Constitutive Counting Functions for Primorials.

Michael Thomas De Vlieger · St. Louis, Missouri · 21 June 2023.

Abstract.

We present several multiplicatively constitutive counting functions applied to primorials. Primorials represent local minima of Euler's totient $\phi(n)$, while they occur among local maxima of the regular counting function $\theta(n)$ = A010846(n).

Introduction.

Consider k and n, nonzero positive integers. Here we are interested only in those $k \le n$.

Consider n as a product of prime power factors p^ε, where $n = 1$ is the empty product, a product of no primes at all.

$$n = \prod_{i=1}^{\omega} p_i^{\varepsilon_i}, \text{ prime } p \mid n,\ \omega = \omega(n). \qquad [1.1]$$

Let RAD(n) = A7947(n) = $\varkappa$ be the squarefree kernel of n as below:

$$\varkappa = \prod_{i=1}^{\omega} p_i, \text{ prime } p \mid n,\ \omega = \omega(n). \qquad [1.2]$$

A totative of n (or reduced residue mod n) is $k < n$ such that $(k, n) = 1$, i.e., k is coprime to n. The set $\check{T}_n$ of totatives of n (or reduced residue system RRS) is defined below:

$$\begin{aligned}\check{T}_n &= \{\, k : k \perp n \wedge k < n \,\}\\ &= \text{row } n \text{ of A038566, i.e., V20.}\end{aligned} \qquad [1.3]$$

We define an n-regular number k as k such that RAD(k) $\mid n$, that is, the squarefree kernel A7947(k) divides n. Then we have a set $\check{R}_n$ of n-regular k such that $k \le n$.

$$\begin{aligned}\check{R}_n &= \{\, k : k \parallel \varkappa \wedge k \le n\}.\\ &= \{\, k \in \bigotimes_{p\mid\varkappa}\{p^\varepsilon : \varepsilon \ge 0\} : \wedge\, k \le n \,\}\\ &= \text{row } n \text{ of A162306, i.e., V30.}\end{aligned} \qquad [1.4]$$

We define an n-semitotative k as k such that $(k, n) > 1$, but there is some factor $q > 1$ such that $q \mid k$ but $q \nmid n$. We may derive the set of n-semitotatives k, $\check{S}_n$, from the union of $\check{R}_n$ and $\check{T}_n$ as follows:

$$\begin{aligned}\check{S}_n &= \{\, k : k \lozenge n \wedge k < n \,\}\\ &= \{\, k : k < n \,\} \setminus (\check{R}_n \cup \check{T}_n)\\ &= \text{row } n \text{ of A272619, i.e., V60.}\end{aligned} \qquad [1.5]$$

It is easy to show that, outside of the empty product 1 being at once n-coprime and n-regular, these multiplicatively constitutive categories are exclusive of one another.

We note that divisors $d \mid n$ are a special case of n-regular k such that $k \le n$. We can write a list $\boldsymbol{D}_n$ = row n of A027750, i.e., V10, of the divisors of n.

As a consequence of RAD(k) $\mid n$, it is clear that $k \mid n^\varepsilon$ such that $\varepsilon \ge 0$. Regarding $k \in \check{R}_n$, we can improve this to have n-regular $k \mid n^{(\tau(k)-1)}$, where $\tau(n)$ is the divisor counting function (see [2.1]). The definition of divisor has $d \mid n$, $0 \le \varepsilon \le 1$. This demonstrates that we can divide n-regular numbers into 2 species, divisors $d \in \boldsymbol{D}$ such that $d \mid n$, and nondivisors $k \in \boldsymbol{Đ}$ such that $k \mid n^\varepsilon$ such that $\varepsilon > 1$. Nondivisor n-regular numbers we call semidivisors. It is clear that all n-regular k such that $k > n$ are semidivisors. It can be shown that semidivisors $k < n$ exist only for n such that $\omega(n) > 1$, that is, $n \in$ A024619. We can list semidivisors k that are no larger than n as in $\check{Đ}_n$, that is, row n of A272618, i.e., V50. Therefore, we have the following:

$$\check{R}_n = \boldsymbol{D}_n \cup \check{Đ}_n. \qquad [1.7]$$

Counting functions.

Define the divisor counting function $\tau(n)$ to be as follows:

$$\begin{aligned}\tau(n) &= \prod_{i=1}^{\omega} (\varepsilon_i + 1)\\ &= \bigotimes_{i=1}^{\omega} \{p_i^{\delta_i} : \delta_i = 0 \ldots \varepsilon_i\}.\end{aligned} \qquad [2.1]$$

Define the totative counting function $\phi(n)$ as follows:

$$\begin{aligned}\phi(n) &= \text{V21}(n) = |\, \check{T}_n \,|\\ &= n \prod_{p\mid n} (1 - 1/p).\end{aligned} \qquad [2.2]$$

This function is better known as the Euler totient function.

