	96 0.990	1
98	$2 \cdot 7^2$	6 0.061	171 1.745	42 0.429	1 2 7
99	$3^2 \cdot 11$	6 0.061	156 1.576	60 0.606	1 3 9
100	$2^3 \cdot 5^2$	9 0.090	217 2.170	40 0.400	1 2 4 5 Ƨ
101	101	2 0.020	102 1.010	100 0.990	1
102	$2 \cdot 3 \cdot 17$	8 0.078	216 2.118	32 0.314	1 2 3 6
103	103	2 0.019	104 1.010	102 0.990	1
104	$2^3 \cdot 13$	8 0.077	210 2.019	48 0.462	1 2 4 8
105	$3 \cdot 5 \cdot 7$	8 0.076	192 1.829	48 0.457	1 3 5 7
106	$2 \cdot 53$	4 0.038	162 1.528	52 0.491	1 2
107	107	2 0.019	108 1.009	106 0.991	1
108	$2^2 \cdot 3^3$	12 0.111	280 2.593	36 0.333	1 2 3 4 6 9
109	109	2 0.018	110 1.009	108 0.991	1
110	$2 \cdot 5 \cdot 11$	8 0.073	216 1.964	40 0.364	1 2 5 Ƨ
111	$3 \cdot 37$	4 0.036	152 1.369	72 0.649	1 3
112	$2^4 \cdot 7$	10 0.089	248 2.214	48 0.429	1 2 4 7 8
113	113	2 0.018	114 1.009	112 0.991	1
114	$2 \cdot 3 \cdot 19$	8 0.070	240 2.105	36 0.316	1 2 3 6
115	$5 \cdot 23$	4 0.035	144 1.252	88 0.765	1 5
116	$2^2 \cdot 29$	6 0.052	210 1.810	56 0.483	1 2 4
117	$3^2 \cdot 13$	6 0.051	182 1.556	72 0.615	1 3 9
118	$2 \cdot 59$	4 0.034	180 1.525	58 0.492	1 2
119	$7 \cdot 17$	4 0.034	144 1.210	96 0.807	1 7
120	$2^3 \cdot 3 \cdot 5$	16 0.133	360 3.000	32 0.267	1 2 3 4 5 6 8 Ƨ
121	11^2	3 0.025	133 1.099	110 0.909	1 ᒣ
122	$2 \cdot 61$	4 0.033	186 1.525	60 0.492	1 2
123	$3 \cdot 41$	4 0.033	168 1.366	80 0.650	1 3
124	$2^2 \cdot 31$	6 0.048	224 1.806	60 0.484	1 2 4
125	5^3	4 0.032	156 1.248	100 0.800	1 5
126	$2 \cdot 3^2 \cdot 7$	12 0.095	312 2.476	36 0.286	1 2 3 6 7 9
127	127	2 0.016	128 1.008	126 0.992	1
128	2^7	8 0.063	255 1.992	64 0.500	1 2 4 8
129	$3 \cdot 43$	4 0.031	176 1.364	84 0.651	1 3
130	$2 \cdot 5 \cdot 13$	8 0.062	252 1.938	48 0.369	1 2 5 Ƨ
131	131	2 0.015	132 1.008	130 0.992	1
132	$2^2 \cdot 3 \cdot 11$	12 0.091	336 2.545	40 0.303	1 2 3 4 6 ᒣ
133	$5 \cdot 17$	4 0.030	160 1.203	108 0.812	1 7
134	$2 \cdot 67$	4 0.030	204 1.522	66 0.493	1 2
135	$3^3 \cdot 5$	8 0.059	240 1.778	72 0.533	1 3 5 9
136	$2^3 \cdot 17$	8 0.059	270 1.985	64 0.471	1 2 4 8
137	137	2 0.015	138 1.007	136 0.993	1
138	$2 \cdot 3 \cdot 23$	8 0.058	288 2.087	44 0.319	1 2 3 6
139	139	2 0.014	140 1.007	138 0.993	1
140	$2^2 \cdot 5 \cdot 7$	12 0.086	336 2.400	48 0.343	1 2 4 5 7 Ƨ
141	$3 \cdot 47$	4 0.028	192 1.362	92 0.652	1 3
142	$2 \cdot 71$	4 0.028	216 1.521	70 0.493	1 2
143	$11 \cdot 13$	4 0.028	168 1.175	120 0.839	1 ᒣ
144	$2^4 \cdot 3^2$	15 0.104	403 2.799	48 0.333	1 2 3 4 6 8 9

Octal
(Base 8)

Ⓣ	4	Ⓣ	2	Ⓣ	4/3	Ⓣ	1
1	2	3	4	5	6	7	10
2	**4**	6	10	12	14	16	20
3	6	**11**	14	17	22	25	30
4	10	14	**20**	24	30	34	40
5	12	17	24	**31**	36	43	50
6	14	22	30	36	**44**	52	60
7	16	25	34	43	52	**61**	70
10	20	30	40	50	60	70	**100**

Full Multiplication Table.

Hexadecimal

(Base 16)

Ⓣ 8 Ⓣ 4 Ⓣ 8/3 Ⓣ 2 Ⓣ 8/5 Ⓣ 4/3 Ⓣ 8/7 Ⓣ 1

1	2	3	4	5	6	7	8	9	Ƨ	ᒣ	ɣ	∂	Ɛ	℮	10
2	**4**	6	8	Ƨ	ɣ	Ɛ	10	12	14	16	18	1Ƨ	1ɣ	1Ɛ	20
3	6	**9**	ɣ	℮	12	15	18	1ᒣ	1Ɛ	21	24	27	2Ƨ	2∂	30
4	8	ɣ	**10**	14	18	1ɣ	20	24	28	2ɣ	30	34	38	3ɣ	40
5	Ƨ	℮	14	**19**	1Ɛ	23	28	2∂	32	37	3ɣ	41	46	4ᒣ	50
6	ɣ	12	18	1Ɛ	**24**	2Ƨ	30	36	3ɣ	42	48	4Ɛ	54	5Ƨ	60
7	Ɛ	15	1ɣ	23	2Ƨ	**31**	38	3℮	46	4∂	54	5ᒣ	62	69	70
8	10	18	20	28	30	38	**40**	48	50	58	60	68	70	78	80
9	12	1ᒣ	24	2∂	36	3℮	48	**51**	5Ƨ	63	6ɣ	75	6Ɛ	87	90
Ƨ	14	1Ɛ	28	32	3ɣ	46	50	5Ƨ	**64**	6Ɛ	78	82	8ɣ	96	Ƨ0
ᒣ	16	21	2ɣ	37	42	4∂	58	63	6Ɛ	**79**	84	8℮	9Ƨ	Ƨ5	ᒣ0
ɣ	18	24	30	3ɣ	48	54	60	6ɣ	78	84	**90**	9ɣ	Ƨ8	ᒣ4	ɣ0
∂	1Ƨ	27	34	41	4Ɛ	5ᒣ	68	75	82	8℮	9ɣ	**Ƨ9**	ᒣ6	ɣ3	∂0
Ɛ	1ɣ	2Ƨ	38	46	54	62	70	6Ɛ	8ɣ	9Ƨ	Ƨ8	ᒣ6	**ɣ4**	∂2	Ɛ0
℮	1Ɛ	2∂	3ɣ	4ᒣ	5Ƨ	69	78	87	96	Ƨ5	ᒣ4	ɣ3	∂2	**Ɛ1**	℮0
10	20	30	40	50	60	70	80	90	Ƨ0	ᒣ0	ɣ0	∂0	Ɛ0	℮0	**100**

Full Multiplication Table with Argam notation for transdecimal bases.

Dozenal
(Base 12)

T	6	4	3	T	2	T	3/2	4/3	6/5	T	1
1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	**4**	6	8	χ	10	12	14	16	18	1χ	20
3	6	**9**	10	13	16	19	20	23	26	29	30
4	8	10	**14**	18	20	24	28	30	34	38	40
5	χ	13	18	**21**	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	**30**	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	**41**	48	53	5χ	65	70
8	14	20	28	34	40	48	**54**	60	68	74	80
9	16	23	30	39	46	53	60	**69**	76	83	90
χ	18	26	34	42	50	5χ	68	76	**84**	92	χ0
Ɛ	1χ	29	38	47	56	65	74	83	92	**χ1**	Ɛ0
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	**100**

Traditional Full Dozenal Multiplication Table.

Decimal

(Base 10)

T	5	T	5/4	2	5/3	T	5/2	T	1
1	2	3	4	5	6	7	8	9	10
2	**4**	6	8	10	12	14	16	18	20
3	6	**9**	12	15	18	21	24	27	30
4	8	12	**16**	20	24	28	32	36	40
5	10	15	20	**25**	30	35	40	45	50
6	12	18	24	30	**36**	42	48	54	60
7	14	21	28	35	42	**49**	56	63	70
8	16	24	32	40	48	56	**64**	72	80
9	18	27	36	45	54	63	72	**81**	90
10	20	30	40	50	60	70	80	90	**100**

Full Multiplication Table.

🅣	ꞓ	∂	℮	ɤ	Ƨ	🅣	℮/2	∂/3	6	🅣	5	🅣	ꞓ/7	4	℮/4	🅣	∂/6	🅣	3	∂/7	ꞓ/7	🅣	5/2	ɤ/5	ꞓ/∂	∂/9	℮/7	🅣	2
1	2	3	4	5	6	7	8	9	Ƨ	Ꞁ	ɤ	∂	ɛ	℮	ʃ	ꝁ	ℓ	ᖴ	∂	7	ƚ	ᒐ	ɤ	ς	ϐ	ɮ	ɛ	ɫ	ꞓ
2	4	6	8	Ƨ	ɤ	ɛ	ʃ	ℓ	∂	ƚ	ɤ	ϐ	ɛ	ꞓ	ɿ	ꝁ	ℓ	ᖴ	ᑫ	ŏ	ᖴ	ϐ	ɤ	ꞓ	ϐ	ℓ	ɜ	Ꙇ	10
3	6	9	ɤ	℮	ℓ	7	ɤ	ɮ	ꞓ	ɀ	ℓ	ɿ	ŏ	ℓ	ɤ	ꝁ	ℓ	ᖴ	10	13	16	19	1ɤ	1℮	1ℓ	17	1ɤ	1ɮ	1ꞓ
4	8	ɤ	ʃ	∂	ɤ	ɛ	ɿ	ℓ	ᑫ	ᖴ	ɤ	ϐ	ɜ	10	14	18	1ɤ	1ʃ	1∂	1ɤ	1ɛ	1ɿ	1ℓ	1ᑫ	1ᖴ	1ɤ	1ϐ	13	20
5	Ƨ	℮	∂	ς	ꞓ	ꞛ	ᑫ	ℓ	ꞓ	ꞓ	10	15	1Ƨ	1℮	1∂	1ς	1ꞓ	1ꞛ	1ᑫ	1ℓ	1ꞓ	1ꞓ	20	25	2Ƨ	2℮	2∂	2ς	2ꞓ
6	ɤ	ℓ	ɤ	ꞓ	ℓ	ŏ	ɤ	ℓ	10	16	1ɤ	1ℓ	1ɤ	1ꞓ	1ℓ	1ŏ	1ɤ	1ℓ	20	26	2ɤ	2ℓ	2ɤ	2ꞓ	2ℓ	2ŏ	2ɤ	2ℓ	30
7	ɛ	7	ɛ	ꞛ	ŏ	ꟻ	ɜ	13	1Ƨ	1ꝁ	1ɤ	1Ꞁ	1ᖴ	1ℓ	1ϐ	1ꟻ	26	2∂	2∂	2ɮ	2ꝁ	2ꟻ	2ɤ	2ꞓ	32	39	3ʃ	3ɿ	3ꞓ
8	ʃ	ɤ	ɿ	ᑫ	ɤ	ɜ	14	1ɤ	1∂	1ɛ	1ℓ	1ᖴ	1ϐ	20	28	2ʃ	2ɤ	2ɿ	2ᑫ	2ɤ	23	34	3ɤ	3∂	3ɛ	3ℓ	3ᖴ	3ϐ	40
9	ℓ	ɮ	ℓ	ℓ	ℓ	13	1ɤ	17	1ꞓ	1ɮ	1ɤ	1ᖴ	26	2℮	2ɤ	2ɀ	2ŏ	2ꝁ	30	39	3ℓ	3ɮ	3ℓ	3ℓ	3ℓ	43	4ɤ	47	4ꞓ
Ƨ	∂	ꞓ	ᑫ	ꞓ	10	1Ƨ	1∂	1ꞓ	1ᑫ	1ꞓ	20	2Ƨ	2∂	2ꞓ	2ᑫ	2ꞓ	30	3Ƨ	3∂	3ꞓ	3ᑫ	3ꞓ	40	4Ƨ	4∂	4ꞓ	4ᑫ	4ꞓ	50
Ꞁ	ƚ	ɀ	ᖴ	Ꞗ	16	1ꝁ	1ɛ	1ɮ	1ꞓ	21	2ɤ	2ɿ	2ꝁ	2ℓ	23	37	3ℓ	3ɀ	3ᑫ	3ꝁ	42	4∂	4ɤ	4ꞛ	4ϐ	4ᖴ	58	5ᖴ	5ꞓ
ɤ	ɤ	ℓ	ɤ	10	1ɤ	1ɤ	1ℓ	1ɤ	20	2ɤ	2ɤ	2ℓ	2ɤ	30	3ɤ	3ɤ	3ℓ	3ɤ	40	4ɤ	4ɤ	4ℓ	4ɤ	50	5ɤ	5ɤ	5ℓ	5ɤ	60
∂	ϐ	ɮ	ϐ	15	1ℓ	1Ꞁ	1ᖴ	1ᖴ	2Ƨ	2ɿ	2ℓ	2ꟻ	32	3℮	3ɛ	3ꟻ	3ℓ	47	4∂	4ɀ	4ϐ	4ꟻ	5ɤ	5ς	5ᖴ	5ꝁ	64	6ꝁ	6ꞓ
ɛ	ɛ	ŏ	ɜ	1Ƨ	1ɤ	1ᖴ	1ϐ	26	2∂	2ꝁ	2ɤ	32	3ʃ	3ꞓ	3ᖴ	3Ꙇ	4ɤ	4ϐ	4ᑫ	4ℓ	58	5ƚ	5ℓ	5ꞓ	64	6ℓ	6ɿ	6ϐ	70
℮	ꞓ	ℓ	10	1℮	1ꞓ	1ℓ	20	2℮	2ꞓ	2ℓ	30	3℮	3ꞓ	3ℓ	40	4℮	4ꞓ	4ℓ	50	5℮	5ꞓ	5ℓ	60	6℮	6ꞓ	6ℓ	70	7℮	7ꞓ
ʃ	ɿ	ɤ	14	1∂	1ℓ	1ϐ	28	2ɤ	2ᑫ	23	3ɤ	3ɛ	3ᖴ	40	4ʃ	4ɿ	4ɤ	54	5∂	5ℓ	5ϐ	68	6ɤ	6ᑫ	63	7ɤ	7ɛ	7ᖴ	80
ꝁ	ꝁ	ꝁ	18	1ꞓ	1ɤ	1ꟻ	2ʃ	2ɀ	2ꞓ	37	3ɤ	3ꟻ	3Ꙇ	4℮	4ɿ	4ꟻ	56	5ɿ	5ᑫ	5ᖴ	6ɜ	6Ꞁ	6ɤ	75	7ƚ	7ɮ	73	8∂	8ꞓ
ℓ	ℓ	ℓ	1ɤ	1ꞓ	1ɤ	26	2ɤ	2ŏ	30	3ℓ	3ℓ	3ℓ	4ɤ	4ꞓ	4ɤ	56	5ɤ	5ŏ	60	6ℓ	6ℓ	6ℓ	7ɤ	7ꞓ	7ɤ	86	8ɤ	8ŏ	90
ᖴ	ᖴ	ᖴ	1ʃ	1ꞛ	1ℓ	2∂	2ɿ	2ꝁ	3Ƨ	3ɫ	3ɤ	47	4ϐ	4ℓ	54	5ɿ	5ŏ	61	6∂	6ɮ	6Ꙇ	7ꝁ	7ℓ	7Ꞗ	89	8ɀ	8ϐ	9Ꞁ	9ꞓ
∂	ᑫ	10	1∂	1ᑫ	20	2∂	2ᑫ	30	3∂	3ᑫ	40	4∂	4ᑫ	50	5∂	5ᑫ	60	6∂	6ᑫ	70	7∂	7ᑫ	80	8∂	8ᑫ	90	9∂	9ᑫ	Ƨ0
7	ŏ	13	1ɤ	1ℓ	26	2ɮ	2ɤ	39	3ꞓ	3ꝁ	4ɤ	4ɀ	4ℓ	5℮	5ℓ	5ᖴ	6ℓ	6ɮ	70	77	7ŏ	83	8ɤ	8ℓ	96	9Ɥ	9ɤ	Ƨ9	Ƨꞓ
ƚ	ᖴ	16	1ɛ	1ꞓ	2ɤ	2ꝁ	23	3ℓ	3ᑫ	42	4ɤ	4ϐ	58	5ꞓ	5ϐ	6ɜ	6ℓ	6Ꙇ	7∂	7ŏ	84	8ϐ	8ɤ	9Ƨ	9ɿ	9ℓ	Ƨʃ	Ƨᖴ	Ƨ0
ᒐ	ϐ	19	1ɿ	1Ꞗ	2ℓ	2ꟻ	34	3ɮ	3ꞓ	4∂	4ℓ	4ꟻ	5ƚ	5ℓ	68	6Ꞁ	6ℓ	7ꝁ	7ᑫ	83	8ϐ	8ꟻ	9ɤ	9ꞛ	9Ꙇ	Ƨ7	Ƨᖴ	Ƨ7	Ƨ7
ɤ	ɤ	1ɤ	1ℓ	20	2ɤ	2ɤ	3ɤ	3ℓ	40	4ɤ	4ɤ	5ɤ	5ℓ	60	6ɤ	6ɤ	7ɤ	7ℓ	80	8ɤ	8ɤ	9ɤ	9ℓ	Ƨ0	Ƨɤ	Ƨɤ	Ƨɤ	Ƨℓ	ɤ0
ς	ꞓ	1℮	1ᑫ	25	2ꞓ	2Ꞗ	3∂	3ℓ	4Ƨ	4ꞛ	50	5ς	5ꞓ	6℮	6ᑫ	75	7ꞓ	7Ꞗ	8∂	8ℓ	9Ƨ	9ꞛ	Ƨ0	Ƨς	Ƨꞓ	Ƨ℮	Ƨᑫ	ɤ5	ɤꞓ
ϐ	ϐ	1ℓ	1ᖴ	2Ƨ	2ℓ	32	3ɛ	3ℓ	4∂	4ϐ	5ɤ	5ᖴ	64	6ꞓ	63	7ƚ	7ɤ	89	8ᑫ	96	9ɿ	9Ꙇ	Ƨɤ	Ƨꞓ	Ƨʃ	Ƨŏ	ɤ8	ɤꝁ	∂0
ɮ	ℓ	17	1ɤ	2℮	2ŏ	39	3ℓ	43	4ꞓ	4ᖴ	5ɤ	5ꝁ	6ℓ	6ℓ	7ɤ	7ɮ	86	8ɀ	90	9Ɥ	9ℓ	Ƨ7	Ƨɤ	Ƨ9	Ƨ7	ɤ9	ɤℓ	∂3	∂ꞓ
ɛ	ɜ	1ɤ	1ϐ	2∂	2ɤ	3ʃ	3ᖴ	4ɤ	4ᑫ	58	5ℓ	64	6ɿ	70	7ɛ	73	8ɤ	8ϐ	9∂	9ɤ	Ƨʃ	Ƨᖴ	Ƨ7	Ƨᑫ	ɤ8	ɤℓ	∂4	∂ɿ	ɛ0
ɫ	Ꙇ	1ɮ	13	2ς	2ℓ	3ɿ	3ϐ	47	4ꞓ	5ᖴ	5ɤ	6ꝁ	6ϐ	7℮	7ᖴ	8∂	8ɤ	9Ꞁ	9ᑫ	Ƨ9	Ƨɀ	Ƨ7	Ƨℓ	ɤ5	ɤꝁ	∂3	∂ɿ	ɛ1	ɛꞓ
ꞓ	10	1ꞓ	20	2ꞓ	30	3ꞓ	40	4ꞓ	50	5ꞓ	60	6ꞓ	70	7ꞓ	80	8ꞓ	90	9ꞓ	Ƨ0	Ƨꞓ	Ƨ0	Ƨꞓ	ɤ0	ɤꞓ	∂0	∂ꞓ	ɛ0	ɛꞓ	℮0
Ꞁ	12	1ɀ	24	2ꞛ	36	3Ɥ	48	4ɮ	5Ƨ	5ꟻ	6ɤ	6Ɒ	7ɛ	7ℓ	8ʃ	8ɀ	9ℓ	9ꟻ	Ƨ∂	Ƨꝁ	Ƨ7	Ƨꝕ	ɤɤ	ɤꞓ	∂ϐ	∂ᖴ	ɛɜ	4ꟻ	℮ꞓ
ɿ	14	1ℓ	28	2ᑫ	3ɤ	3ᖴ	4ʃ	4ɤ	5∂	5ϐ	6ɤ	63	7ɛ	80	8ɿ	94	9ℓ	Ƨ8	Ƨᑫ	ɤ7	Ƨᖴ	ɤʃ	ɤɤ	∂∂	∂ℓ	ɛɤ	ɛ3	℮9	ʃ0
ɀ	16	1ɮ	2ɤ	2ℓ	3ℓ	3ꝁ	4ɤ	4ᖴ	5ꞓ	63	6ℓ	79	7ŏ	8℮	8ɤ	97	9ℓ	Ƨɮ	Ƨ0	ɤ3	ɤ6	ɤ9	∂ɤ	∂ℓ	ɛℓ	ɛꝁ	℮ɤ	℮ᖴ	ʃꞓ
ꝁ	18	1ŏ	2ʃ	2ꞓ	3ɤ	3Ꙇ	4ɿ	56	5ᑫ	6ɜ	6ɤ	7ƚ	73	8ꞓ	94	9ɀ	Ƨŏ	Ƨϐ	Ƨ9	Ƨℓ	ɤɛ	∂2	∂ℓ	ɛ3	℮ᖴ	℮ℓ	℮ϐ	ʃϐ	ꝁ0
ꞛ	1Ƨ	1ℓ	2∂	2Ꞗ	3ꞓ	45	4ᑫ	5℮	5ꞓ	6ς	70	7ꞛ	8Ƨ	8ℓ	9∂	9Ꞗ	Ƨꞓ	Ƨ5	Ƨᑫ	ɤᑫ	ɤꞓ	∂ς	ɛ0	ɛꞛ	℮9	℮ℓ	ʃ∂	ʃꞗ	ꝁꞓ
ℓ	1ɤ	1ɤ	2ɤ	30	3ℓ	4ɤ	4ɤ	5ɤ	60	6ℓ	7ɤ	7ɤ	8ɤ	90	9ℓ	Ƨɤ	Ƨɤ	Ƨɤ	ɤ0	ɤℓ	∂ɤ	∂ɤ	ɛɤ	℮0	℮ℓ	ʃɤ	ʃɤ	ꝁɤ	ℓ0
Ɥ	1ɛ	1ꝁ	2ɛ	35	3ŏ	4ᖴ	43	5ɀ	6Ƨ	6ɀ	7ɤ	81	8ᖴ	9℮	9ϐ	Ƨɫ	Ƨ9	ƧƉ	ɤ∂	ɤᖴ	∂ꝁ	ɛ73	ɛɤ	℮ς	ʃ2	ʃɮ	ꝁʃ	ꝁꝕ	ℓꞓ
ᖴ	1ʃ	1ℓ	2ɿ	3Ƨ	3ɤ	4ϐ	54	5ŏ	6∂	6Ꙇ	7ℓ	8ɜ	8ϐ	9ꞓ	8ɜ	Ƨϐ	Ƨ7	ɤ2	ɤᑫ	∂ℓ	∂3	ɛꝁ	℮ɤ	℮℮	ʃɛ	ꝁ6	ꝁᖴ	ℓƚ	ᖴ0
ɮ	1ℓ	1ᖴ	2ℓ	3℮	3ℓ	4ɀ	5ŏ	5ꝁ	6ꞓ	79	7ɤ	8ɮ	96	9ℓ	Ƨɤ	Ƨ3	Ƨ7	ɤ7	∂0	∂ɮ	ɛ8	ɛƧ	℮ℓ	ʃ℮	ʃℓ	ꝁɀ	ℓɤ	ℓꝁ	ᖴꞓ
ᑫ	1∂	20	2ᑫ	3∂	40	4ᑫ	5∂	60	6ᑫ	7∂	80	8ᑫ	9∂	Ƨ0	Ƨᑫ	Ƨ∂	ɤ0	ɤᑫ	∂∂	ɛ0	ɛᑫ	℮∂	ʃ0	ʃᑫ	ꝁ∂	ℓ0	ℓᑫ	ᖴ∂	∂0
ꟻ	1ƚ	23	2ᖴ	3ς	46	4Ƨ	5ɛ	69	6ꞓ	7Ꞁ	8ɤ	8ꝕ	9ꝁ	Ƨ℮	Ƨ3	ƧꞪ	ɤℓ	ɤꟻ	∂ᑫ	ɛ7	℮2	℮Ɒ	ʃɤ	ꝁ5	ꝁϐ	ℓɮ	ᖴ8	ᖴꟻ	∂ꞓ
ŏ	1ɤ	26	2ɤ	3ꞓ	4ɤ	4ℓ	5ℓ	6ℓ	70	7ŏ	8ɤ	96	9ɤ	Ƨꞓ	Ƨŏ	Ƨℓ	ɤℓ	∂ℓ	ɛ0	ɛŏ	℮ɤ	ʃ6	ʃɤ	ꝁꞓ	ℓɤ	ℓℓ	ᖴℓ	∂ℓ	70
Ɒ	1ϐ	29	2ϐ	3ꞛ	4ℓ	51	5ᖴ	6ɮ	7Ƨ	7ꝕ	8ℓ	9ᖴ	Ƨ2	Ƨℓ	ɛ7	ɤƧ	ɤℓ	∂Ɥ	ɛ3	℮3	℮ϐ	ʃɫ	ꝁɤ	ꝁꞗ	ℓɤ	ᖴ7	∂4	∂ɀ	7ꞓ
ᖴ	1ɛ	2ɤ	23	3ᑫ	4ɤ	58	5ϐ	6ℓ	7∂	84	8ɤ	9ɿ	Ƨʃ	Ƨ0	℮7	ɤɛ	∂ɤ	∂3	ɛᑫ	℮ɤ	ʃ8	ʃϐ	ꝁℓ	ℓ∂	ᖴ4	ᖴɤ	∂ɿ	7ʃ	ƚ0
ℓ	1ꞓ	2℮	30	3ℓ	4ꞓ	5℮	60	6ℓ	7ꞓ	8℮	90	9ℓ	Ƨꞓ	Ƨ℮	ɤ0	ɤℓ	∂ꞓ	℮3	℮0	℮ℓ	ʃꞓ	ꝁ℮	ℓ0	ℓℓ	ᖴꞓ	∂℮	70	7ℓ	ƚꞓ
ϐ	1ɿ	2ℓ	34	3ꞓ	4ℓ	5ƚ	68	6ℓ	7ᑫ	8ϐ	9ɤ	9Ꙇ	Ƨᖴ	Ƨ7	ɤʃ	∂2	∂ɤ	ɛꝁ	℮∂	ʃ6	ʃϐ	ꝁɀ	ℓɤ	ᖴƧ	ᖴ3	∂ŏ	7ɛ	ƚɛ	ɿ0
ɀ	1ꝁ	27	38	3Ꞗ	4ŏ	5ɫ	6ʃ	73	7ꞓ	8Ɥ	9ɤ	Ƨ7	Ƨ5	Ƨℓ	ɤɿ	∂ᖴ	ɛ6	ɛꝕ	℮ᑫ	ʃɮ	ꝁɛ	ℓ1	ℓɤ	ᖴꞛ	∂ƚ	79	73	ƚƉ	ɿꞓ
ɤ	1ℓ	2ɤ	3ɤ	40	4ɤ	5ℓ	6ɤ	7ɤ	80	8ɤ	9ℓ	Ƨɤ	Ƨɤ	ɤ0	ɤɤ	∂ℓ	ɛɤ	℮ɤ	ʃ0	ʃɤ	ꝁℓ	ℓɤ	ᖴɤ	∂0	∂ɤ	7ℓ	ƚɤ	ɿɤ	ɤ0
ꟻ	1ᖴ	2ɮ	3ʃ	45	4ℓ	5Ɒ	6ɿ	77	8Ƨ	8ꟻ	9ɤ	ƧꞪ	Ƨϐ	ɤ℮	∂4	∂ꝕ	ɛ3	℮ℓ	ʃ∂	ꝁ9	ꝁꙆ	ℓɀ	ᖴℓ	∂ς	7ɛ	ƚ3	ƚϐ	ɿꟻ	ɤꞓ
ꞓ	1ᑫ	2ꞓ	3∂	4Ƨ	50	5ꞓ	6ᑫ	7ꞓ	8∂	9ꞓ	Ƨ0	Ƨꞓ	Ƨᑫ	ɤꞓ	∂∂	ɛꞓ	℮0	℮ꞓ	ʃᑫ	ꝁꞓ	ℓ∂	ᖴꞓ	∂0	∂ꞓ	7ᑫ	ƚꞓ	ɿ∂	ɤꞓ	ς0
ꝁ	1ŏ	2ɀ	3ɤ	4℮	56	5ᖴ	6ŏ	7ɮ	8ꞓ	97	Ƨɤ	Ƨ3	Ƨℓ	ɤℓ	∂ℓ	ɛɮ	℮ℓ	ʃ9	ꝁ0	ꝁꝁ	ℓŏ	ᖴɀ	∂ɤ	7℮	ƚ6	ƚᖴ	ɿɤ	ɤɮ	ςꞓ
ϐ	1ᖴ	2ℓ	3ɛ	4∂	5ɤ	64	63	7ɤ	8ᑫ	9ɿ	Ƨɤ	Ƨʃ	ɤ8	∂0	∂ϐ	℮3	℮ℓ	ʃɛ	ꝁϐ	ℓɤ	ᖴ4	ᖴ3	∂ɤ	7ᑫ	ƚɿ	ɿɤ	ɤʃ	ς8	ϐ0
ꝕ	1ϐ	2ɮ	3ɿ	4ς	5ℓ	6Ꞁ	74	7ᖴ	8ꞓ	9Ɒ	Ƨℓ	Ƨɫ	ɤƚ	∂℮	ɛ8	℮1	℮ℓ	ʃɀ	ꝁᑫ	ℓɀ	ᖴϐ	ᖴɀ	7ɤ	ƚ5	ƚꙆ	ɿꝁ	ɤᖴ	ςꞪ	ϐꞓ
ℓ	1ɤ	2ŏ	3ℓ	4ꞓ	5ɤ	6ℓ	7ɤ	86	90	9ℓ	Ƨɤ	Ƨŏ	ɤℓ	∂ꞓ	ɛɤ	℮ℓ	ʃɤ	ꝁ6	ℓ0	ℓℓ	ᖴɤ	∂ŏ	7ℓ	ƚꞓ	ɿɤ	ɤℓ	ςɤ	ϐ6	ɮ0
Ꞗ	1ꞓ	2ℓ	3ᑫ	4ꞛ	5ꞓ	6ς	7∂	8℮	9Ƨ	Ƨ5	Ƨ0	Ƨꞓ	Ƨ7	ɤꞓ	∂ℓ	ɛᑫ	℮ꞛ	ʃꞓ	ꝁς	ℓ∂	ᖴ℮	∂Ƨ	75	ƚ0	ƚꞓ	ɿꞓ	ɤℓ	ςᑫ	ϐꞛ	ɮꞓ
ɜ	1ϐ	2ɤ	3ᖴ	4ᑫ	5ℓ	6ɿ	7ɛ	8ɤ	9∂	Ƨʃ	Ƨɤ	ɤ8	∂4	ɛ0	ɛ3	℮ϐ	ʃɤ	ꝁᖴ	ℓᑫ	ᖴℓ	∂ɿ	7ɛ	ƚɤ	ɿ∂	ɤʃ	ςɤ	ϐ8	ɮ4	ɛ0
ᖴ	1ℓ	2ꝁ	3ɤ	4ℓ	5ŏ	6ɮ	7ℓ	8ɀ	9ꞓ	Ƨɮ	Ƨ7	ɤ7	∂ℓ	℮3	℮ɤ	ʃ9	ꝁ6	ℓ3	ᖴ0	ᖴℓ	∂ℓ	7ꝁ	ƚɤ	ɿℓ	ɤŏ	ςꞓ	ϐℓ	ɮɀ	ɛꞓ
Ꙇ	13	2ℓ	3ϐ	4ꞓ	5ɤ	6ϐ	7ᖴ	8ŏ	9ᑫ	Ƨɀ	Ƨ7	ɤꝁ	∂ɿ	ɛ3	℮ɮ	ʃϐ	ꝁɤ	ℓƚ	ᖴ∂	∂ℓ	7ʃ	ƚɛ	ɿɤ	ɤƧ	ς8	ϐ6	ɮ4	ɛ2	ɫ0
ꟻ	1Ꙇ	2ᖴ	33	4ꞓ	5ℓ	6ꝕ	7ϐ	8ꝁ	9ꞓ	Ƨꟻ	Ƨ7	ɤɀ	∂ϐ	ɛℓ	℮ᑫ	ʃƉ	ꝁꞓ	ℓꝁ	ᖴᑫ	∂ɮ	7ɤ	ƚꞪ	ɿℓ	ɤꞛ	ςꝁ	ϐɀ	ɮɿ	ɛɿ	ɫꞓ
10	20	30	40	50	60	70	80	90	Ƨ0	Ꞁ0	ɤ0	∂0	ɛ0	℮0	ʃ0	ꝁ0	ℓ0	ᖴ0	∂0	70	ƚ0	ɿ0	ɤ0	ς0	ϐ0	ɮ0	ɛ0	ɫ0	ꞓ0