Define the regular counting function $\theta(n)$ as follows:

$$\begin{aligned}\theta(n) &= \text{A010846}(n) = \text{V31}(n) = |\, \check{R}_n \,|\\ &= \sum_{t<n}^{t\perp n} \mu(t) \times \lfloor n/t \rfloor,\ t \in \check{T}_n.\end{aligned} \qquad [2.3]$$

(usage of θ comes from Granville).

Define the semidivisor counting function to be as follows:

$$\begin{aligned}\xi_d(n) &= \text{A243822}(n) = \text{V51}(n) = |\, \check{Đ}_n \,|\\ &= \theta(n) - \tau(n).\end{aligned} \qquad [2.4]$$

Define the semitotative counting function to be as shown below:

$$\begin{aligned}\xi_t(n) &= \text{A243823}(n) = \text{V61}(n) = |\, \check{S}_n \,|\\ &= n - \phi(n) - \theta(n) + 1.\end{aligned} \qquad [2.5]$$

Since both semidivisors and semitotatives are neither divisors of n nor are coprime to n, we say they are n-neutral. Since primes p must either divide or be coprime to n, $p \notin \check{Đ}_n$ and $p \notin \check{S}_n$. This is to say that composite numbers comprise both $\check{Đ}_n$ and $\check{S}_n$. For this reason we may define the following:

$$\begin{aligned}\Xi_n &= \{\, k : 1 < (k, n) < \text{MIN}(k, n) \wedge k \le n\}.\\ &= \check{Đ}_n \cup \check{S}_n\\ &= \text{row } n \text{ of A133995, i.e., V40.}\end{aligned} \qquad [2.6]$$

$$\begin{aligned}\xi(n) &= \text{A045763}(n) = \text{V41}(n) = |\, \Xi_n \,|\\ &= \xi_d(n) + \xi_t(n)\\ &= n - \phi(n) - \tau(n) + 1.\end{aligned} \qquad [2.7]$$

Application of Counting Functions to Primorials.

Let the primorial $\mathcal{P}(n)$ = A2110(n) be the product of the smallest n primes as shown below:

$$\mathcal{P}(n) = \text{A2110}(n) = \text{V0111}(n) = \prod_{i=1}^{n} \text{PRIME}(n). \qquad [3.1]$$

Primorials are interesting as they minimize $\phi(n)$ and set records in $\theta(n)$. This means to say that primorials represent local minima of Euler's totient, and occur among local maxima of the regular counting function.

Therefore we define the following constitutive counting function sequences applied to primorials:

$$\begin{aligned}\text{A5867}(n) &= \text{V2111}(n) = \phi(\mathcal{P}(n)) = \text{A10}(\text{A2110}(n)),\\ \text{A363061}(n) &= \text{V3111}(n) = \theta(\mathcal{P}(n)) = \text{A010846}(\text{A2110}(n)),\\ \text{A363844}(n) &= \text{V6111}(n) = \xi_t(\mathcal{P}(n)) = \text{A243823}(\text{A2110}(n))\\ &= \text{A2110}(n) - \text{A5867}(n) - \text{A363061}(n) + 1.\end{aligned}$$

We may also regard A363061(n) as the number of PRIME(n)-smooth $k \le \mathcal{P}(n)$.

$$
\begin{aligned}
\text{V5111}(n) &= \xi_d(\mathcal{P}(n)) = \text{A243822}(\text{A2110}(n))\\
&= \text{A010846}(\text{A2110}(n)) - \text{A5}(\text{A2110}(n)).\\
&= \text{A010846}(\text{A2110}(n)) - 2^n.\\
\text{V4111}(n) &= \xi(\mathcal{P}(n)) = \text{A045763}(\text{A2110}(n))\\
&= \xi_d(\mathcal{P}(n)) + \xi_t(\mathcal{P}(n))\\
&= \text{A243822}(\text{A2110}(n)) + \text{A243823}(\text{A2110}(n))\\
&= \text{V5111}(n) + \text{A363844}(n)\\
&= \text{A2110}(n) - \text{A5867}(n) - \text{A5}(\text{A2110}(n)) + 1\\
&= \text{A2110}(n) - \text{A5867}(n) - 2^n + 1.
\end{aligned}
$$

At this time, we have decided that V4111 and V5111 are not of general interest.