Sexagesimal (Base 60)

Ⓣ	6/8	∂/ɾ	6/₭	ɤ/7	5/3	Ⓣ	6/7	∂/∂	3/2	Ⓣ	7/7	Ⓣ	6/ɾ	4/3	6/ɓ	Ⓣ	5/4	Ⓣ	6/5	∂/₭	6/∂	Ⓣ	7/9	ɤ/ɾ	6/ɛ	∂/7	6/4	Ⓣ	1
ɾ	ƨ	ⱻ	₭	ឧ	ℓ	Ŋ	₴	ϑ	٩	ⱶ	ɕ	ᗜ	ᑫ	ℓ	ɓ	Ƶ	ɤ	ɬ	ʖ	₭	ɞ	ⴼ	ϱ	Ƽ	ɜ	Ʀ	Ⅹ	ⱶ	10
12	14	16	18	1Ꞇ	1ɤ	1Ɛ	1ʃ	1ϱ	1∂	1ɬ	1ɤ	1ɞ	1Ɛ	1ɕ	1ƨ	1₭	1ϱ	1₴	1٩	1ɕ	1٩	1ɓ	1ɤ	1ɕ	1ɞ	1ϱ	13	1Ⅹ	20
1ⱻ	1ℓ	1ϑ	1ɕ	1ℓ	1ɤ	1₭	1ϱ	1Ʀ	20	23	26	29	2ɤ	2ɛ	2ϱ	27	2ɤ	2ʖ	2ɕ	2ⱻ	2ℓ	2ϑ	2ɕ	2ℓ	2ɤ	2₭	2ϱ	2Ʀ	30
24	28	2ɤ	2ʃ	2∂	2ɤ	2Ɛ	2ƨ	2ℓ	2٩	2ɛ	2ɤ	2ɞ	23	30	34	38	3ɤ	3ʃ	3∂	3ɤ	3Ɛ	3ƨ	3ℓ	3٩	3ɛ	3ɤ	3ɞ	33	40
2ឧ	2٩	2ℓ	2Ꞇ	2Ƽ	30	35	3Ꞇ	3ɛ	3∂	3ɕ	3ɕ	3ឧ	3٩	3ℓ	3Ꞇ	3Ƽ	40	45	4Ꞇ	4ɛ	4∂	4ɕ	4ɕ	4ឧ	4٩	4ℓ	4Ꞇ	4Ƽ	50
36	3ɤ	3ϱ	3ɤ	3ɕ	3ℓ	3ɕ	3ɤ	3ϱ	40	46	4ɤ	4ϱ	4ɤ	4ɕ	4ℓ	4ɕ	4ɤ	4ϱ	50	56	5ɤ	5ϱ	5ɤ	5ɕ	5ℓ	5ɕ	5ɤ	5ϱ	60
3Ŋ	3ɛ	3₭	3Ⅹ	45	4ɤ	47	4ɞ	4ⱻ	4٩	4Ƶ	4ϱ	51	58	5ɛ	5ɬ	5ɬ	5ℓ	5ᗜ	5ɕ	5Ʀ	64	6Ꞇ	6ϱ	6ɕ	6ƨ	6ϑ	6ɤ	6ⴼ	70
48	4ʃ	4ɤ	4ƨ	4٩	4ɤ	43	54	5ɤ	5∂	5Ɛ	5ℓ	5ɛ	5ɞ	60	68	6ʃ	6ɤ	6ƨ	6٩	6ɤ	63	74	7ɤ	7∂	7Ɛ	7ℓ	7ɛ	7ɞ	80
4ϑ	4ɤ	4Ʀ	56	5ɛ	5ɤ	5ⱻ	5ɕ	5₭	60	69	6ϱ	6ʖ	6ℓ	6ℓ	6ϱ	73	7ɤ	77	7ɕ	7ϑ	7ɤ	7Ʀ	86	8ɛ	8ɤ	8ⱻ	8ɕ	8₭	90
5Ꞇ	5∂	5ɕ	5٩	5ɕ	60	6Ꞇ	6∂	6ɕ	6٩	6ɕ	70	7Ꞇ	7∂	7ɕ	7٩	7ɕ	80	8Ꞇ	8∂	8ɕ	8٩	8ɕ	90	9Ꞇ	9∂	9ɕ	9٩	9ɕ	Ꞇ0
5ⱶ	5ɞ	63	6Ɛ	6ɕ	6ℓ	6Ƶ	6Ⅹ	79	7∂	7ɾ	7ɤ	7ⴼ	84	8ɛ	8ɞ	8Ƶ	8ɤ	8ⱶ	9Ꞇ	97	9ƨ	9ᗜ	9ϱ	Ꞇ5	Ꞇʃ	Ꞇʖ	Ꞇ₴	Ꞇɬ	0Ꞇ
6ɤ	6ɤ	6ℓ	6ɤ	70	7ɤ	7ɤ	7ℓ	7ɤ	80	8ɤ	8ɤ	8ℓ	8ɤ	90	9ɤ	9ɤ	9ℓ	9ɤ	0Ꞇ	ɤꞆ	ɤꞆ	ℓꞆ	ɤꞆ	0Ꞇ	ɤꞆ	ɤꞆ	ℓꞆ	ɤꞆ	ɤ0
6ᗜ	63	79	7ɬ	7ឧ	7ɤ	81	8Ɛ	8ʖ	8٩	8ⴼ	96	97	9ƨ	9ℓ	9Ⅹ	ƧꞆ	ƨꞆ	ŊꞆ	ƧꞆ	ƐꞆ	ʃꞆ	ɬꞆ	ƼꞆ	ƼꞆ	ɤ8	ɤ7	ɤ₭	ɤƵ	0∂
7Ɛ	7Ɛ	7ɕ	73	8Ꞇ	8ɤ	8₴	8ɞ	96	9∂	9₭	9ɤ	ƧꞆ	ʃꞆ	ƧꞆ	ɛꞆ	ⅩꞆ	ɤꞆ	ɛꞆ	9Ꞇ	ƧꞆ	ɤ8	ɤ₭	ɤℓ	ɤƧ	ɤ4	ɤ6	ɤ∂	ɤ∂	Ɛ0
7ℓ	80	8ɛ	8ɕ	8ℓ	90	9ɛ	9ɕ	9ℓ	Ꞇ0	ƧꞆ	ƧꞆ	ƧꞆ	0Ꞇ	ɛꞆ	ɛꞆ	ƧꞆ	ɤ0	ɤ9	ɤƧ	ɤℓ	0∂	9∂	ɕ∂	ℓ∂	0Ɛ	ɛƐ	ɛƐ	ƐƐ	0ɛ
8ʃ	8ƨ	8ɤ	94	9∂	9ℓ	9ɞ	Ƨ8	ƧƧ	Ꞇ٩	ƐƧ	ƤƧ	ƐƧ	ƤƧ	0ɤ	ʃɤ	ƨɤ	ɤɤ	∂4	∂∂	∂ℓ	∂ɞ	Ɛ8	ƐƐ	Ɛ٩	ƐƐ	ɕƐ	ɛƐ	ɛƤ	ʃ0
8ⱻ	94	97	9₴	9Ƽ	ƧƧ	ɬƧ	ɞƧ	ƐƧ	ƤƧ	ŊƧ	ƱƧ	ƤƧ	ƐƧ	ℓƧ	∂2	∂Ƥ	∂ℓ	∂ⴼ	ƐƐ	ʖƐ	ƐƤ	ɛ1	ɛϱ	ɛឧ	ɛɞ	ʃ9	ʃɞ	ʃᗜ	₭0
9ϱ	9ℓ	9ϱ	ƧƧ	ƼƧ	ƧƧ	Ƨ6	ƤƧ	ƐƧ	0Ƨ	ϱƧ	ℓƧ	ϱƧ	∂ɤ	∂ɕ	∂ɤ	ɤ6	ƐƐ	ƐƐ	0ɛ	ɛϱ	ɛℓ	ɛϱ	ʃɤ	ʃɕ	ʃɤ	₭6	₭ɤ	₭ɕ	ϱ0
9ɬ	8Ƨ	ℓʖ	ƐƧ	Ƨ5	ƤƧ	ᗜƧ	ƧƐ	ɤƧ	٩Ƨ	ⱶƧ	ϱɤ	Ŋɤ	Ɛɤ	Ɛɤ	₭Ɛ	ⴼƐ	ϑɛ	Ɩɛ	ϑ5	ϑʃ	ⱻʃ	Ƶʃ	ϑ₭	ϱ₭	ɛ₭	Ɛϱ	ɬϱ	₭ϱ	0ϱ
ɕƐ	ɕƧ	0Ꞇ	ƐƧ	ƤƧ	0Ɛ	ɤ∂	ɤƧ	0∂	∂∂	٩∂	0Ɛ	∂Ɛ	ƐƐ	0ɛ	ɛ∂	ɛƧ	0ʃ	ɕʃ	٩ʃ	0₭	₭∂	₭٩	0ϱ	ϱƐ	ϱƧ	0ᑫ	ᑫ∂	ᑫƤ	0∂
ɕ₭	ƐƧ	ⱻƧ	ϱƧ	ɤɤ	ϱɤ	ƦƐ	ϱ∂	ϑ∂	0Ɛ	ƐƐ	ƐƐ	ƐƐ	ƧƐ	ℓƐ	ʃɛ	ʖʃ	ƨʃ	9₭	ɕ₭	₭₭	ɤϱ	ⱻϱ	ϱϱ	ɛᑫ	ℓᑫ	Ʀᑫ	ϱ∂	ϑ∂	70
ɬƧ	ɛƐ	ɤɤ	Ɛɤ	ɕɤ	∂ɤ	₭∂	Ɛ∂	8Ɛ	ƐƧ	ɛ2	ƐƤ	ɞƐ	8ʃ	ʃɕ	ɞʃ	ƐƐ	ɕ₭	ⅩƐ	ϱɤ	ɤƐ	ƤƤ	ɞƤ	ƦƐ	ɤ∂	ƨ∂	ϱ∂	ʃ7	ƦƤ	ɬ0
ⴼƐ	ʃƐ	ϑƐ	∂2	ɕ∂	ɤ∂	ƐƐ	ƐƐ	ƐƐ	ɛƐ	ᗜɛ	ɛʃ	ɬʃ	ɞʃ	ɛ₭	₴₭	1ϱ	ɤϱ	Ƶϱ	ƤƧ	ⱻƤ	ƐƤ	Ƥ∂	ɕ∂	75	ƐƤ	₭Ƥ	Ɛɬ	Ŋɬ	0ɬ
ɤɤ	ɤɤ	ɕ∂	ℓ∂	0Ɛ	ƐƐ	ƐƐ	ɛɤ	ℓɛ	0ʃ	ɤʃ	ɤʃ	ɤ₭	ℓ₭	0ϱ	ɤϱ	ɤϱ	ɤƤ	ℓƤ	0∂	ɤ∂	ɤ∂	ɤƤ	ℓƤ	0ɬ	ɤɬ	ɤɬ	ɤɕ	ℓɕ	0ɤ
ƼƐ	∂∂	ℓ∂	ƐƐ	ឧƐ	0ɛ	ɕɛ	ɕɛ	ɛʃ	٩ʃ	₭5	ɕ₭	₭Ƽ	ϱϱ	ℓϱ	ƼƤ	ឧƤ	0∂	ɕ∂	ɕ∂	ɛ7	٩7	ɬ5	ɕɬ	Ƽɬ	∂ɕ	ℓɕ	Ƽɤ	ឧɤ	0ɕ
ɞ∂	ϱ∂	ƐƐ	ƤƐ	ƧƐ	ℓɛ	ʃ2	ƐƐ	ϱʃ	ɛ₭	₭ɓ	ɤϱ	ⱻϱ	74	ɕƤ	73	ɬ∂	ɤ∂	Ɛ7	٩7	6ɬ	ƨɬ	Ⅹɬ	ɤɕ	ʖɕ	ʃɤ	ɤɤ	8ɕ	₭ɕ	0ɞ
Ʀ∂	ɤƐ	₭Ɛ	ɛƐ	ℓɛ	ɤʃ	ϑʃ	6₭	ⱻ₭	0ϱ	ʖϱ	77	ɞƤ	ɤƤ	ɛ∂	ƨ∂	79	ℓ7	Ɛɬ	ɕɬ	Ʀɬ	ɤɕ	₭ɕ	ɤϱ	ℓɤ	ɤƐ	ϑƐ	6ɞ	Ƶɞ	0ʖ
ƐƐ	ƐƐ	ɛɤ	ɞɛ	ϑʃ	ɤʃ	ʃ₭	ɛ₭	ɤϱ	٩ϱ	ɤ8	ℓƤ	4∂	ƨ∂	07	ƐƐ	73	ɤɬ	ɞɬ	∂ɕ	ɤɕ	ʃɤ	ɛɤ	Ƨɤ	٩Ƨ	8ɞ	ℓɞ	ʖ4	ƨʖ	0Ɛ
4ⱶ	ϑƐ	ƤƐ	ɞʃ	Ƽʃ	ɤ₭	ⴼ₭	ɬϱ	₭ϱ	ϑƤ	Ɛ٩	ϱ∂	Ƶ∂	ʃ7	ℓ7	ƐƐ	ᗜɬ	ɤɕ	ƕɕ	Ƨɤ	ϑɤ	8Ƨ	ŊƧ	6ɞ	ឧɞ	ʖ4	ⱻʖ	2Ɛ	ɾƐ	0ɬ
ɛϑ	0ʃ	ɕʃ	0₭	ɕ₭	0ϱ	ɕϱ	0Ƥ	ɕƤ	0∂	ɕ∂	70	ɕ7	0ɬ	ɕɬ	0ɕ	ɕɕ	0ɤ	ɕɤ	0Ƨ	ɕƧ	0ɞ	ɕɞ	0ʖ	ɕʖ	0Ɛ	ɕƐ	0ɬ	ɕɬ	0Ƨ
ʃ1	ʃƨ	₭3	₭₭	Ƥ5	ϱℓ	Ƥ7	Ƥɤ	∂9	∂٩	7Ꞇ	7ɕ	ɬ∂	ɬƐ	ɕɛ	ɕɓ	ɤ₭	ɤɤ	ƧƤ	ƧƼ	ɞ7	ɞɞ	ʖʖ	ʖϱ	ƐƧ	Ɛ3	ɬʖ	ɬⅩ	ƧƼ	0ɾ
ʃƨ	₭4	₭ℓ	ϱ8	Ƥ٩	ƤƧ	Ƥɛ	∂ʃ	∂ɤ	7∂	7ɞ	ɬɤ	ɬ3	ɕƐ	ɤ0	ɤƨ	Ƨ4	Ƨℓ	ɞ8	ɞ٩	ʖɤ	ʖɛ	Ɛʃ	Ɛɤ	ɬ∂	ɬɞ	Ƨɤ	Ƨ3	Ɛɾ	0ƨ
₭3	₭ℓ	ϱ9	ϱƧ	Ƥɛ	ƤƧ	∂7	∂ϱ	7ʖ	ɬ0	ɬⱻ	ɕ6	ɕϑ	ɤɤ	ɤℓ	Ƨϱ	Ƨ₭	ɞɤ	ɞƦ	ʖƐ	Ɛ3	Ɛℓ	ɬ9	ɬƧ	Ƨ9	Ƨɤ	ɾ7	ɾϱ	ƨʖ	0ⱻ
₭₭	ϱ8	ϱƧ	Ƥʃ	ƤƧ	∂ɤ	∂Ⅹ	7ƨ	ɬ6	ɬ٩	ɕƐ	ɕɤ	ɤɬ	ɤ3	ƧƼ	ɞ4	ɞ₴	ʖɤ	ʖɓ	Ɛ9	Ɛϱ	ɬƐ	Ƨ2	Ƨℓ	ɾƧ	ɾ٩	ƨϱ	ƨɞ	ⱻɞ	0₭
Ƥ5	Ƥ٩	Ƥɛ	ƤƧ	∂Ƽ	70	7ឧ	ɬƧ	ɬℓ	ɕ∂	ɕƼ	ɤƼ	Ƨ5	Ƨ٩	ɞɛ	ɞƐ	ʖƼ	Ɛ0	Ɛឧ	ɬƧ	ɬℓ	Ƨɛ	ƧƼ	ɾƧ	ƨ5	ƨ٩	ⱻɛ	ⱻƧ	₭Ƽ	0ឧ
ϱℓ	Ƥɤ	Ƥɤ	∂ɤ	70	7ℓ	ɬɤ	ɬɤ	ɕɤ	ɤ0	ɤℓ	Ƨɤ	Ƨɤ	ɞɤ	ʖ0	ʖℓ	Ɛɤ	Ɛɤ	ɬɤ	Ƨ0	Ƨℓ	ɾɤ	ɾɤ	ƨɤ	ⱻ0	ⱻℓ	₭ɤ	₭ɤ	ឧɤ	ℓ0
Ƥ7	Ƥɛ	∂7	∂Ⅹ	7ឧ	ɬɤ	ɬɬ	ɕɾ	ɤʖ	ɤ٩	Ƨ₭	Ƨϱ	ɞɾ	ʖ8	ʖℓ	Ɛɬ	Ɛⱶ	ɬℓ	Ƨɕ	ƧƼ	ɾʖ	ƨ4	ƨⱶ	ⱻϱ	ⱻƼ	₭ʖ	ឧ9	ឧɓ	ℓʖ	Ŋ0
ƤƧ	∂ʃ	∂ϱ	7ƨ	ɬƧ	ɬɤ	ɕɕ	ɤƐ	ɤƧ	Ƨ9	ƧⅩ	ɞℓ	ʖƐ	ʖɞ	Ɛⱻ	ɬ8	ɬɓ	Ƨɤ	ɾ2	ɾ٩	ƨϱ	ƨ3	ⱻ₭	₭ɤ	₭Ƽ	ឧƐ	ℓ6	ℓɛ	Ŋɬ	₴0
∂9	∂ɤ	7ʖ	ɬ6	ɬℓ	ɕɤ	ɤʖ	ɤƧ	Ƨ7	ɞ0	ɞϑ	ʖϱ	ʖƦ	Ɛℓ	ɬɛ	ɬϱ	ƧƵ	ɾɤ	ɾ₭	ƨƐ	ⱻ9	ⱻɤ	₭ʖ	ឧ6	ឧℓ	ℓɤ	Ŋ3	ŊƧ	₴7	ϑ0
∂٩	7∂	ɬ0	ɬ٩	ɕ∂	ɤ0	ɤ٩	Ƨ∂	ɞ0	ɞ٩	ʖ∂	Ɛ0	Ɛ٩	ɬ∂	Ƨ0	Ƨ٩	ɾ∂	ƨ0	ƨ٩	ⱻ∂	₭0	₭٩	ឧ∂	ℓ0	ℓ٩	Ŋ∂	₴0	₴٩	ϑ∂	٩0
7Ꞇ	7ɞ	ɬⱻ	ɕƐ	ɕƼ	ɤℓ	Ƨ₭	ƧⅩ	ɞϑ	ʖ∂	Ɛ1	ƐƧ	ɬɕ	Ƨ4	Ƨℓ	ɾɞ	ƨ7	ƨɤ	ⱻɬ	₭Ƨ	₭₭	ឧƨ	ℓ∂	ℓϱ	Ŋឧ	₴ʃ	₴Ʀ	ϑ₴	٩Ʀ	ⱶ0
7ɕ	ɬɤ	ɕ6	ɕɤ	ɤƼ	Ƨɤ	Ƨϱ	ɞℓ	ʖϱ	Ɛ0	ƐƧ	ɬɤ	Ƨ6	ƧƧ	ɾƧ	ƨɤ	ƨϱ	ⱻℓ	₭ϱ	ɤ0	ឧƧ	ℓɤ	Ŋ6	Ŋɤ	₴Ƽ	ϑɤ	ϑϱ	٩ℓ	ⱶϱ	Ƨ0
ɬ∂	ɬ3	ɕϑ	ɤɬ	Ƨ5	ƧƧ	ɞɾ	ʖƐ	ʖƦ	Ɛ٩	ɬɕ	Ƨ6	Ƨ٩	ɾƨ	ƨɛ	ƨⅩ	ⱻⱶ	₭ɤ	ɤ7	ឧɕ	ℓⱻ	Ŋʃ	Ŋⱶ	₴Ƨ	ϑƐ	٩8	٩₭	ⱶ₭	Ƨ₭	ᗜ0
ɬ٩	ɕƐ	ɤɤ	ɤ3	Ƨ٩	ɞɤ	ʖ8	ʖɞ	Ɛℓ	ɬ∂	Ƨ4	Ƨɤ	ɾƨ	ƨʃ	ⱻ0	ⱻɛ	₭Ɛ	ឧɤ	ឧ3	ℓ٩	Ŋɤ	₴8	₴ɞ	ϑℓ	٩∂	ⱶ4	ⱶɤ	Ƨƨ	ᗜʃ	ɛ0
ɕɛ	ɤ0	ɤℓ	ƧƼ	ɞɛ	ʖ0	ʖℓ	ƐƐ	ɬɛ	Ƨ0	Ƨℓ	ɾƧ	ƨɛ	ⱻ0	ⱻℓ	₭Ƨ	ឧɛ	ℓ0	ℓℓ	ŊƧ	₴ɛ	ϑ0	ϑℓ	٩Ƨ	ⱶɛ	Ƨ0	Ƨℓ	ᗜƧ	ɛɛ	ℓ0
ɕɓ	ɤƨ	ƧƐ	ɞ4	ɞƧ	ʖℓ	Ɛɬ	ɬ8	ɬϱ	Ƨ٩	ɾɞ	ƨɤ	ƨⅩ	ⱻɛ	₭Ƨ	ឧʃ	ℓ2	ℓɤ	Ŋ₭	ɤɤ	ϑ6	ϑɞ	٩₴	ⱶɤ	ƧƐ	Ƨ3	ᗜƧ	ɛƐ	ƐƐ	ɞ0
ɤ₭	Ƨ4	Ƨ₭	ɞ₴	ʖƼ	Ɛɤ	Ɛⱶ	ɬɓ	ƧƵ	ɾϑ	ƨ7	ƨƧ	ⱻⱶ	₭Ɛ	ឧɛ	ℓ2	ℓ٩	Ŋϱ	₴ɬ	ϑƐ	ϑƦ	٩ɛ	ⱶƨ	ƐƧ	Ƨ5	ᗜɞ	ɛɛ	ɞɤ	ɞƧ	ƨ0
ɤɤ	Ƨℓ	ɞɤ	₭ɤ	Ɛ0	Ɛɤ	ɬℓ	ɬƧ	Ƨ0	٩0	ƨɤ	ⱻℓ	₭ɤ	ឧɤ	ℓ0	ℓɤ	Ŋℓ	Ʀɤ	ϑɤ	٩0	٩ɤ	ⱶℓ	Ƨɤ	ᗜɤ	ɛ0	ɛɤ	ɞℓ	ɞɤ	ƧƐ	Ƨ0
Ƨ7	ɞ8	ɞƦ	ʖɓ	Ɛឧ	ɬɤ	Ƨɛ	ɾƧ	ɾ₭	ƨ٩	Ɛ1	₭ƨ	ɓ7	ឧ3	ℓℓ	Ŋ₭	₴ɬ	ϑɤ	91	٩Ƨ	ⱶ9	ƧƐ	ᗜ₭	ɛɕ	ɛƼ	ɞɤ	ɞƦ	Ƶɬ	ƧꞆ	0ⱻ
ƧƼ	ɞ٩	ʖƧ	Ɛɛ	ɬƧ	ƨ0	ƨƧ	ⱻ٩	₭Ƨ	ⱻɛ	₭Ƨ	ឧ0	ឧƧ	ℓ٩	Ŋ5	₴ɤ	ϑƧ	٩0	٩Ƨ	ⱶ٩	ƧƧ	ᗜ0	ɛƧ	ɛ٩	ɛƧ	ɞ٩	ɞƧ	ⴼƧ	ⱻƧ	0Ƽ
ɞ7	ʖɤ	Ɛ3	Ɛℓ	ɬℓ	ɕℓ	ɾʖ	ƨɤ	ⱻ9	₭0	₭₭	ឧƧ	ℓⱻ	Ŋɤ	₴ɛ	96	ϑƦ	٩Ƥ	ⱶϑ	ƧƧ	ᗜ7	ɛɛ	ɛᗜ	ɛϱ	ɞℓ	Ƶℓ	ƨʖ	ⱻⱶ	ɛƐ	₭0
ɞɞ	ʖɛ	Ɛℓ	ɬƐ	Ƨ∂	ɾɤ	ƨ4	ƨ3	ⱻɤ	₭٩	ឧƨ	ℓɤ	Ŋʃ	₴8	90	ϑ6	٩ɛ	ⱶ₭	ƐƧ	ᗜ∂	ɛɤ	ɛɛ	ɛ3	ɞɤ	Ƶ٩	ƨɤ	ⱻɤ	ɕʃ	₭8	ɞ0
ʖʖ	Ɛʃ	ɬ9	Ƨ2	ƧƼ	ɾɤ	ƨⱶ	ⱻ₭	₭ʖ	ឧ∂	ℓ∂	Ŋ6	Ŋⱶ	₴Ƨ	ϑ6	٩ɛ	ⱶƧ	ƨɤ	ᗜ₭	Ƨ9	ɛᗜ	ɛ3	ɞɤ	ƵƧ	ƨɤ	ⱻɤ	Ƨɾ	₭Ɛ	ɛ7	ⴼ0
ʖϱ	Ɛɤ	ɬƧ	Ƨℓ	ɾƧ	ƨɤ	ⱻϱ	₭ɤ	ឧ6	ℓ0	ℓϱ	Ŋɤ	₴Ƨ	ϑϱ	٩Ƨ	ⱶɤ	Ƨϱ	ᗜɤ	ɛ6	ɛ0	ɛϱ	ɞɤ	ƵƧ	ƨℓ	ⱻƧ	₭Ƨ	₭ϱ	ɛɤ	ⴼ6	ϱ0
ƐƧ	ɬ∂	Ƨɛ	ɾƧ	ƨ5	ⱻ0	ⱻƼ	₭Ƨ	ឧℓ	ℓ٩	Ŋឧ	₴Ƨ	ϑƤ	٩∂	ⱶ٩	ƧƐ	ᗜ5	ɛ0	ɛƧ	ɛƧ	ɞℓ	ƵƧ	ƨឧ	ⱻƧ	ƧƼ	₭Ƨ	ɛϑ	ⴼƧ	ϱ5	Ƽ0
Ɛ3	ɬɞ	ƧƐ	ɾ٩	ƨ٩	ⱻℓ	₭ƨ	ឧƐ	ℓɤ	Ŋ∂	₴ʃ	ϑɤ	٩8	ⱶ4	Ƨ0	ᗜ3	ɛɞ	ɛϱ	ɛϱ	ɞ٩	Ƶℓ	ƨƧ	ⱻɛ	ƧƤ	₭∂	ɛʃ	ⴼɤ	ϱ8	Ƽ4	30
ɬʖ	ƧƐ	ɾ7	ƨϱ	ⱻɛ	₭Ƨ	ឧ9	ℓ6	Ŋ3	₴0	ϑƦ	٩ϱ	ⱶ٩	ƧƧ	ᗜℓ	ɛƧ	ɛϑ	ɛℓ	ɞƧ	ƵƧ	ƨɤ	ⱻɤ	Ƨ7	₭ϱ	ɛϑ	ⴼɤ	29	Ƽ6	33	Ʀ0
ɬⅩ	Ƨ3	ɾϱ	ƨɞ	ⱻƧ	₭Ƨ	ឧɤ	ℓϱ	ŊƧ	₴٩	ϑƦ	٩ϱ	ⱶ₭	ƧƐ	ᗜƧ	ɛƤ	ɛℓ	ɛɤ	ƵƐ	ƨɤ	ⱻϱ	Ƨƨ	₭Ɛ	ɛɤ	ⴼƧ	ϱ8	Ƽ6	34	Ʀ2	Ⅹ0
ƧƼ	ɾƐ	ƨʖ	ⱻɞ	₭Ƨ	ឧɤ	ℓʖ	Ŋⱶ	₴7	ϑ∂	٩Ʀ	ⱶϱ	ƧƐ	ᗜʃ	ɛɛ	ɛƐ	ɛƐ	ƵƧ	ƨƐ	ⱻɤ	Ƨ9	₭8	ɛɤ	ⴼ6	ϱ5	Ƽ4	33	Ʀ2	Ⅹ1	ⱶ0
ɾ0	ƨ0	ⱻ0	₭0	ឧ0	ℓ0	Ŋ0	₴0	ϑ0	٩0	ⱶ0	Ƨ0	ᗜ0	ɛ0	ℓ0	ɞ0	Ƶ0	ɤ0	ɬ0	Ƨ0	₭0	ɞ0	ⴼ0	ϱ0	Ƽ0	30	Ʀ0	Ⅹ0	ⱶ0	100

Octal
(Base 8)

T	4	T	2	T	4/3	T	1
1	2	3	4	5	6	7	10
2	4	6	10	12	14	16	20
3	6	11	14	17	22	25	30
4	10	14	20	24	30	34	40
5	12	17	24	31	36	43	50
6	14	22	30	36	44	52	60
7	16	25	34	43	52	61	70
10	20	30	40	50	60	70	100

Hexadecimal

(Base 16)

T	8	T	4	T	8/3	T	2	T	8/5	T	4/3	T	8/7	T	1
1	2	3	4	5	6	7	8	9	↊	ᒣ	ɣ	∂	↋	ɘ	10
2	4	6	8	↊	ɣ	↋	10	12	14	16	18	1↊	1ɣ	1↋	20
3	6	9	ɣ	ɘ	12	15	18	1ᒣ	1↋	21	24	27	2↊	2∂	30
4	8	ɣ	10	14	18	1ɣ	20	24	28	2ɣ	30	34	38	3ɣ	40
5	↊	ɘ	14	19	1↋	23	28	2∂	32	37	3ɣ	41	46	4ᒣ	50
6	ɣ	12	18	1↋	24	2↊	30	36	3ɣ	42	48	4↋	54	5↊	60
7	↋	15	1ɣ	23	2↊	31	38	3ɘ	46	4∂	54	5ᒣ	62	69	70
8	10	18	20	28	30	38	40	48	50	58	60	68	70	78	80
9	12	1ᒣ	24	2∂	36	3ɘ	48	51	5↊	63	6ɣ	75	7↋	87	90
↊	14	1↋	28	32	3ɣ	46	50	5↊	64	6↋	78	82	8ɣ	96	↊0
ᒣ	16	21	2ɣ	37	42	4∂	58	63	6↋	79	84	8ɘ	9↊	↊5	ᒣ0
ɣ	18	24	30	3ɣ	48	54	60	6ɣ	78	84	90	9ɣ	↊8	ᒣ4	ɣ0
∂	1↊	27	34	41	4↋	5ᒣ	68	75	82	8ɘ	9ɣ	↊9	ᒣ6	ɣ3	∂0
↋	1ɣ	2↊	38	46	54	62	70	7↋	8ɣ	9↊	↊8	ᒣ6	ɣ4	∂2	↋0
ɘ	1↋	2∂	3ɣ	4ᒣ	5↊	69	78	87	96	↊5	ᒣ4	ɣ3	∂2	↋1	ɘ0
10	20	30	40	50	60	70	80	90	↊0	ᒣ0	ɣ0	∂0	↋0	ɘ0	100

Multiplication Table Analysis.