The following is a table of the first two dozen terms of each of these counting functions. Keep in mind that $\theta(\mathcal{P}(n))$ is maximized as $\phi(\mathcal{P}(n))$ minimized. This table illustrates the sparsity of the set of regular numbers compared to those of totatives and semitotatives. $\mathcal{P}(4) = 210$ represents a turning point where $\mathcal{P}(n)$-regular numbers no longer dominate the range $\{1 \dots \mathcal{P}(n)\}$, while $\mathcal{P}(5) = 2310$ represents the point where primorials become semitotative-dominant.

n	A2110(n)	A363061(n)	A363844(n)	A5867(n)
0	1	1	0	1
1	2	2	0	1
2	6	5	0	2
3	30	18	5	8
4	210	68	95	48
5	2310	283	1548	480
6	30030	1161	23110	5760
7	510510	4843	413508	92160
8	9699690	19985	8020826	1658880
9	223092870	83074	186514437	36495360
10	6469693230	349670	5447473481	1021870080
11	200560490130	1456458	169902931273	30656102400
12	7420738134810	6107257	6317112341154	1103619686400

Conclusion.

We have explored the application of multiplicatively constitutive counting functions to primorials $\mathcal{P}(n)$, since primorials represent local minima of Euler's totient and appear among local maxima of the regular counting function A010846. We have put in place several sequences that help explore the nature of numbers $k \le \mathcal{P}(n)$.

Note.

In this paper we employ OEIS sequences, which also have V-numbers; we prefer OEIS numbers over V-numbers, but use V-numbers for those sequences that perhaps do not merit entry in OEIS. V-numbers pertain to sequences having to do with constitutive aspects, meaning set theoretic comparisons between sets of prime divisors with multiplicity for nonzero positive integers k and n.

References:

[1] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, retrieved June 2023.

[2] Eric W. Weisstein, "Regular Number", *MathWorld*, retrieved June 2023.

[3] *Wikipedia*, "Regular number", retrieved June 2023.

[4] Michael Thomas De Vlieger, Constitutive Basics, *Simple Sequence Analysis*, 20230125.

Code:

[C1] Efficiently calculate regular counting function $\theta(n)$:

```
rcf[1] = 1;
rcf[n_] :=
 Function[w,
   ToExpression@
      StringJoin["Block[{n = ", ToString@ n,
       ", k = 0}, Flatten@ Table[k++, ",
       Most@ Flatten@ Map[{#, ", "} &, #], "]; k]"] &@
    MapIndexed[
     Function[p,
       StringJoin["{", ToString@ Last@ p, ", 0, Log[",
        ToString@ First@ p, ", n/(",
        ToString@
         InputForm[
          Times @@ Map[Power @@ # &,
          Take[w, First@ #2 - 1]]],
        ")]}"]]@w[[First@#2]] &, w] ]@
    Map[{#,
      ToExpression["p" <> ToString@ PrimePi@ #]} &,
   FactorInteger[n][[All, 1]]]
```

[C2] Generate A363061:

```
a363061 = Map[rcf,
 FoldList[Times, 1, Prime@ Range@ 9] ] ];
```

[C3] Generate A363844:

```
a363844 =
   Array[#2 - EulerPhi[#2] - a363061[[#1 + 1]] + 1 &
   @@ {#, Product[Prime[i], {i, #}]} &, 9, 0]
```

Concerns sequences:

A000005: Divisor counting function $\tau(n)$.
A000010: Euler totient function $\phi(n)$.
A000040: Prime numbers.
A000079: Powers of 2.
A002110: Primorials $\mathcal{P}(n)$.
A005867: $\phi(\mathcal{P}(n)) = \text{A10}(\text{A2110}(n))$.
A010846: Regular counting function $\theta(n)$.
A027750: List of divisors of n, $\boldsymbol{D}_n$.
A038566: List of n-totatives $\check{\boldsymbol{T}}_n = \{\, k : k \perp n \wedge k < n \,\}$.
A133995: List n-neutral $\boldsymbol{\Xi}_n = \{\, k : 1 < (k, n) < \text{MIN}(k, n) \wedge k \le n \}$.
A162306: List of n-regular $\check{\boldsymbol{R}}_n = \{\, k \in \bigotimes_{p|n} \{p^\varepsilon : \varepsilon \ge 0\} : \wedge\, k \le n \,\}$.
A243822: Semidivisor counting function $\xi_d(n)$.
A243823: Semitotative counting function $\xi_t(n)$.
A272618: List n-semidivisors $\check{\boldsymbol{D}}_n = \check{\boldsymbol{R}}_n \setminus \boldsymbol{D}_n$.
A272619: List n-semitotatives $\check{\boldsymbol{S}}_n = \{\, k : k < n \,\} \setminus (\check{\boldsymbol{R}}_n \cup \check{\boldsymbol{T}}_n)$
A363061: $\theta(\mathcal{P}(n)) = \text{A010846}(\text{A2110}(n))$.
A363844: $\xi_t(\mathcal{P}(n)) = \text{A243823}(\text{A2110}(n))$.

Document Revision Record:

2023 0628: Version 1.