Circles indicate periods. Phases are shaded triangular areas. The width of the phases relates to the magnitude of the unit digit. Multiphase cycles feature a phase which begins on a period and one which ends on a period. Cycles which start and end on a period and include one uptrending phase are divisors of the base.

Notation above the table indicates the integral reciprocal divisor for divisor factors. A "T" in a circle marks totative factors. The simplified ratio $d' / r = d / m_n$ appears above non totative non divisor factors. The periods of totative factors 1 and $r - 1$ are indicated, but all other totative factors are marked with a simple gray stroke.

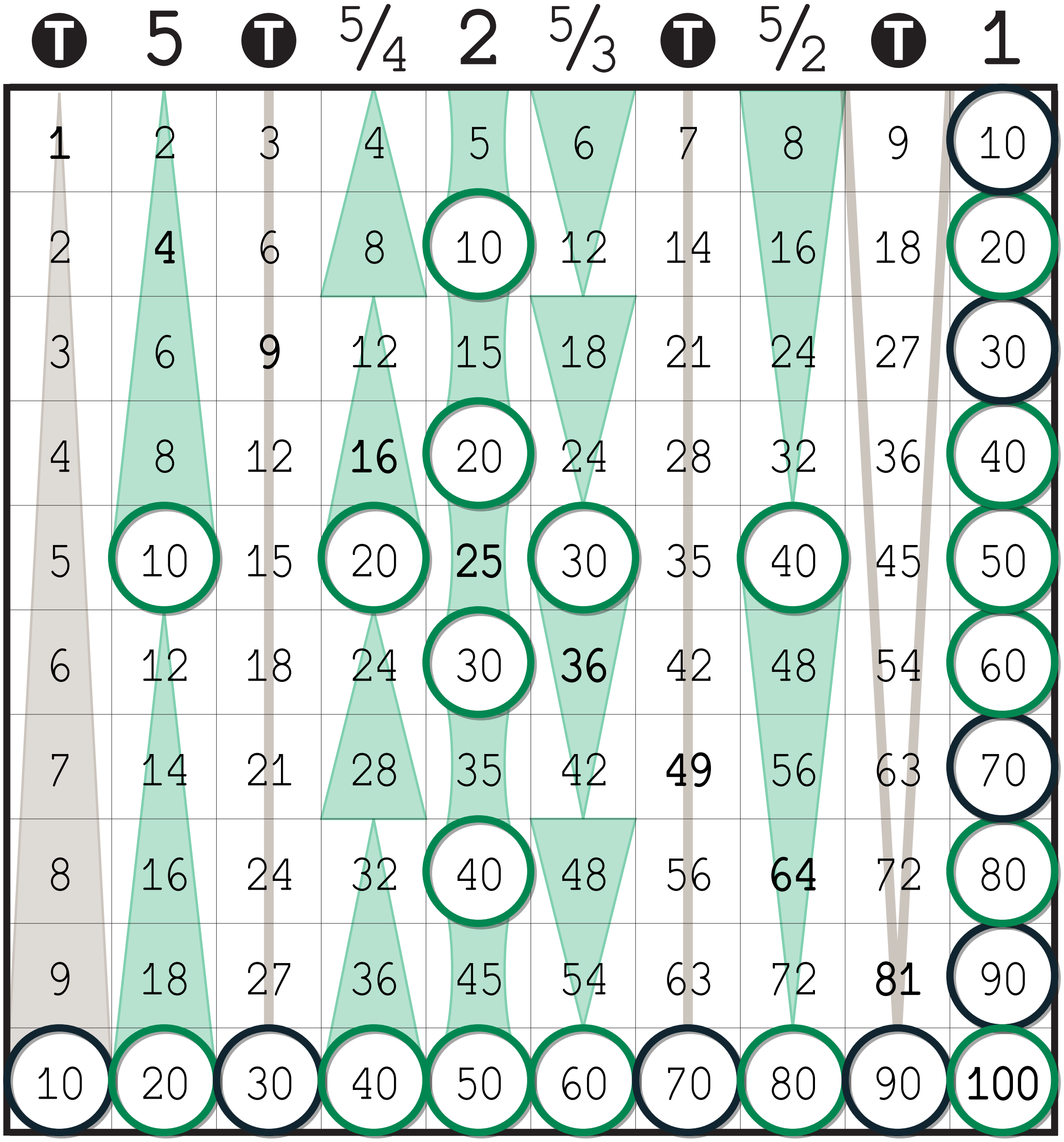
Dozenal

(Base 12)

T	6	4	3	T	2	T	3/2	4/3	6/5	T	1
1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	4	6	8	χ	10	12	14	16	18	1χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	χ	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
χ	18	26	34	42	50	5χ	68	76	84	92	χ0
Ɛ	1χ	29	38	47	56	65	74	83	92	χ1	Ɛ0
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	100

Decimal

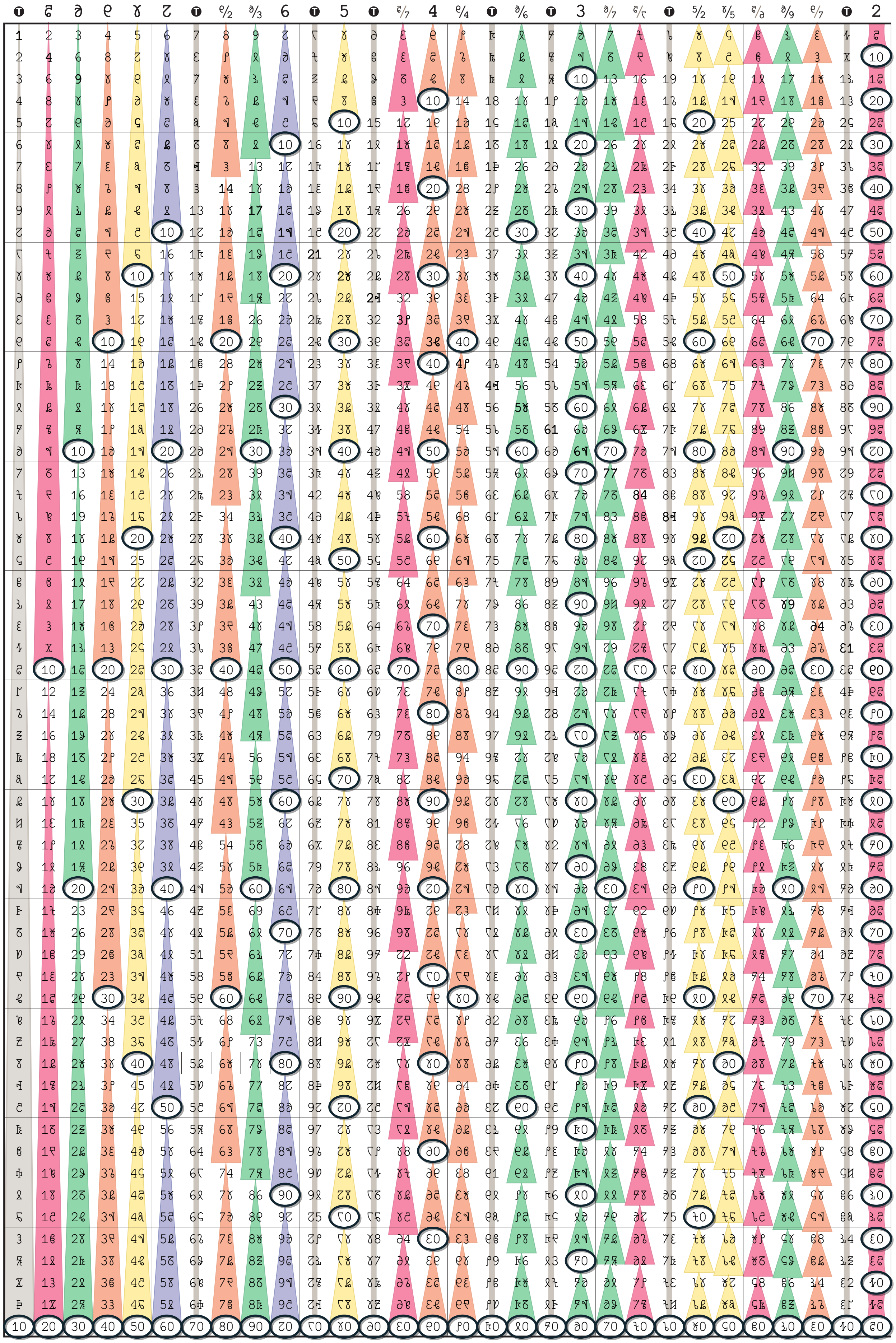
(Base 10)



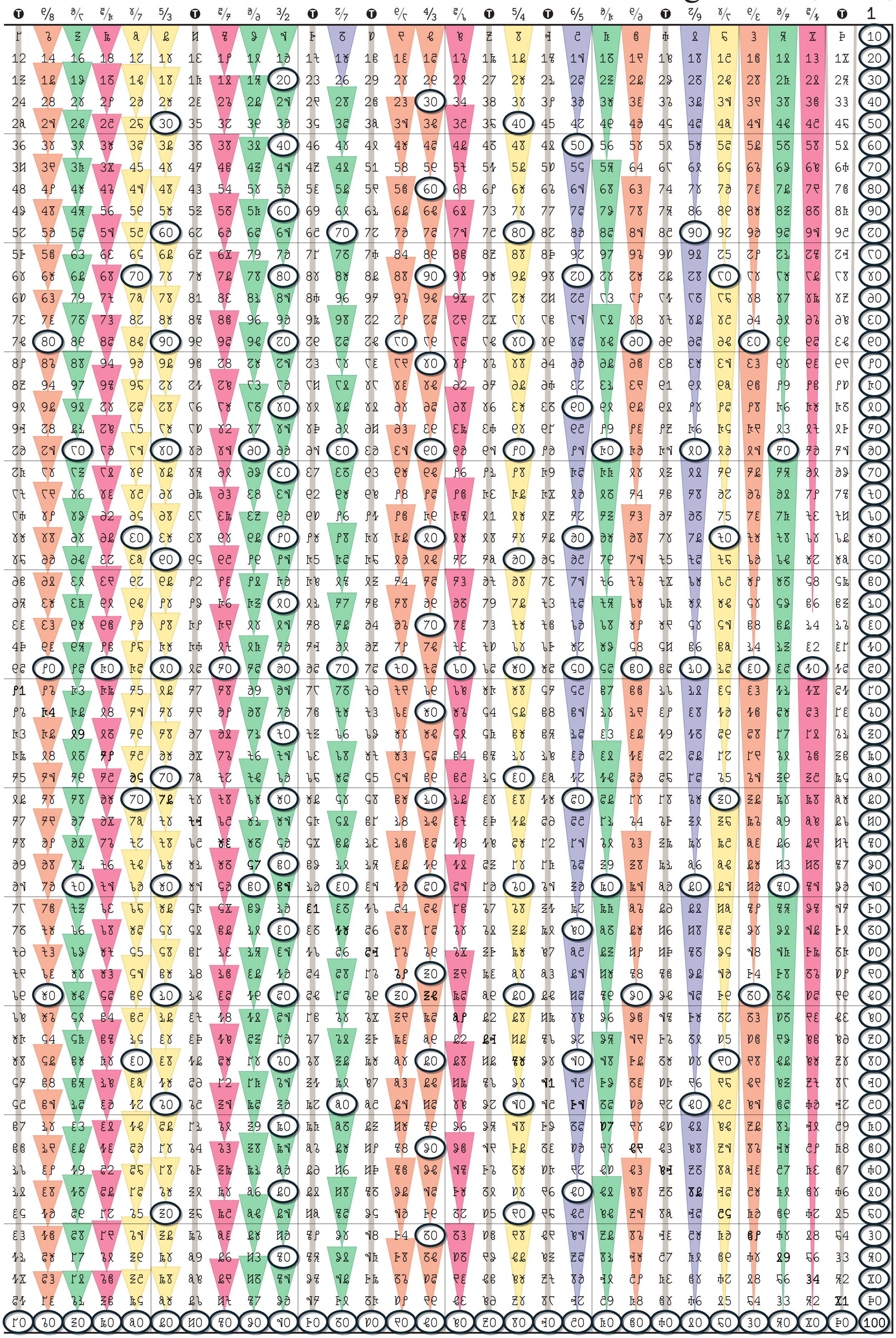
Multiplication Table Analysis.

Circles indicate periods. Phases are shaded triangular areas. The width of the phases relates to the magnitude of the unit digit. Multiphase cycles feature a phase which begins on a period and one which ends on a period. Cycles which start and end on a period and include one uptrending phase are divisors of the base.

Notation above the table indicates the integral reciprocal divisor for divisor factors. A “T” in a circle marks totative factors. The simplified ratio $d' / r = d / m_n$ appears above non totative non divisor factors. The periods of totative factors 1 and $r - 1$ are indicated, but all other totative factors are marked with a simple gray stroke.



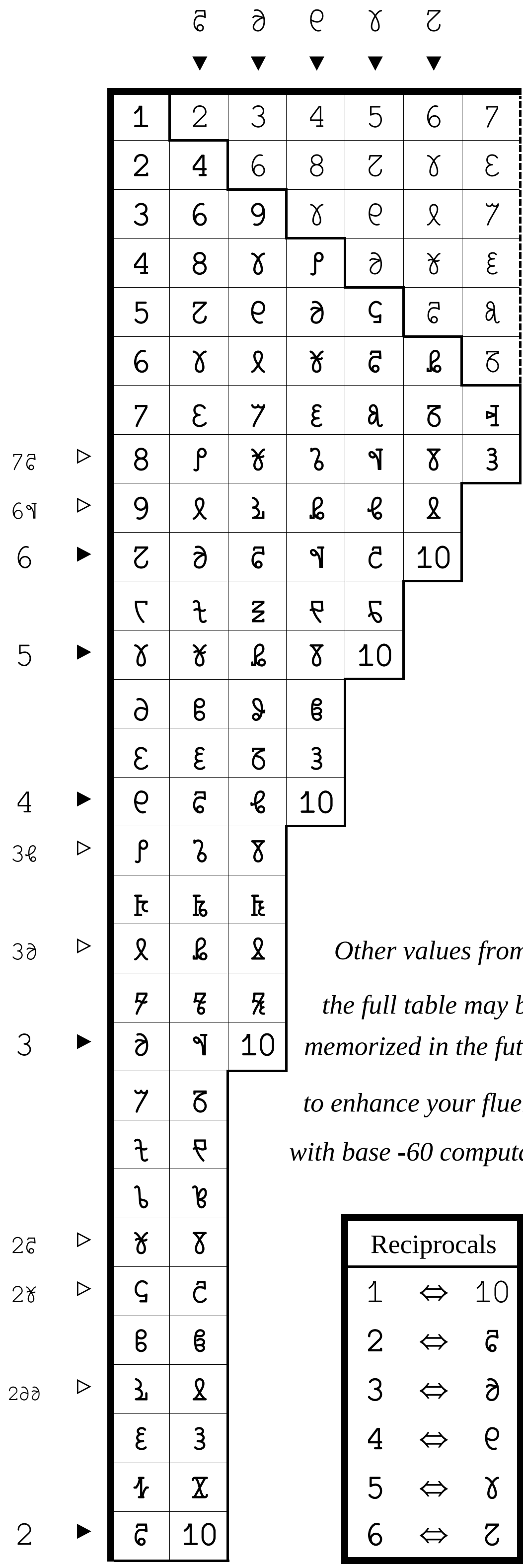
Sexagesimal (Base 60)



		30 ▼	20 ▼	15 ▼	12 ▼	10 ▼	
	1	2	3	4	5	6	7
	2	4	6	8	10	12	14
	3	6	9	12	15	18	21
	4	8	12	16	20	24	28
	5	10	15	20	25	30	35
	6	12	18	24	30	36	42
	7	14	21	28	35	42	49
	8	16	24	32	40	48	56
	9	18	27	36	45	54	
6 ►	10	20	30	40	50	60	
	11	22	33	44	55		
5 ►	12	24	36	48	60		
	13	26	39	52			
	14	28	42	56			
4 ►	15	30	45	60			
	16	32	48				
	17	34	51				
	18	36	54				
	19	38	57				
3 ►	20	40	60				
	21	42					
	22	44					
	23	46					
	24	48					
	25	50					
	26	52					
	27	54					
	28	56					
	29	58					
2 ►	30	60					

Reciprocals		
1	⇔	60
2	⇔	30
3	⇔	20
4	⇔	15
5	⇔	12
6	⇔	10

Sexagesimal Multiplication Table using Reciprocal Divisors



Other values from the full table may be memorized in the future to enhance your fluency with base -60 computation

Reciprocals		
1	⇔	10
2	⇔	{kinex}
3	⇔	{score}
4	⇔	{trick}
5	⇔	{zen}
6	⇔	{dess}

Argam Numeral Set and Names appropriate for Base-60.

0 zero	score	kinoct
1 one	tress	alume
2 two	dell	exeff
3 three	flore	silick
4 four	cadex	cadell
5 five	quint	kinove
6 six	dithe	diore
7 seven	trine	foss
8 eight	cadeff	exoct
9 nine	neve	effent
dess	kinex	kiness
ell	sode	trizote
zen	cadoct	cadithe
thise	trell	sull
zeff	dizote	exove
trick	kineff	kinell
tess	exent	sevoct
zote	mack	triax
dine	dax	dineve
ax	trithe	clore

1 Class I computations involve factors whose product is represented in the table. An example is 9*5 = {kinove}

2 Class II computations involve the usage of reciprocal divisors for operations involving at least one divisor in order to obtain carries for products greater than 10. Example is:
{score}*{zen} = 10({zen}/3) = 40

3 Class III computations extend the basic set of reciprocal divisors to their simple multiples by splitting the operation into two parts. An example is:
{flore}*{kinoct} = 10({flore}/3)*2 = 7{kinoct}*2 = {trick}{score}.

4 Class IV computations use addition to reach factors that are near the divisors:
{clore}*{silick} = ({kinoct}*{silick})+{silick} = (10({silick}/3)*2)+{silick} = ({zeff}{score}*2)+{silick} = {cadeff}{kinoct}+{silick} = {neve}{flore}.

Base 16

			8		4
			▼		▼
ʆ		1	2	3	4
		2	4	6	8
		3	6	9	ɣ
4	▶	4	8	ɣ	10
		5	ᘔ	ɘ	
		6	ɣ		
		7	Ɛ		
2	▶	8	10		

	ʆ	
4	⇔	4
2	⇔	8

Base 18

			9	6	
			▼	▼	
ℓ		1	2	3	4
		2	4	6	8
		3	6	9	ɣ
		4	8	ɣ	ʆ
		5	ᘔ	ɘ	
3	▶	6	ɣ	10	
		7	Ɛ		
		8	ʆ		
2	▶	9	10		

	ℓ	
3	⇔	6
2	⇔	9

Base 20

			ᘔ		5
			▼		▼
∂		1	2	3	4
		2	4	6	8
		3	6	9	ɣ
		4	8	ɣ	ʆ
4	▶	5	ᘔ	ɘ	10
		6	ɣ	ℓ	
		7	Ɛ		
		8	ʆ		
		9	ℓ		
2	▶	ᘔ	10		

	∂	
4	⇔	5
2	⇔	ᘔ

Base 24

			ɣ	8	6
			▼	▼	▼
ɤ		1	2	3	4
		2	4	6	8
		3	6	9	ɣ
		4	8	ɣ	ʆ
		5	ᘔ	ɘ	∂
4	▶	6	ɣ	ℓ	10
		7	Ɛ	ʔ	
3	▶	8	ʆ	10	
		9	ℓ		
		ᘔ	∂		
		Ɩ	ɫ		
2	▶	ɣ	10		

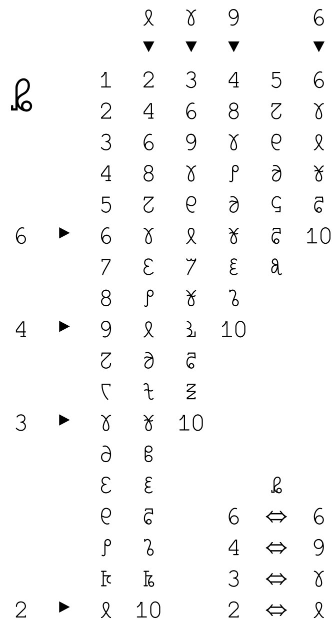
	ɤ	
4	⇔	6
3	⇔	8
2	⇔	ɣ

Base 30

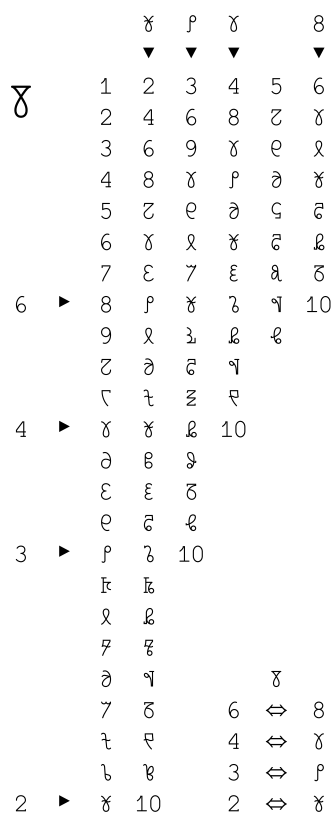
			ɘ	ᘔ		6
			▼	▼		▼
ɞ		1	2	3	4	5
		2	4	6	8	ᘔ
		3	6	9	ɣ	ɘ
		4	8	ɣ	ʆ	∂
		5	ᘔ	ɘ	∂	ς
5	▶	6	ɣ	ℓ	ɤ	10
		7	Ɛ	ʔ	ε	
		8	ʆ	ɤ		
		9	ℓ	ȝ		
3	▶	ᘔ	∂	10		
		Ɩ	ɫ			
		ɣ	ɤ			
		∂	ɞ			
		Ɛ	ε			
2	▶	ɘ	10			

	ɞ	
5	⇔	6
3	⇔	ᘔ
2	⇔	ɘ

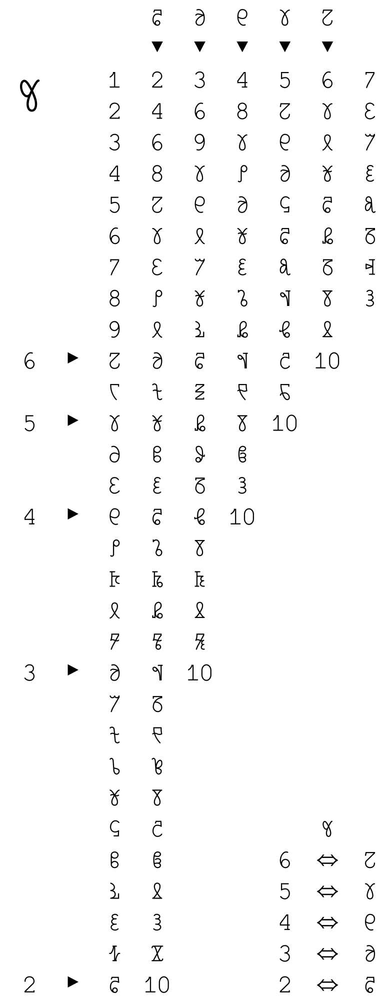
Base 36



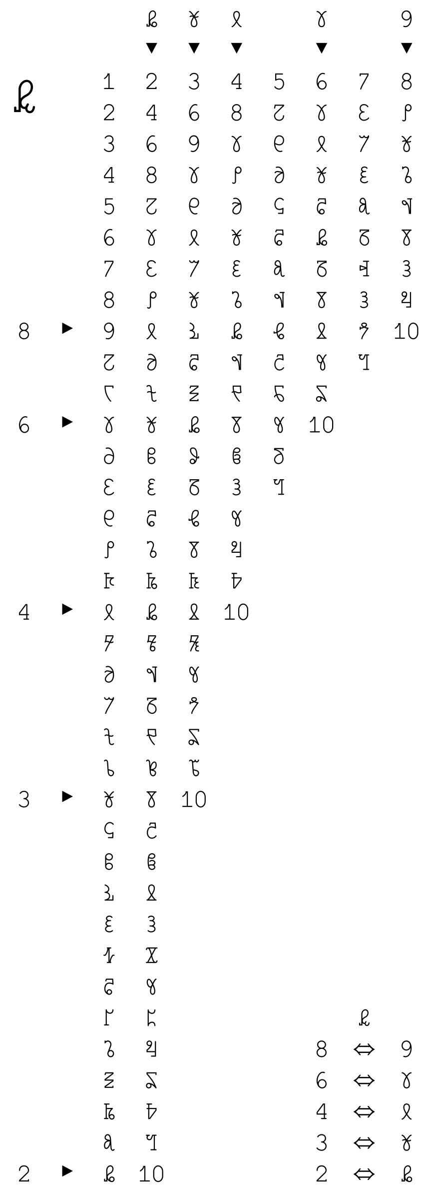
Base 48



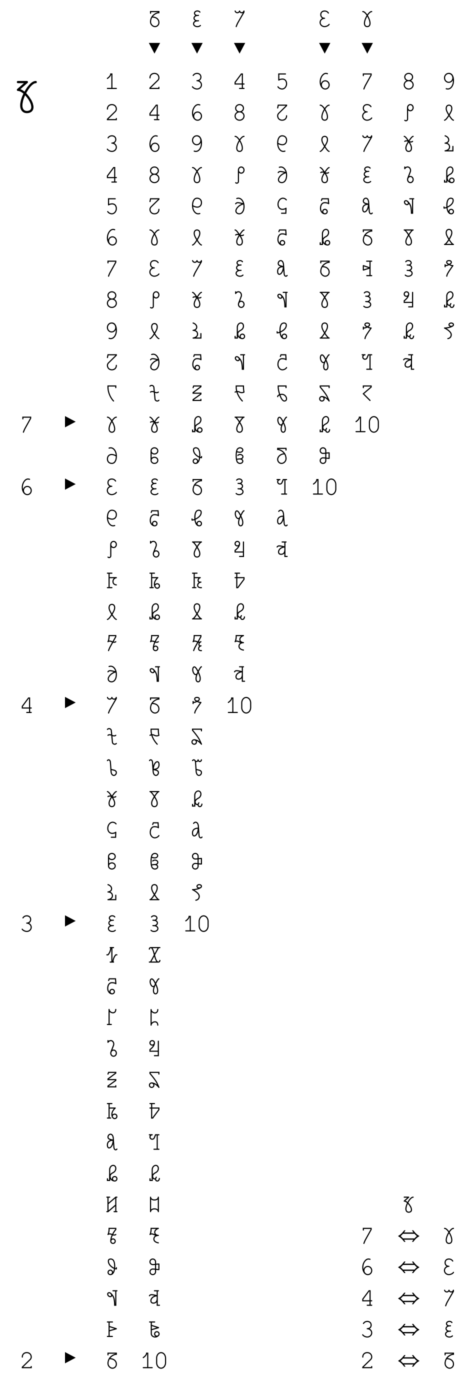
Base 60



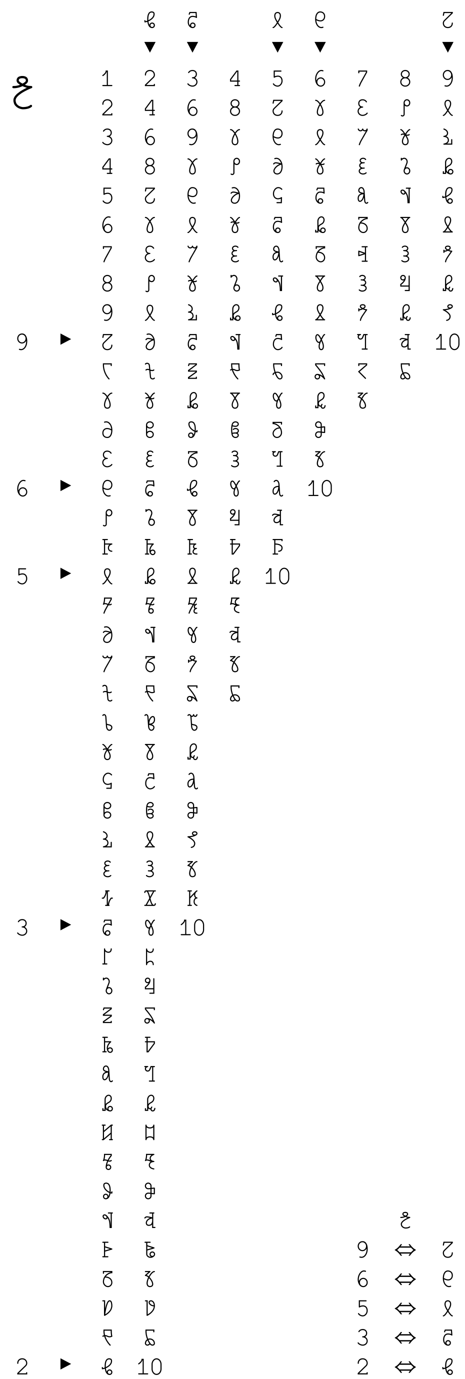
Base 72



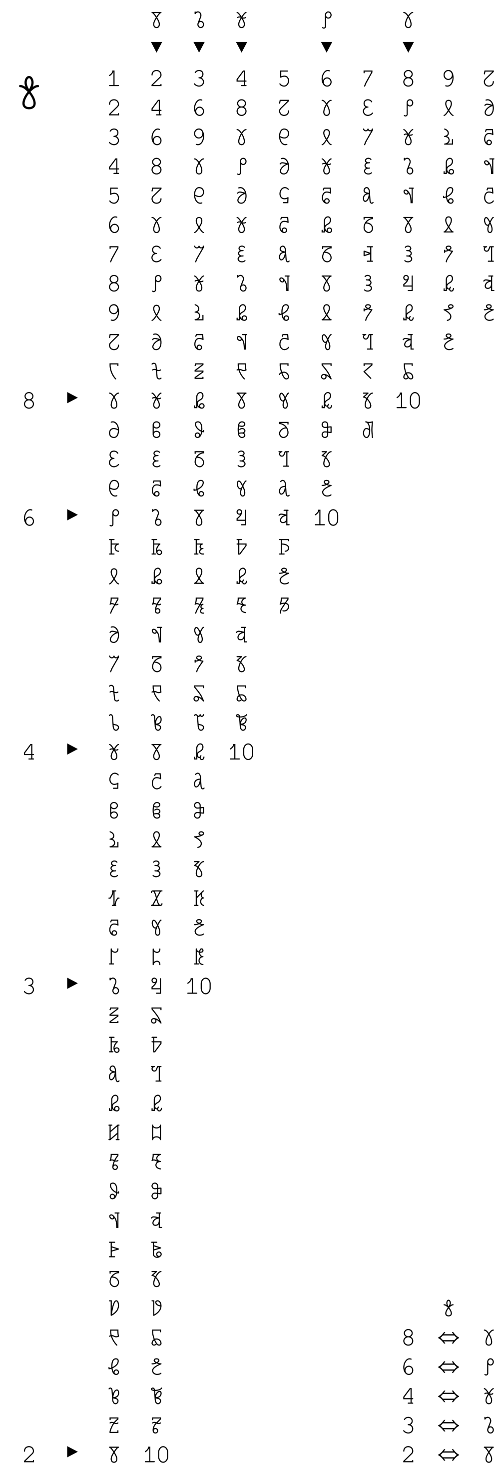
Base 84



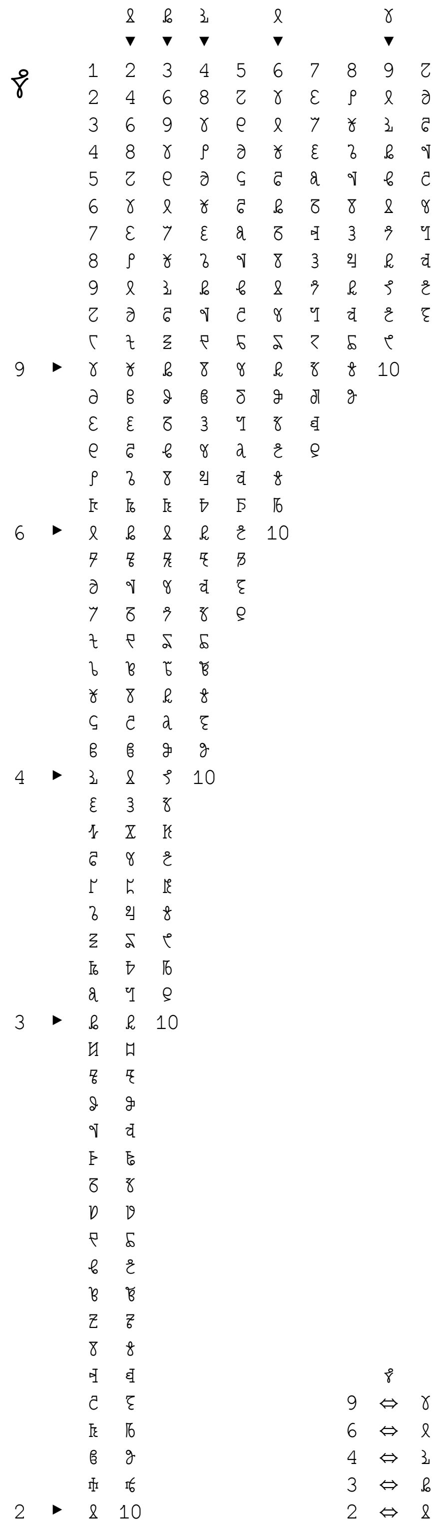
Base 90



Base 96



Base 108



Base 120

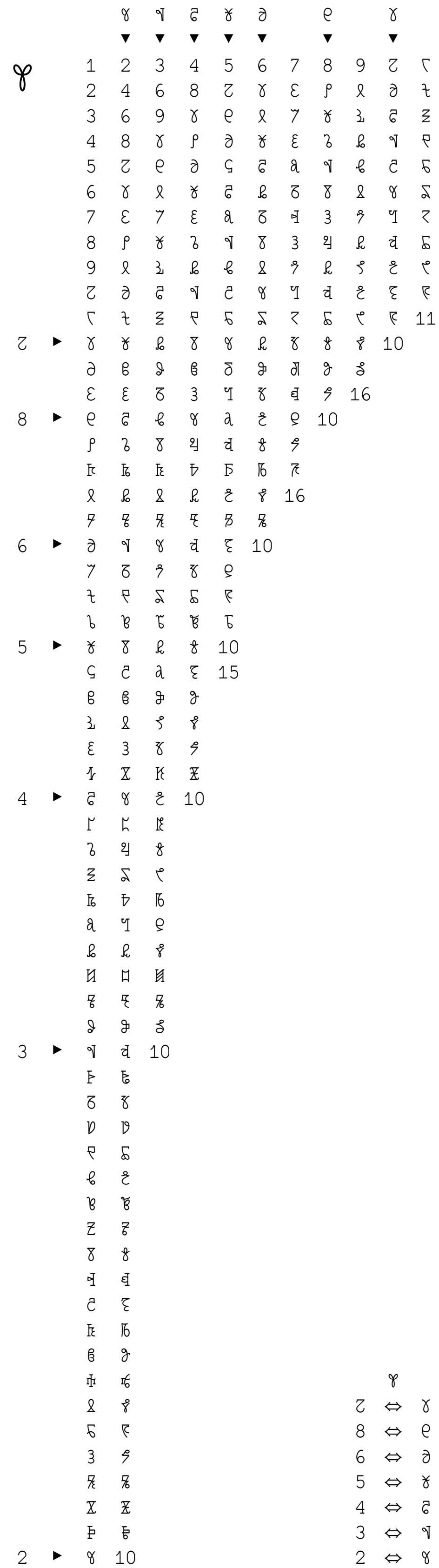


Table of the First 120 Argam and Selected Highly Composite Integers

Name	Number
zero	0
one	1
two	2
three	3
four	4
five	5
six	6
seven	7
eight	8
nine	9
dess	10
ell	11
zen	12
thise	13
zeff	14
trick	15
tess	16
zote	17
dine	18
ax	19
score	20
tress	21
dell	22
flore	23
cadex	24
quint	25
dithe	26
trine	27
cadeff	28
neve	29
kinex	30
sode	31
cadoct	32
trell	33
dizote	34
kineff	35
exent	36
mack	37
diax	38
trithe	39
kinoct	40
alume	41
exeff	42
silick	43
cadell	44
kinove	45
diore	46
foss	47
exoct	48
effent	49
kiness	50
trizote	51
cadithe	52
sull	53
exove	54
kinell	55
sevoct	56
triax	57
dineve	58
clore	59
shock	60
ark	61
disode	62
senove	63
octent	64
kinithe	65
exell	66
kale	67
cazote	68
triore	69
sevess	70
calse	71
octove	72
scand	73
dimack	74
kinchick	75
cadax	76
sevell	77
exithe	78
tite	79
kintess	80
novent	81
dilume	82
van	83
sezen	84
kinote	85
disill	86
trineve	87
octell	88
crome	89
novess	90
sevithe	91
caflore	92
trisode	93
difoss	94
kinax	95
extess	96
mang	97
effendi	98
novell	99
kent	100
ferr	101
exote	102
cobe	103
octithe	104
sechick	105
disull	106
nick	107
catrine	108
cupe	109
dessell	110
trimack	111
setess	112
zinn	113
exax	114
kinore	115
cateve	116
novithe	117
diclore	118
sevote	119
hund	120
cadexeff	168
catkinove	180
kinexoct	240
exsevoct	336
kinoctove	360
exsevess	420
exoctess	480
sevoctove	504
cadexquint	600
senovess	630
exdessell	660
exeftess	672
octovess	720
seveszen	840
noveszen	1080
catkisenove	1260
novestess	1440
sechitess	1680
kintestrine	2160
kinsevoctove	2520
sevoctovess	5040
sevoctovessell	55440
sevoctovessellithe	720720

Argam Test Page, sort by superior highly composite, A002201 |18 September 2007|

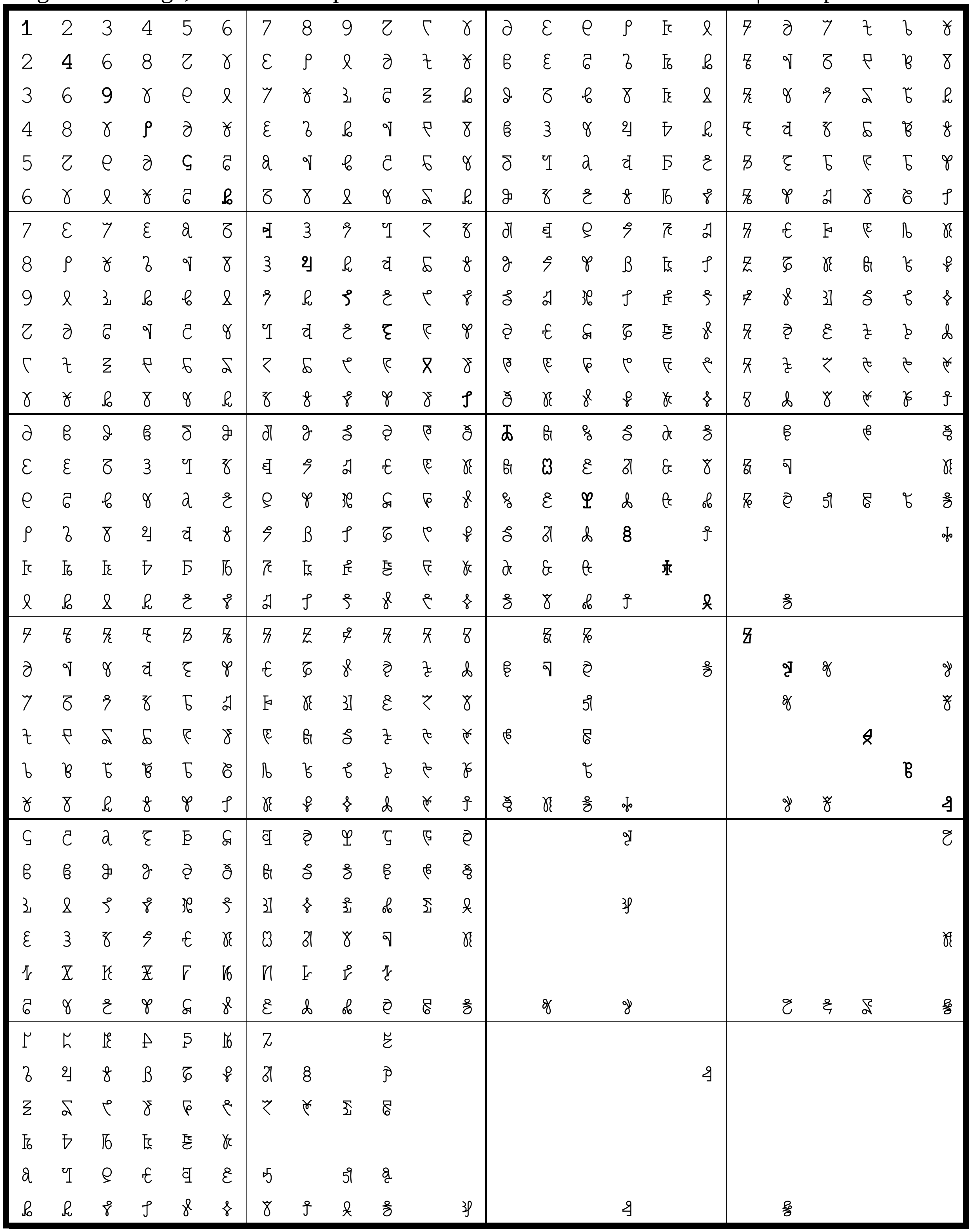
0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	348
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	277	[illegible]	301	313	325	[illegible]	349
2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	278	[illegible]	302	314	326	338	[illegible]
3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	279	291	303	[illegible]	327	339	351
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	268	[illegible]	[illegible]	304	[illegible]	328	340	352
5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	269	281	293	305	317	329	341	353
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	282	294	306	318	[illegible]	342	354
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	271	283	295	307	319	331	[illegible]	355
8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	272	284	[illegible]	308	[illegible]	332	344	356
9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	273	285	[illegible]	309	321	[illegible]	[illegible]	357
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	274	[illegible]	298	[illegible]	322	334	346	358
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	299	311	323	335	347	359

[illegible]	372	[illegible]	396	408	[illegible]	[illegible]	444	456	468	[illegible]	492	[illegible]	516	528	540	552	564	[illegible]	588	[illegible]	612	624	636	648	[illegible]	[illegible]	684	696	708
[illegible]	[illegible]	385	397	409	421	433	445	457	469	481	493	505	517	[illegible]	541	553	565	577	589	601	613	[illegible]	637	649	661	673	685	697	709
362	374	386	398	410	422	434	446	458	470	482	494	506	518	530	542	554	566	578	590	602	614	626	638	650	662	674	686	698	710
363	375	387	399	411	423	435	447	459	471	483	495	507	519	531	543	555	567	579	591	603	615	627	639	651	663	675	687	699	711
364	376	388	[illegible]	412	424	436	448	460	472	484	496	508	520	532	544	556	568	580	592	604	616	628	640	652	664	676	688	700	712
365	377	389	401	413	425	437	449	461	473	485	497	509	521	533	545	557	569	581	593	605	617	629	641	653	665	677	689	701	713
366	378	390	402	414	426	438	450	462	474	486	498	510	522	534	546	558	570	582	594	606	618	[illegible]	642	654	666	678	690	702	714
367	379	391	403	415	427	439	451	463	475	487	499	511	523	535	547	559	571	583	595	607	619	631	643	655	667	679	691	703	715
368	380	392	404	416	428	440	452	464	476	488	500	[illegible]	524	536	548	560	572	584	596	608	620	632	644	656	668	680	692	704	716
369	381	393	405	417	429	441	453	465	477	489	501	513	525	537	549	561	573	585	597	609	621	633	645	657	669	681	693	705	717
[illegible]	382	394	406	418	430	442	454	466	478	490	502	514	526	538	550	562	574	586	598	610	622	634	646	658	670	682	694	706	718
371	383	395	407	419	431	443	455	467	479	491	503	515	527	539	551	563	575	587	599	611	623	635	647	659	671	683	695	707	719

[illegible]	732	744	756	768	780	792	804	816	828	[illegible]	852	864	876	888	900	912	924	936	948	[illegible]	972	984	996	1008	1020	1032	1044	1056	1068
721	733	745	757	769	781	793	805	817	829	841	853	865	877	889	901	913	925	937	949	961	973	985	997	1009	1021	1033	1045	1057	1069
722	734	746	758	770	782	794	806	818	830	842	854	866	878	890	902	914	926	938	950	962	974	986	998	1010	1022	1034	1046	1058	1070
723	735	747	759	771	783	795	807	819	831	843	855	867	879	891	903	915	927	939	951	963	975	987	999	1011	1023	1035	1047	1059	1071
724	736	748	760	772	784	796	808	820	832	844	856	868	880	892	904	916	928	940	952	964	976	988	[illegible]	1012	1024	1036	1048	1060	1072
725	737	749	761	773	785	797	809	821	833	845	857	869	881	893	905	917	929	941	953	965	977	989	1001	1013	1025	1037	1049	1061	1073
726	738	750	762	774	786	798	810	822	834	846	858	870	882	894	906	918	930	942	954	966	978	990	1002	1014	1026	1038	1050	1062	1074
727	739	751	763	775	787	799	811	823	835	847	859	871	883	895	907	919	931	943	955	967	979	991	1003	1015	1027	1039	1051	1063	1075
728	740	752	764	776	788	800	812	824	836	848	860	872	884	896	908	920	932	944	956	968	980	992	1004	1016	1028	1040	1052	1064	1076
729	741	753	765	777	789	801	813	825	837	849	861	873	885	897	909	921	933	945	957	969	981	993	1005	1017	1029	1041	1053	1065	1077
730	742	754	766	778	790	802	814	826	838	850	862	874	886	898	910	922	934	946	958	970	982	994	1006	1018	1030	1042	1054	1066	1078
731	743	755	767	779	791	803	815	827	839	851	863	875	887	899	911	923	935	947	959	971	983	995	1007	1019	1031	1043	1055	1067	1079

2 6 [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

2 3 4 6 8 [illegible]
[illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]



Sexagesimal Powers of Popular Bases

	8 (eight)	Ƨ (dess = 10)	ɣ (zen = 12)	ʃ (tess = 16)
ɤ	2 Ƨ9ƽƧ ꞓ&ɤɣ 6ꜰ8ʃ	7ꝺ ɓИƧ7 ꝺ4ꜰŧ ꟻИꝺ٩	Ƨ8٩ ǝƧꝺ7 ꜰɑ&ɑ Ɛŧꞓ&	2 ɣƽ99 ɓИ34 ꝶꝺɑŧ ꜰꜰɣʃ
ɓ	ʃʃƐƐ Ƨƽ31 &ɣ8ɣ	ꞓ 37ꝺ0 ꞓ&ꝺɣ ŧꝶꝺ٩	ƧꝺƐ 7٩ǝƧ 3ꝺXƧ ꝶɤƐɣ	ƧɼŧꝺƐ ɣꝺИ9 ꝶɑƐɣ &ŧƐʃ
ŧ	221ɣ &ǝ9ꞓ ƽ414	4 ɑꝶƧꝶ 4ɑ&٩ 3ꜰꝺ٩	4ǝ &ɤꜰ9 Ɛƽ7ꜰƧ ƐƧƧɤ	ꝺꝺɑ ꝺƧꝺꜰ ƽɤꝺ1 ꜰ1ɤʃ
ꝛ	999 ꜰɣꝛƽ ǝ9ƽ8	ꝺƐꟻ1 ɣꝺɣ٩ ꝺИꝺ٩	7 843ꞔ ꝺǝꝛꝺ Ƨǝƽɣ	2Ɛ5 ꝶ3ꞔǝ ꞔ92ꜰ 73ǝʃ
ǝ	1ꝶɤ Ƨꟻ3ꜰ ꞔ13ʃ	2ꞓŧꝶ ƧƧƧʃ 2ꝶꝺ٩	1 ꞓ٩ǝꜰ 8ꜰ7ꝶ 939&	99 ŧŧʃꜰ ɑꝶꝶƽ 57ʃʃ
ꝶ	Ɛꝶ 1ƐИꜰ ƽИ٩ɓ	ʃɣꜰ ꞔ54ɼ &ꜰꝺ٩	8ɣ7ꜰ ꞔ٩9ꝶ Ƨꞓ٩ɣ	ꜰ ƽꝶŧɣ 9ɣꝺꞔ 3ꜰɣʃ
ꝶ	1ꝶ 9Ƨ4٩ ꜰꝺǝ4	1ꝺǝ ٩ƽƽꝺ 9Иꝺ٩	٩1ɣ ꝺ8ꜰꝶ Ƨ3ꝶɤ	2 Ƨ9ƽƧ ꞓ&ɤɣ 6ꜰ8ʃ
ꜰ	ǝ ɤɓŧ5 Иꞔꝶ8	9ƽ ŧꝺ32 ƽꝶꝺ٩	3٩9 29ꝺ8 ƽꝶꝺɣ	887Ɛ ꞔ71ƽ ɣʃ4ʃ
ʃ	1 ٩ɣꜰ8 ɣƧ٩ʃ	ꜰ ɣƐƽꝶ ʃꜰꝺ٩	ꝶǝ ꞓƧꝶИ ɣꜰꝶ&	ƽƽꝺ 953Ƨ ٩ɼ0ʃ
9	ɣꜰ7ɓ ɼɼǝɓ	5 ꝶǝƐǝ ꟻИꝺ٩	1ɼ ꝺꞓ3ꝶ 62ꝺɣ	1ꝶɤ Ƨꟻ3ꜰ ꞔ13ʃ
Ɛ	1ꜰ99 88ꞔ4	ɑꝺǝꟻ ŧꝶꝺ٩	7 ꝶꝶꟻƽ ꞓ3ƽɤ	79 0٩ꝶꜰ ꝶɣ6ʃ
ǝ	ɼꝺꜰ ꞔɣꝶ8	3ꜰǝ4 3ꜰꝺ٩	ꝶǝƐ8 ꝶƧɼɣ	6 ɣƧꝺƧ 3ꜰɣʃ
ɣ	1Ɛŧ ꞔꝺɣʃ	7ɣ0 ꝺИꝺ٩	3ƧǝƧ ƽɤꝶ&	1 ٩ɣꜰ8 ɣƧ٩ʃ
ꝛ	Ƨ2 ɣɣ3ɓ	28& 2ꝶꝺ٩	9ƽƽ ŧꜰ4ɣ	6ꜰ3ꜰ ꞓꞓ٩ʃ
Ƨ	1ŧ ꜰ1И4	ɣꝶ &ꜰꝺ٩	1ꝶИ ꜰꝺƧɤ	ɓꝶX ꜰ&&ʃ
9	Ƨ 7ŧƽ8	1ꜰ 9Иꝺ٩	6ꝶ 76ꝶɣ	1Ɛŧ ꞔꝺɣʃ
8	1 ꜰ٩ǝʃ	7 ƽꝶꝺ٩	ꝶ Ƨ97&	5ɼ ɤ6Ɛʃ
7	9ƽɣɓ	ꝺꜰꝺ٩	2 ꞓŧʃɣ	ǝ ƽꞓɤʃ
6	1ɣꟻ4	4Иꝺ٩	ǝꟻ6ɤ	1 ꜰ٩ǝʃ
5	968	ꝺꝺ٩	197ɣ	4ꜰʃʃ
4	18ʃ	2ꝺ٩	5ꞓ&	ꝶɣʃ
3	8ɓ	ʃ٩	Ɛɣ	18ʃ
2	14	1٩	2ɤ	4ʃ
1	8	Ƨ	ɣ	ʃ
0	1	1	1	1
−1	0.7ƽ	0.6	0.5	0.3ꞓ
−2	0.039	0.0&	0.0ꞔ	0.0Ɛ3ꞓ
−3	0.0716 ƽ	0.03&	0.025	0.006٩ 3ꞓ
−4	0.006٩ 3ꞓ	0.007&	0.00Ƨꞔ	0.003ꜰ ꞓƐ3ꞓ
−5	0.006ɑ ƽƐ7ƽ	0.0029 &	0.0006 5	0.000ɣ 7ɤИ٩ 3ꞓ
−6	0.000ꟻ 6ꝶƽ3 9	0.000ɣ ꝶ&	0.0004 ǝꞔ	0.0000 ꝺǝꝶꜰ ƽƐ3ꞓ
−7	0.0006 ƧƧꝶ6 16ƽ	0.0001 ꜰꞓ&	0.0000 7ƽ5	0.0000 2ŧɣꞔ ꝺ8ŧ٩ 3ꞓ
−8	0.0000 ꝺǝꝶꜰ ƽƐ3ꞓ	0.0000 7ꝺꝶ&	0.0000 1ɣƽꞔ	0.0000 0Ƨꜰꝺ &6ꝺꞔ 9Ɛ3ꞓ

Study of the third rank divisors versus prime factorization shape

six
23

2	3	6		2	3
		1	1000	3	3
1		2	300	2	3
	1	3	200	3	2
2		4	130	1	3
1	1	10	100	2	2
3		12	43		3
	2	13	40	3	1
2	1	20	30	1	2

dess (10)
25

2	5	Ƨ		2	5
		1	1000	3	3
1		2	500	2	3
2		4	250	1	3
	1	5	200	3	2
3		8	125		3
1	1	10	100	2	2
2	1	20	50	1	2
	2	25	40	3	1

zeff (14)
27

2	7	Ɛ		2	7
		1	1000	3	3
1		2	700	2	3
2		4	370	1	3
	1	7	200	3	2
3		8	137		3
1	1	10	100	2	2
2	1	20	70	1	2
	2	37	40	3	1

zen (12)
2^2 3

2	3	Ɣ		2	3
		1	1000	6	3
1		2	600	5	3
	1	3	400	6	2
2		4	300	4	3
1	1	6	200	5	2
3		8	160	3	3
	2	9	140	6	1
2	1	10	100	4	2
4		14	90	2	3
1	2	16	80	5	1
3	1	20	60	3	2
	3	23	54	6	
5		28	46	1	3
2	2	30	30	4	1

score (20)
2^2 5

2	5	∂		2	5
		1	1000	6	3
1		2	Ƨ00	5	3
2		4	500	4	3
	1	5	400	6	2
3		8	2Ƨ0	3	3
1	1	Ƨ	200	5	2
4		ʃ	150	2	3
2	1	10	100	4	2
	2	15	ʃ0	6	1
5		1ɣ	ɣƧ	1	3
3	1	20	Ƨ0	3	2
1	2	2Ƨ	80	5	1
6		34	65	1	3
4	1	40	50	2	2

cadeff (28)
2^2 7

2	7	ɛ		2	7
		1	1000	6	3
1		2	Ɛ00	5	3
2		4	700	4	3
	1	7	400	6	2
3		8	3Ɛ0	3	3
1	1	Ɛ	200	5	2
4		ʃ	170	2	3
2	1	10	100	4	2
5		14	ɤƐ	1	3
	2	17	ʃ0	6	1
3	1	20	Ɛ0	3	2
6		28	ɣ7		3
1	2	3Ɛ	80	5	1
4	1	40	70	2	2

exent (36)
2^2 3^2

2	3	ɓ		2	3
		1	1000	6	6
1		2	ℓ00	5	6
	1	3	ɣ00	6	5
2		4	900	4	6
1	1	6	600	5	5
3		8	4ℓ0	3	6
	2	9	400	6	4
2	1	ɣ	300	4	5
4		ʃ	290	2	6
1	2	ℓ	200	5	4
3	1	ɤ	1ℓ0	3	5
	3	ʓ	1ɣ0	6	3
5		ƨ	14ℓ	1	6
2	2	10	100	4	4
4	1	1ɣ	ʓ0	2	5
1	3	1ℓ	ɤ0	5	3
6		1ɛ	∂9		6
3	2	20	ℓ0	3	4
	4	29	ʃ0	6	2
5	1	2ɤ	∂ℓ	1	5
2	3	30	ɣ0	4	3
4	2	40	90	2	4
1	4	4ℓ	80	5	2
6	1	5ɣ	6ʓ		5
3	3	60		3	3

kent (100)
2^2 5^2

2	5	Ꜫ		2	5
		1	1000	6	6
1		2	Ƹ00	5	6
2		4	ϛ00	4	6
	1	5	∂00	6	5
3		8	ɣƸ0	3	6
1	1	Ƨ	Ƨ00	5	5
4		ʃ	6ϛ0	2	6
2	1	∂	500	4	5
	2	ϛ	400	6	4
5		ƨ	3ɣƸ	1	6
3	1	ɋ	2Ƹ0	3	5
1	2	Ƹ	200	5	4
6		ყ	13ϛ		6
5	1	ժ	1ϛ0	1	5
2	2	10	100	4	4
	3	1ϛ	ժ0	6	3
5	1	1ɤ	ʗƸ	1	5
3	2	20	Ƹ0	3	4
1	3	2Ƹ	ɋ0	5	3
6	1	3∂	ɼϛ		5
4	2	40	ϛ0	2	4
2	3	50	∂0	4	3
	4	6ϛ	ʃ0	6	2
5	2	80	ɣƸ	1	4
3	3	Ƨ0			

zeffent (196)
2^2 7^2

2	7	Ʊ		2	7
		1	1000	6	6
1		2	ɗ00	5	6
2		4	ꟻ00	4	6
	1	7	ɛ00	6	5
3		8	ɤɗ0	3	6
1	1	Ɛ	Ɛ00	5	5
4		ʃ	ɣꟻ0	2	6
2	1	ɛ	700	4	5
5		ƨ	6ɤɗ	1	6
	2	ꟻ	400	6	4
3	1	ʒ	3ɗ0	3	5
6		ყ	3ɣꟻ		6
1	2	ɗ	200	5	4
4	1	ϟ	1ꟼ0	2	5
2	2	10	100	4	4
5	1	1ɛ	ϟɗ	1	5
	3	1ꟼ	ϟ0	6	3
3	2	20	ɗ0	3	4
6	1	2ʒ	ƃꟼ		5
1	3	3ɗ	ʒ0	5	3
4	2	40	ꟻ0	2	4
2	3	70	ɛ0	4	3
5	2	80	ɤɗ	1	4
	4	ɣꟻ	ʃ0	6	2
3	3	Ɛ0			

Study of the third rank divisors versus prime factorization shape

trick (15)

35

3	5	ϱ		3	5
		1	1000	3	3
1		3	500	2	3
	1	5	300	3	2
2		9	1Ƨ0	1	3
1	1	10	100	2	2
	2	1Ƨ	90	3	1
3		1ɣ	85		3
2	1	30	50	1	2

tress (21)

37

3	7	ʔ		3	7
		1	1000	3	3
1		3	500	2	3
	1	5	300	3	2
2		9	1Ƨ0	1	3
1	1	10	100	2	2
	2	1Ƨ	90	3	1
3		1ɣ	85		3
2	1	30	50	1	2

kinove (45)

3^2 5

3	5	ɞ		3	5
		1	1000	6	3
1		3	ϱ00	5	3
	1	5	900	6	2
2		9	500	4	3
1	1	ϱ	300	5	2
	2	ϛ	1ɞ0	6	1
3		ʓ	1ϭ0	3	3
2	1	10	100	4	2
1	2	1ϭ	ʓ0	5	1
4		1ɞ	ϛ0	2	3
	3	2ɑ	ʃ9	6	
3	1	30	ϱ0	3	2
2	2	50	90	4	1
5		5ℓ	8ϱ	1	3

senove (63)

3^2 7

3	7	ʔ		3	7
		1	1000	6	3
1		3	700	5	3
	1	7	900	6	2
2		9	700	4	3
1	1	ʔ	300	5	2
3		ʓ	270	3	3
	2	ꟾ	1ℓ0	6	1
2	1	10	100	4	2
4		1ℓ	ꟾ0	2	3
1	2	27	ʓ0	5	1
3	1	30	70	3	2
5		3ℓ	ʃ7	1	3
2	2	70	90	4	1
6		Γɞ	5ɛ		3

Study of the third rank divisors versus prime factorization shape

kinex (30)

2 3 5

2	3	5	Ƨ		2	3	5
			1	1000	3	3	3
1			2	e00	2	3	3
	1		3	Ƨ00	3	2	3
2			4	7e0	1	3	3
		1	5	600	3	3	2
1	1		6	500	2	2	3
3			8	3ɫe		3	3
	2		9	3Ƨ0	3	1	3
1		1	Ƨ	300	2	3	2
2	1		ɣ	2e0	1	2	3
	1	1	e	200	3	2	2
1	2		ℓ	1∂0	2	1	3
2		1	∂	1e0	1	3	2
3	1		ɤ	17e		2	3
		2	ς	160	3	3	1
	3		ʅ	13Ƨ	3		3
1	1	1	10	100	2	2	2
2	2		16	ς0	1	1	3
3		1	1Ƨ	ɫe		3	2
	2	1	1e	∂0	3	1	2
1		2	1∂	ℓ0	2	3	1
1	3		1ɤ	ʃ∂	2		3
2	1	1	20	e0	1	2	2
	1	2	2e	ɣ0	3	2	1
1	2	1	30	Ƨ0	2	1	2
2		2	3Ƨ	90	1	3	1
3	1	1	40	7e		2	2
		3	4e	6∂	3	3	
1	1	2	50	60	2	2	1

shock (60)

2^2 3 5

2	3	5	ɣ		2	3	5
			1	1000	6	3	3
1			2	Ꙅ00	5	3	3
	1		3	∂00	6	2	3
2			4	e00	4	3	3
		1	5	ɣ00	6	3	2
1	1		6	Ƨ00	5	2	3
3			8	7Ꙅ0	3	3	3
	2		9	6ꝗ0	6	1	3
1		1	Ƨ	600	5	3	2
2	1		ɣ	500	4	2	3
	1	1	e	400	6	2	2
4			ʃ	3ɕ0	2	3	3
1	2		ℓ	3∂0	5	1	3
2		1	∂	300	4	3	2
3	1		ɤ	2Ꙅ0	3	2	3
		2	ς	2ɤ0	6	3	1
	3		ʅ	2∂∂	6		3
1	1	1	Ꙅ	200	5	2	2
5			ʓ	1ßꙄ	1	3	3
2	2		ß	1ꝗ0	4	1	3
3		1	ꝗ	1Ꙅ0	3	3	2
	2	1	ɕ	1∂0	6	1	2
4	1		ȣ	1e0	2	2	3
1		2	Ɛ	1ɣ0	5	3	1
1	3		ℓ	16ꝗ	5		3
2	1	1	10	100	4	2	2
6			14	3e		3	3
3	2		1ɣ	Ɛ0	3	1	3
	1	2	1e	ȣ0	6	2	1
4		1	1∂	ɕ0	2	3	2
1	2	1	1Ꙅ	ꝗ0	5	1	2
5	1		1ß	ꞪꙄ	1	2	3
2		2	1ꝗ	ß0	4	3	1
2	3		1ȣ	Ʒ∂	4		3
3	1	1	20	Ꙅ0	3	2	2
		3	25	Ɛȣ	6	3	
4	2		2ɤ	ς0	2	1	3
2	1	2	2Ꙅ	ɤ0	4	2	1
2	2	1	30	∂0	4	1	2
3		2	3∂	ℓ0	3	3	1
	2	2	3ɕ	ʃ0	6	1	1
4	1	1	40	e0	2	2	2
2	1	2	50	ȣ0	4	2	1
3	2	1	60	Ƨ0	3	1	2
4		2	6ꝗ	90	2	3	1
1	2	2	7Ꙅ	80	5	1	1

Study of the factorization of integers which set or tie records for the number of divisors. The notation below the argam indicates the exponents of the primes which compose the integer. The notation situates two at the right, zeros holding places for skipped primes.

σ Number of Divisors A002183	Highly Composite A005179							
two 2	two 2 1	three 3 10						
three 3	four 4 2							
four 4	six 6 11	eight 8 3	dess (10) 101					
six 6	zen (12) 12	dine (18) 21	score (20) 102					
eight 8	cadex (24) 13	kinex (30) 111						
nine 9	exent (36) 22							
dess (10)	exoct (48) 14							
zen (12)	shock (60) 112	octove (72) 23	sezen (84) 1012	novess (90) 121	extess (96) 15	catrine (108) 32		
tess (16)	hund (120) 113	cadexeff (168) 1013						
dine (18)	catkinove (180) 122							
score (20)	kinexoct (240) 114	exsevoct (336) 1014						
cadex (24)	kinoctove (360) 123	exsevess (420) 1112	exodess (480) 115	sevoctove (504) 1023	cadexquint (600) 213	senovess (630) 1121	exdessell (660) 10112	exeftess (672) 1015
kinex (30)	octovess (720) 124							
cadoct (32)	seveszen (840) 1113	noveszen (1080) 133						
exent (36)	catkisenove (1260) 1122	novestess (1440) 125						
kinoct (40)	sechitess (1680) 1114	kintestrine (2160) 134						
exoct (48)	kinsevoctove (2520) 1123							

Study of the divisors of the first integers which either set or meet records for total number of divisors.

two

2 | 1 / 2

three

3 | 1 / 3

four

4 | 1 / 4 | 2